

Trivial Colouring and Non-Trivial Colouring for Graph's Edges

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ABSTRACT: In this paper we prove some results of edge's colouring (trivial colouring and nontrivial colouring). We prove some results of edge's colouring for many graphs connected by bridges. In many examples we determine minimum colours to colour edges of the graphs only by calculation when we know number of the edges, number of the vertices and largest degree of vertex, whatever the graph is huge. We used the concept of trivial colouring and nontrivial colouring to prove results of edge's colouring for trees, bipartite graphs and complete graphs.

KEYWORDS: local minimum families of disjoint sets, parallel edges, set labeled by families of disjoint sets, trivial colouring, nontrivial colouring.

I. INTRODUCTION

In paper [1] we introduced the concept of colouring of families of disjoint sets technique as a topic of set theory. We also introduce definitions of degree of intersection, meeting and standard of meeting, minimum number of families of disjoint sets and maximum number of families of disjoint sets. In paper [2] we modify some definitions and examples of paper [1]. In paper [3] we introduce some concepts in graph theory to introduce new method for colouring edges and vertices of graph. This method has two types first related to vertices of the graph and second related to cycles. This method depend on the theory in our paper "Family of disjoint sets and Its Applications", with suitable changes. In paper [4] we prove minimum number of families of disjoint sets with known degree of intersection (minimum number of colours) and we introduce another proof for maximum number of families of disjoint sets (maximum number of colours). Also in that paper we introduce the concept types of families of disjoint sets (types of colouring) and prove some results related to this concept, and we introduce the concept restricted families of disjoint sets (restricted colouring), restricted colouring is colouring in which we add a condition. Suppose we have small identical cubes, we want to colour these cubes such that any two adjacent small cubes coloured by two different colours, and any big cube consists of twenty seven small identical cubes coloured by twenty seven different colours, how many colours will be used. This problem of cubes can be solved by using restricted colouring. In paper[5] we assume both sets and graphs are finite. Results in that paper use colouring of families of disjoint sets technique for finite sets and finite graphs. Using results in that paper we can compare between colouring of families of disjoint sets technique and colouring of graphs in graph theory.

In paper [6] we prove some results in graph theory using colouring of families of disjoint sets technique. We introduced the concept of set labeled by families of disjoint sets and the degree of this set, as generalization to the degree of intersection and proved some results related to this concept. We introduce sorts of families of disjoint sets (sorts of colour classes) as a concept related to set labeled by families of disjoint sets, also we compare between familiar and unfamiliar sorts of families of disjoint sets and explain that by some examples.

In this paper we prove some results of edge's colouring (trivial colouring and nontrivial colouring). We study k graphs each graph is finite and each graph with largest vertex degree n , then these k graphs connected by bridges (cycle or path) such that largest degree n not changed. We used the concept of trivial colouring and nontrivial colouring to prove results of edge's colouring for trees, bipartite graph and complete graph. We solve problem of minimum colours to

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colour edges of the graph only by calculation when we know number of the edges, number of the vertices and largest degree of vertex, whatever the graph is huge, and give many examples.

II. RELATED WORK

In this section we recall some definitions and results related to colouring in graph theory and colouring of families of disjoint sets technique.

Definition 2.1. If we have m families of disjoint sets and we have a set all element of this set represent some of these m families of disjoint sets, such that any two elements of this set represent two different families of disjoint sets and there is a relation between these elements of the set, then we call this set, set labeled by families of disjoint sets [6].

Definition 2.2. Suppose we have m families of disjoint sets and many sets labeled by families of disjoint sets, if maximum number of elements for each set labeled by families of disjoint sets is n where $n < m$, then n is called the degree of set labeled by families of disjoint sets. If $n = m$ then the degree is called trivial degree or trivial case [6].

Proposition 2.3. Suppose we have finite sets with k intersections (these k intersections have different degrees of intersection), and maximum degree of intersection equal n , if $\chi_1, \chi_2, \chi_3, \dots, \chi_m$ any families of disjoint sets any two of them meet each other at these intersections $X_1, X_2, X_3, \dots, X_w$. If the difference in degrees of intersection at intersections

$X_1, X_2, X_3, \dots, X_w$ equal $\frac{n(w-1)}{2}$ for w odd, and equal $\frac{n(w-2)}{2}$ for w even, then m is trivial families of disjoint sets [1].

Theorem 2.4. Minimum number of families of disjoint sets with degree of intersection n equal $n + 1$ [4].

Theorem 2.5. Let $w_1 < w_2$, m_1 is local minimum families of disjoint sets at $X_1, X_2, X_3, \dots, X_{w_1}$ (nonempty intersection each of degree equal n), and m_2 is local minimum families of disjoint sets at $X_1, X_2, X_3, \dots, X_{w_2}$ (nonempty intersection each of degree equal n), $m_1 = \lfloor \frac{nw_1}{w_1-1} \rfloor$ and $m_2 = \lfloor \frac{nw_2}{w_2-1} \rfloor$, where $\lfloor x \rfloor$ is integral part of x , such that $m_1(w_1 - 1) \leq nw_1$, and

$m_2(w_2 - 1) \leq nw_2$, and then $m_1(w_1 - 1) \leq nw_1$ [1].

Remark 2.6. In this paper we mean by different families of disjoint sets, different values of m from relation $\sum_{i=1}^m t_i = wn$,

where n is constant and both w and t_i are variables, and we mean by different types of families of disjoint sets,

different values of m from relation $\sum_{i=1}^m t_i = wn$, where n is variable, both w and t_i can be variables or constant [5].

Proposition 2.7. For the regular graph has k vertices (and k edges) where k is even number then colouring of edges is trivial colouring [6].

Proposition 2.8. For the regular graph has k vertices (and k edges) where k is odd number then colouring of edges is non trivial colouring [6].

Corollary 2.9. The number of colour classes used to colour edges of the graph K_{2n-1} equal $2n - 1$ [6].

Proposition 2.10. If the number of colour classes used to colour edges of the regular graph of degree k , is trivial where $k = 1, 2, 3, \dots$ then each colour class appears $\frac{k}{2}$ times at edges [6].

Definition 2.11. A bridge is an edge whose removal increases the number of components.

Theorem 2.12. The sum of degrees of vertices equals twice the number of edges [12].

Definition 2.13. If the vertex set of a graph G can be split into two disjoint sets A and B so that each edge of G joins a vertex of A and a vertex of B then G is a bipartite graph. Alternatively, a bipartite graph is one whose vertices can be coloured black and white in such a way that each edge joins a black vertex (in A) and a white vertex (in B). A complete bipartite graph is a bipartite graph in which each vertex in A is joined to each vertex in B by just one edge [8].

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Definition 2.14. Let $1, 2, 3, \dots, m$ be m colours used for colouring edges, $v_1, v_2, v_3, \dots, v_w$ be w vertices each of degree n , $1 \leq i \leq m, 1 \leq j \leq m$, then i, j , called two different colours at w vertices if whatever we exchange the numbers $1, 2, 3, \dots, m$, (which labelled the vertices, $v_1, v_2, v_3, \dots, v_w$) there exists a vertex represented (labelled) by i, j . We called i, j same colour class (same colour) if we exchange the numbers $1, 2, 3, \dots, m$ (which labelled the vertices edges, $v_1, v_2, v_3, \dots, v_w$) there exists a vertex not represented (labelled) by i, j [3].

Definition 2.15. Let $\chi_1, \chi_2, \chi_3, \dots, \chi_m$ be m families of disjoint sets, $X_1, X_2, X_3, \dots, X_w$ be w nonempty intersections each of degree n , then m families of disjoint sets is called local minimum families of disjoint sets at w intersections with t families of disjoint sets appear at each intersection, if $m(w-1) + t = nw$ where $0 \leq t < w$ such that $m \neq n$. When $m = n$, then m families of disjoint sets is called local minimum families of disjoint sets at w intersections with t families of disjoint sets appear at each intersection if $m(w-k) + t = nw$ where $0 \leq t < w, k = 2, 3, 4, \dots$ and $m(w-k) \leq nw < m(w-k+1)$ such that $m \neq n$. Minimum of all locals minimum families of disjoint sets called minimum families of disjoint sets [2].

Proposition 2.16. Let $1, 2, 3, \dots, m$ be m colours used for colouring edges at vertices $v_1, v_2, v_3, \dots, v_w$, the colour i appear w_i times at w vertices and the colour j appear w_j time at these w vertices where $1 \leq i \leq m, 1 \leq j \leq m$, then the colour i is different from the colour j at these w vertices if and only if $w + 1 \leq w_i + w_j$ [3].

III. CYCLE OF INTERSECTIONS AND CYCLE OF SETS LABELED BY N SETS

Definition 3.1. If $S_1, S_2, S_3, \dots, S_{k-1}$ are sets (families of disjoint) each labeled by one label such that $S_i \neq S_{i+1}$, where $1 \leq i \leq k-2, k \geq 3$, and any set S_i represent family of disjoint sets χ_i , and if $X_1, X_2, X_3, \dots, X_k$ are sets labeled by l labels, such that $2 \leq l \leq n$, at least there is one of degree n , and X_i labeled by l labels include label of S_{i-1} and include label of S_i , where $2 \leq i \leq k-1, X_1$ labeled by l labels include label of S_1 , and X_k labeled by l labels include label of S_{k-1} , then $X_1, S_1, X_2, S_2, X_3, S_3, \dots, X_{k-1}, S_{k-1}, X_k$ is called a sequence of intersections and sets.

The following definition is generalization of Definition 3.1.

Definition 3.2. If $S_1, S_2, S_3, \dots, S_{k-1}$ are sets (families of disjoint) such that $S_i \neq S_{i+1}$, where $1 \leq i \leq k-2, k \geq 3$, and any set S_i represent family of disjoint sets χ_i , and if $X_1, X_2, X_3, \dots, X_k$ are intersections each of degree l such that $2 \leq l \leq n$, at least there is one of degree n , and X_i intersection of l sets include S_{i-1} and S_i , where $2 \leq i \leq k-1, X_1$ can be intersection of l sets include S_1 , and X_k can be intersection of l sets include S_{k-1} , then $X_1, S_1, X_2, S_2, X_3, S_3, \dots, X_{k-1}, S_{k-1}, X_k$ is called a sequence of sets labeled by n families of disjoint sets or sequence of sets labeled by n labels.

Remark 3.3. From definition of sequence of intersections and sets (families of disjoint sets) two successive sets are different, but two successive intersections no need to be different.

Definition 3.4. A walk of intersections and sets is an alternating sequence $X_1, S_1, X_2, S_2, X_3, S_3, \dots, X_k, S_k$ of intersections and sets where $S_1, S_2, S_3, \dots, S_k$ are sets such that $S_i \neq S_{i+1}, S_k \neq S_1$, and any set S_i represent family of disjoint sets χ_i where $1 \leq i \leq k$, and $X_1, X_2, X_3, \dots, X_k$ are intersections each of degree n such that X_i is intersection of n sets include S_{i-1} and S_i , where $2 \leq i \leq k$ and X_1 is intersection of n sets include S_1 and S_k . The walk of intersections and sets can also denoted by $X_1, X_2, \dots, X_{k-1}, X_k$. A walk of k intersections $S_0, X_1, S_1, X_2, S_2, X_3, S_3, \dots, X_k, S_k$ is called cycle of intersections if any two intersections meet each other at these k intersections, $S_0 = S_k$ and $k \geq 3$.

Definition 3.5. If $S_0, X_1, S_1, X_2, S_2, X_3, S_3, \dots, X_k, S_k$ is a sequence of sets labeled by n families of disjoint sets or sequence of sets labeled by n labels, then these k sets labeled by n labels called cycle of sets labeled by n labels if $S_0 = S_k$ and $k \geq 3$.

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IV. SOME RESULTS OF EDGE'S COLOURING

In this section we prove some results of edge's colouring (trivial colouring and nontrivial colouring).

Remark 4.1. If the degree of vertex v equal n , and these edges incident with the vertex v labeled by $1, 2, 3, \dots, n$. We labeled the vertex v by $(1, 2, 3, \dots, n)$. Let G be a simple graph with largest vertex degree n , and if minimum number of colours to colour edges of the graph G equal $n+1$, (nontrivial case) any vertex labeled by n labels of the set $\{1, 2, 3, \dots, n, n+1\}$. If minimum number of colours to colour edges of the graph G equal n , (trivial case) any vertex labeled by $(1, 2, 3, \dots, n)$ or labeled by labels less than n of the set $\{1, 2, 3, \dots, n\}$.

Corollary 4.2. Let G be a simple graph with even number of vertices, and largest vertex-degree equal n , then minimum number of colours to colour edges of this graph equal n , (it is trivial colouring).

Proof: Let us have regular graph G has t vertices and t edges, where t is even number, each vertex of degree $t-1$, then $G = K_t$. Using Proposition 2.7. then colouring edges of this graph is trivial colouring. From proof of Proposition 2.7. if we remove many edges from the graph G such that largest degree of vertex equal n , labeled by $(1, 2, 3, \dots, n)$, and any vertex less than n labeled by some elements of the set $\{1, 2, 3, \dots, n\}$. Therefore minimum number of colours to colour edges of this graph equal n , (it is trivial colouring).

Proposition 4.3. If we take any three vertices of maximum degree in the graph G , and if it include a vertex of degree one, then colouring edges of this graph is trivial colouring.

Proof: Using Proposition 2.1. the rule $\sum_{i=1}^m t_i = wn$ can be written as $\sum_{i=1}^m t_i = \sum_{j=1}^w d_j$, where d_j is degree of the vertex v_j , we suppose the graph is nontrivial colouring and degree of colouring equal n , for w odd we take three vertices, two degree are given, $d(v_1) = n$, $d(v_2) = 1$, and suppose the third is maximum, let $d(v_3) = n$, since we suppose it is nontrivial and using $m(\frac{w+1}{2}) \leq wn$, we have $2(n+1) \leq \sum_{i=1}^{n+1} t_i = \sum_{j=1}^3 d_j = n+2$ a contradiction, therefore the graph is trivial colouring.

Corollary 4.4. Colouring edges of the graph G which consists one vertex of degree n and n vertices each of degree one, is trivial colouring.

Proof: If we take three vertices, two degree are given, $d(v_1) = n$, $d(v_2) = 1$, $d(v_3) = 1$, using Proposition 4.3. therefore the graph is trivial colouring.

The following proposition has been proved in paper [6], here we using Definition 2.15. to prove this proposition.

Remark 4.5. Each colour class represented by an edge or number of edges, we label edges at vertices, all labels equal the sum of all degrees of vertices, since sum of degrees of vertices equals twice the number of edges, therefore a colour class represented by even number of labels.

Proposition 4.6. For the regular graph has k vertices (and k edges) where k is even number, then minimum number to colour edges of this graph is trivial colouring.

Proof: Suppose minimum number to colour edges of this graph is nontrivial colouring, means $m = k$, and $w = k$, from Definition 2.15. we have $k(k-1) \leq k(k-1)$, where $w-1 = k-1$, and $w-1$ should be even number, it is odd, a contradiction. Therefore minimum number to colour edges of this graph is trivial colouring.

Proposition 4.7. For the regular graph has k vertices (and k edges) where k is number number, then minimum number to colour edges of this graph is nontrivial colouring.

Proof: Suppose minimum number to colour edges of this graph is trivial colouring, means $m = k-1$, and $w = k$, then sum of all labels of vertices equal $k(k-1)$, since m is number of colour classes, let $m = 2$ we have to label all vertices by $(1, 2)$, since k is odd it is contradiction, because two edges labeled by 1 can not be adjacent or two edges labeled by

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2 can not be adjacentor Definition 2.15. we have $k(k-1) \leq k(k-1)$, where $w-1 = k-1$, and $w-1$ should be even number, it is odd, a contradiction. Therefore minimum number to colour edges of this graph is trivial colouring.

Proposition 4.8. For the regular graph has k vertices (and k edges) where k is odd number, then minimum number to colour edges of this graph is nontrivial colouring.

Proof: : Suppose minimum number of colours to colour edges of this graph is trivial colouring, let m be minimum number of colours to colour edges of this graph, since it is trivial colouring every vertex will be labeled by $(1, 2, 3, \dots, m)$, suppose $m = 2$, then all vertices be labeled by $(1, 2)$, since the number of edges is odd, there exists two adjacent edges labeled by 1, or there exists two adjacent edges labeled by 2, it is a contradiction, because k is not multiple of m . Suppose $m = 3$, then all vertices be labeled by $(1, 2, 3)$, since the number of edges is odd and equal k , we labeled by $1, 2, 1, 2, 1, 2, \dots$ till the edge number $k-1$, then last one labeled by 3, it is a contradiction if we labeled by $1, 2, 3, 1, 2, 3, 1, 2, 3, \dots$ then all vertices be labeled by $(1, 2), (2, 3), (3, 1), \dots$ it is a contradiction. Therefore it is a contradiction for case m is even and m is odd, if m is even, m can not be factor of odd number, if m is odd, m can not be a multiple of $m-1$.

V. SYCLE OF FINITE GRAPHS

In this section we will study k graphs each graph is finite and each graph with largest vertex degree n , then these k graphs connected by bridges (cycle or path) such that largest degree n not changed.

Proposition 5.1. If we have k finite graphs $G_1, G_2, G_3, \dots, G_k$, each graph with largest vertex degree n , then number of colour to colour edges of the graph G_i is trivial colouring (equal n), where $1 \leq i \leq k$, and these k finite graphs connected by edges without changing largest degree n , where $n \geq 3$, then colouring edges of this cycle of k graphs and k edges (k bridges) is trivial colouring.

Proof: Suppose k is even, for the cycle of k graphs there are k edges connected these k graphs. If we labeled these edges by $1, 2, 1, 2, 1, 2, \dots$, for all i , where $1 \leq i \leq k$, every an edge labeled by $3, 4, 5, \dots, n-1, n$, and belongs to G_i is adjacent only to an edge belongs to G_i itself, then all vertices of degree n can be labeled by $(1, 2, 3, \dots, n)$, therefore colouring edges of this cycle is trivial colouring. Suppose k is odd, for the cycle of k graphs there are k edges connected these k graphs. If we labeled these edges by $1, 2, 1, 2, 1, 2, \dots, 1, 2, 3$, for all i , where $1 \leq i \leq k$, every an edge labeled by $4, 5, 6, \dots, n-1, n$, and some edges labeled by 3, belongs to G_i is adjacent only to an edge belongs to G_i itself, then all vertices of degree n can be labeled by $(1, 2, 3, \dots, n)$, therefore colouring edges of this cycle is trivial colouring.

Proposition 5.2. If we have k finite graphs $G_1, G_2, G_3, \dots, G_k$, each graph with largest vertex degree n , then number of colour to colour edges of the graph G_i is trivial colouring (equal n), where $1 \leq i \leq k$, and these k finite graphs connected by edges without changing largest degree n , where $n \geq 3$, then minimum number of colours to colour edges of this path of k graphs and $k+1$ edges (bridges) is trivial colouring (equal n).

Proof: Suppose we have a path of k graphs, (for all k odd or even) there are $k+1$ edges connected these k graphs. If we labeled these edges by $1, 2, 1, 2, 1, 2, \dots$, for all i , where $1 \leq i \leq k$, every an edge labeled by $3, 4, 5, \dots, n-1, n$, and belongs to G_i is adjacent only to an edge belongs to G_i itself, then all vertices of degree n can be labeled by $(1, 2, 3, \dots, n)$, therefore colouring edges of this cycle is trivial colouring.

Proposition 5.3. If we have k finite graphs $G_1, G_2, G_3, \dots, G_k$, each graph with largest vertex degree n , the minimum number of colour to colour edges of the graph G_i is nontrivial colouring (equal $n+1$), where $1 \leq i \leq k$, and these k finite graphs connected by edges without changing largest degree n , where $n \geq 3$, then colouring edges of this cycle of k graphs and k edges is nontrivial colouring.

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Proof: Since each graph G_i with largest vertex degree n , vertex degree remain unchanged, and number of colour to coloured edges of the graph G_i is nontrivial colouring (equal $n + 1$), the proof is same as in proof Proposition 5.1. only changing n by $n + 1$, therefore colouring edges of this cycle equal $n + 1$.

Proposition 5.4. If we have k finite graphs $G_1, G_2, G_3, \dots, G_k$, each graph with largest vertex degree n , then number of colour to coloured edges of the graph G_i is nontrivial colouring (equal $n + 1$), where $1 \leq i \leq k$, and these k finite graphs connected by edges without changing largest degree n , where $n \geq 3$, then colouring edges of this path of k graphs and $k + 1$ edges is nontrivial colouring.

Proof: Same proof of Proposition 5.3. only changing n by $n + 1$, therefore colouring edges of this cycle equal $n + 1$.

Corollary 5.5. Colouring edges of the tree is trivial colouring.

Proof: We consider a tree a path of k finite graphs $G_1, G_2, G_3, \dots, G_k$, each graph with largest vertex degree n , each graph G_i consists one vertex of degree n and n vertices each of degree one, using Corollary 4.4. graph G_i is trivial colouring, for all i , where $1 \leq i \leq k$, using Proposition 5.3. Colouring edges of the tree is trivial colouring.

Corollary 5.6. If removal of one bridge or two bridges from a graph G does not change value of largest degree of vertex n , where $n \geq 3$, and if colouring edges of the graph G after removal of one bridge or two bridges equal n , then colouring edges of the graph G before removal equal n .

Proof: Suppose we have one graph G with largest degree of vertex n and two edges, colouring edges of the graph G after removal of one bridge or two bridges equal n , using Proposition 5.2. then colouring edges of the graph G before removal equal n .

Corollary 5.7. If removal of one bridge or two bridges from a graph G does not change value of largest degree of vertex n , where $n \geq 3$, and if colouring edges of the graph G before removal of one bridge or two bridges equal $n + 1$, then colouring edges of the graph G after removal equal $n + 1$.

Proof: Suppose we have one graph G with largest degree of vertex n and two edges, colouring edges of the graph G after removal of one bridge or two bridges equal $n + 1$, using Proposition 5.4. then colouring edges of the graph G before removal equal $n + 1$.

Remark 5.8. Removal and addition of any number of bridges in a graph G whose does not change value of largest degree of vertex n , where $n \geq 3$, and does not change value of minimum colours to coloured edges of this is graph G .

Proposition 5.9. If we have k finite graphs $G_1, G_2, G_3, \dots, G_k$, each graph with largest vertex degree n , the minimum number of colours to coloured edges of the graph G_i is trivial colouring (equal n), where $1 \leq i \leq k$, and these k finite graphs connected by edges changing largest degree from n to $n + 1$, where $n \geq 2$, then minimum colours to coloured edges of this cycle (or path) of k graphs and edges (bridges) equal $n + 1$.

Proof: The largest vertex degree becomes $n + 1$ after connecting the bridge, we have only two bridges connected to two vertices each of degree equal $n + 1$, since all vertices (except the two bridges), labeled by one of the set $\{1, 2, 3, \dots, n\}$ and the two bridges are adjacent, we can label any bridge by the label $n + 1$, therefore the minimum colours to coloured edges of this cycle (or path) of k graphs and edges (bridges) equal $n + 1$.

Proposition 5.10. If we have k finite graphs $G_1, G_2, G_3, \dots, G_k$, each graph with largest vertex degree n , then number of colour to coloured edges of the graph G_i is nontrivial colouring (equal $n + 1$), where $1 \leq i \leq k$, and these k finite graphs connected by edges (bridges), changing largest degree from n to $n + 1$, where $n \geq 2$, then minimum colours to coloured edges of this cycle (or path) of k graphs and edges (bridges) equal $n + 1$.

Proof: Before connecting the two bridges, minimum colours to coloured edges of each graph equal $n + 1$, after connecting the two bridges, every edge labeled by $n + 1$ is not adjacent to any one of the two bridges, therefore the minimum colours to coloured edges of this cycle (or path) of k graphs and edges (bridges) equal $n + 1$.

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Corollary 5.11. In a graph G if we add a bridge to two graphs whose change value of largest degree of vertex from n to $n + 1$ where $n \geq 2$, then minimum number of colours to colour edges of these two graphs and bridge equal $n + 1$.

Proof: First suppose minimum number of colours to colour edges of this graph before addition of the bridge equal n , using Proposition 5.9. therefore minimum number of colours to colour edges of this graph equal $n + 1$. Second suppose minimum number of colours to colour edges of this graph before addition of any bridge equal $n + 1$, using Proposition 5.10. therefore minimum number of colours to colour edges of this graph equal $n + 1$.

We can prove Corollary 5.11. without using Proposition 5.9. and Proposition 5.10, and introduce the corollary as a proposition.

Proposition 5.12. In a graph G if we add a bridge to two graphs whose change value of largest degree of vertex from n to $n + 1$ where $n \geq 2$, then minimum number of colours to colour edges of these two graphs and bridge equal $n + 1$.

Proof: First suppose minimum number of colours to colour edges of this graph before addition of the bridge equal n , since it is trivial colouring, at vertex of degree n the edges labeled by $1, 2, 3, \dots, n$ and every edge adjacent to the bridge, so the bridge labeled by $n + 1$, therefore minimum number of colours to colour edges of this graph equal $n + 1$. Second suppose minimum number of colours to colour edges of this graph before addition of the bridge equal $n + 1$, since before addition of bridge, suppose at vertex v of degree n there are n edges labeled by $1, 2, 3, \dots, n$ and the edge labeled by $n + 1$ not incident at v , after connecting the bridge to this vertex v the degree of v become $n + 1$ and the bridge adjacent only to the n edges labeled by $1, 2, 3, \dots, n$ and not adjacent to the edge labeled by $n + 1$, so the bridge can be labeled by $n + 1$, therefore minimum number of colours to colour edges of this graph equal $n + 1$.

Corollary 5.13. In a graph G if a bridge connected to two vertices one of them is largest degree of vertex and it is equal n , then minimum colours to colour all edges of the graph G equal n .

Proof: Suppose we remove the bridge in the graph G and we have two graphs G_1, G_2 , the largest degree of vertex equal $n - 1$, in both G_1, G_2 , if $n - 1$ is trivial colouring, using Proposition 5.9. we have minimum colours to colour all edges of the graph G equal n . If $n - 1$ is nontrivial colouring, using Proposition 5.10. we have minimum colours to colour all edges of the graph G equal n .

Example 5.14. Suppose we have two separated triangles connected only by a bridge, the degree of each vertex equal two, $n = 2$, minimum colours to colour all edges of each triangle is three ($n + 1 = 3$), after connecting the two triangles by a bridge minimum colours to colour all edges of the two triangles and the bridge equal three ($n + 1 = 3$), which satisfies Proposition 5.12.

Example 5.15. Suppose we have two separated graphs G_1 and G_2 , the where equal $G_1 = G_2 = K_4$, minimum colours to colour all edges of K_4 equal three, all vertices of K_4 are of degree three, colouring edges of K_4 is trivial colouring. If we connect G_1 and G_2 by a bridge, the largest degree become four, from Corollary 5.11. minimum colours to colour all edges of G_1 and G_2 after be connected by a bridge equal four.

Example 5.16. Suppose we have two separated graphs G_1 and G_2 , the where equal $G_1 = G_2 = K_5$, minimum colours to colour all edges of K_5 equal five, all vertices of K_5 are of degree four, colouring edges of K_5 is nontrivial colouring. If we connect G_1 and G_2 by a bridge, the largest degree become five, from Corollary 5.11. minimum colours to colour all edges of G_1 and G_2 after be connected by a bridge equal five.

Corollary 5.17. If the largest degree of vertex connected to bridge then minimum colours to colour all edges of this graph is trivial colouring.

Proof: Let G_1 be a graph, its vertex of largest degree equal n , this vertex connected to a bridge, largest degree become $n + 1$, using Proposition 5.12. then minimum number of colours to colour edges of G_1 and the bridge equal $n + 1$. Since the largest degree equal $n + 1$, and minimum number of colours to colour edges of G_1 and the bridge equal $n + 1$, therefore it is trivial colouring.

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Corollary 5.18. In the graph G if the number of vertices is even then minimum colours to colour all edges of the graph is trivial colouring.

Proof: If the graph G has a bridge connected to the vertex of largest degree, then from Corollary 5.17. minimum colours to colour all edges of the graph is trivial colouring. If the graph G has no bridge, let G be a graph which consists of k vertices, where k is even number, and largest degree of vertex equal n , and all vertices of degree equal n , then using Proposition 4.2. colouring edges of the graph G is trivial colouring. If G has k vertices, of different degrees and largest degree n , we draw a graph G_1 which consists of k vertices, where k is even number, and largest degree of vertex equal n , and all vertices of degree equal n , all vertices in graph G_1 labeled by $(1, 2, 3, \dots, n)$, if the degree equal t , where equal $t < n$, and then the vertex labeled by t elements of the set $\{1, 2, 3, \dots, n\}$, therefore minimum colours to colour all edges of the graph is trivial colouring.

VI. GRAPHS WITH ODD NUMBER OF VERTICES

In this section we used the concept of parallel edges to determine number of edges belong to one colour class, we used number of the vertices and largest degree of vertex, whatever the graph is huge.

Definition 6.1. Two edges are said to be parallel if have no common vertex. Two edges are parallel even if they intersect each other.

Remark 6.2. In a graph any number of edges belong to one colour class are parallel edges.

Remark 6.3. From Corollary 2.19. the number of colour classes used to colour edges of the graph K_{2n-1} equal $2n-1$, and then each vertex of degree equal $2n-2$, and each colour class appears $\frac{2n-2}{2}$ times at edges. Suppose we have regular graph has k vertices

(and k edges) where k is odd number then colouring of edges is nontrivial colouring, equal k , each vertex of degree equal $k-1$, and each colour class appears $\frac{k-1}{2}$ times (there are $\frac{k-1}{2}$ parallel edges). From Proposition 2.10. the number of colour classes used to

colour edges of the graph K_{2n} equal $2n-1$, and then each vertex of degree equal $2n-1$, and each colour class appears $\frac{2n}{2} = n$

times (there are n parallel edges). Suppose we have regular graph has k vertices (and k edges) where k is even number then colouring of edges is trivial colouring, equal $k-1$, each vertex of degree equal $k-1$, and each colour class appears $\frac{k}{2}$ times (there

are $\frac{k}{2}$ parallel edges).

Example 6.4. Suppose we have graph G with five vertices, $d(v_1) = 4$, $d(v_2) = 3$, $d(v_3) = 3$, $d(v_4) = 2$, $d(v_5) = 2$, this graph consists of seven edges and five vertices, from the graph K_5 each colour class appears $\frac{5-1}{2}$ times (there are two parallel edges),

number of colour classes equal $\frac{7}{2}$, from greatest degree of vertex, number of colour classes not less than four. Therefore we have four colour classes.

Example 6.5. Suppose we have graph G with five vertices, $d(v_1) = d(v_2) = d(v_3) = d(v_4) = 3$, $d(v_5) = 2$, this graph consists of seven edges and five vertices, from the graph K_5 each colour class appears $\frac{5-1}{2}$ times (there are two parallel edges),

number of colour classes equal $\frac{7}{2}$, from greatest degree of vertex, number of colour classes not less than three. Therefore we have four colour classes.

Remark 6.6. To check Example 6.5. we use Definition 2.14, we label the four colour classes by 1, 2, 3, 4, the three colour classes labeled by 1, 2, 3 each appear four times at these five vertices, and the colour class labeled by 4 appear twice at these five vertices. Therefore these four colour classes any two meet each other at these five vertices.

Example 6.7. Suppose we have graph G with five vertices, $d(v_1) = d(v_2) = d(v_3) = 2$, $d(v_4) = d(v_5) = 3$, this graph consists of six edges and five vertices, from the graph K_5 each colour class appears $\frac{5-1}{2}$ times (there are two parallel edges), number

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of colour classes equal $\frac{6}{2}$, from greatest degree of vertex, number of colour classes not less than three. Therefore we have three colour classes.

Remark 6.8. To check Example 6.7. we use Definition 2.14, we label the three colour classes by 1, 2, 3, the three colour classes labeled by 1, 2, 3 each appear four times at these five vertices. Therefore these three colour classes any two meet each other at these five vertices.

Example 6.9. Suppose we have graph G with nine vertices, four vertices of degree five, and five vertices of degree four. This graph consists of 20 edges and nine vertices, from the graph K_9 , each colour class appears $\frac{9-1}{2}$ times (there are four parallel edges),

number of colour classes equal $\frac{20}{4}$, from greatest degree of vertex, number of colour classes not less than five. Therefore we have five colour classes.

Remark 6.10. To check Example 6.9. we use Definition 2.15, we label the five colour classes by 1, 2, 3, 4, 5, the sum of vertex degrees equal 40, the five colour classes each appear eight times at these nine vertices. Therefore these five colour classes any two meet each other at these nine vertices.

Example 6.11. Suppose we have graph G with nine vertices, six vertices of degree five, and three vertices of degree four. This graph consists of 21 edges and nine vertices, from the graph K_9 , each colour class appears $\frac{9-1}{2}$ times (there are four parallel edges),

number of colour classes equal $\frac{21}{4}$, from greatest degree of vertex, number of colour classes not less than five. Therefore we have six colour classes.

Remark 6.12. To check Example 6.11. we use Definition 2.15, the six colour classes labeled by 1, 2, 3, 4, 5, 6, the sum of vertex degrees equal 42, four colour classes each appear eight times at these nine vertices, one colour class appear six times at these nine vertices, and one colour class appear four times at these nine vertices. Therefore these six colour classes any two meet each other at these nine vertices.

We will use definition of parallel edges to prove the two following propositions.

Example 6.13. Suppose we have graph G with 19 vertices, 91 edges, largest degree of vertex equal of 10, to find minimum number of colour classes to colour edges of the graph, we use from Definition 2.15. $m(w-1)+t = nw$ where $0 \leq t < w$ such that $m \neq n$, m is minimum number of colour classes to colour edges of the graph w is number of vertices, n is largest degree of vertex, nw is sum of all degrees of vertices, therefore $m = 11$, 9 colour classes each its labels appear at 18 vertices, and 2 colour classes each its labels appear at 10 vertices. Using Proposition 2.16. any two colour classes meet each other at these 19 vertices because their appearance at these 19 vertices more than 19. Table (1) explain the colour of these 91 edges, where these eleven colours labeled by 1, 2, 3, ..., 11.

The colour label	Number of vertices in this colour
1	9
2	9
3	9
4	9
5	9
6	9
7	9
8	9
9	9
10	5
11	5

Table (1)

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Example 6.14. Suppose we have graph G with 113 vertices, largest degree of vertex equal of 20, find minimum number of edges in this graph such that minimum number of colour classes to colour edges of this graph equal 21, and find maximum number of edges in this graph such that minimum number of colour classes to colour edges of this graph equal 21.

Solution: From Definition 2.15. $m(w-1)+t=nw$ where $0 \leq t < w$ such that $m \neq n$, m is minimum number of colour classes to colour edges of the graph, w is number of vertices, $w = 113$, n is largest degree of vertex $n = 20$, wn is sum of all degrees of vertices, sum of all degrees of vertices equal number of colour classes multiple by maximum number of edges have same colour (maximum number of parallel edges). Here maximum number of parallel edges equal $\frac{w-1}{2} = 56$, sum of all degrees of vertices such that there are minimum number of edges for 21 colour classes equal $(20)(56)(2) + (1)(1)(2) = 2242$, minimum number of edges equal $\frac{2242}{2} = 1121$, and maximum number of edges equal $\frac{(21)(56)(2)}{2} = 1176$.

Remark 6.15. Number of edges of the graphs equal maximum number of parallel edges (maximum number of edges belong to one colour class) multiple by minimum number of colour classes to colour edges of the graph.

Proposition 6.16. Number of minimum colour classes to colour edges of the graph K_k , where k is even, equal $k-1$.

Proof: Number of vertices is even and equal k , maximum number of parallel edges equal $\frac{k}{2}$, (maximum number of edges belong to one colour class equal $\frac{k}{2}$), number of edges of the graph K_k equal $\frac{k(k-1)}{2}$, if we divide $\frac{k(k-1)}{2}$ by $\frac{k}{2}$ equal $k-1$. Therefore minimum colour classes to colour edges equal $k-1$.

Proposition 6.17. Number of minimum colour classes to colour edges of the graph K_k , where k is odd, equal k .

Proof: Number of vertices is odd and equal k , maximum number of parallel edges equal $\frac{k-1}{2}$, (maximum number of edges belong to one colour class equal $\frac{k-1}{2}$), number of edges of the graph K_k equal $\frac{k(k-1)}{2}$, if we divide $\frac{k(k-1)}{2}$ by $\frac{k-1}{2}$ equal k . Therefore minimum colour classes to colour edges equal k .

Proposition 6.18. If $K_{r,s}$ is bipartite graph where $r < s$, then maximum number of parallel edges equal r .

Proof: From definition of bipartite graph and definition of parallel edges, we have maximum number of parallel edges equal r .

Proposition 6.19. If G is bipartite graph with largest vertex-degree s then minimum colour classes to colour edges of this graph equal s .

Proof: Let $K_{r,s}$ be bipartite graph where $r < s$, then largest vertex-degree equal s , number of edges of the graphs equal rs , maximum number of parallel edges equal r (maximum number of edges belong to one colour class equal r).

Therefore minimum colour classes to colour edges equal $\frac{rs}{r} = s$.

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