

# Lyapunov-type Inequality and Eigen value Analysis for a Fractional Problem of order $\alpha$

Nimisha Pathak

Assistant Professor, Department of Applied Mathematics, The M. S. University of Baroda, Vadodara, Gujarat, India

**ABSTRACT:** In this work, we establish a Lyapunov-type inequality for a Riemann-Liouville fractional boundary value problem for  $2 < \alpha \leq 3$ , where  $\alpha$  is the order of the derivative. We consider both integer and fractional order eigenvalue problems, determine a lower bound for the smallest eigenvalue using a Lyapunov-type inequality, and improve this bound using a semi maximum norm and Cauchy-Schwarz inequality. We use the improved lower bounds to obtain intervals where a certain Mittag-Leffler function has no real zeros. Further, for both the fractional and the integer order eigenvalue problems, we give a comparison between the smallest eigenvalue and its lower bounds obtained from the semi maximum norm and Lyapunov-type and Cauchy-Schwarz inequalities. Results show that the Lyapunov-type inequality gives the worse and the Cauchy-Schwarz inequality gives the best lower bound estimates for the smallest eigenvalues.

**KEYWORDS:** Lyapunov inequality, fractional derivative, fractional Green's function, Mittag-Leffler function.

## I. INTRODUCTION

The Lyapunov inequality was first obtained by Lyapunov in [1] which has proved to be very useful in the study of spectral properties of ordinary differential equations [2][3]. This inequality can be stated as follows:

**Theorem 1.1** (See [1]) A necessary condition for the boundary value problem (BVP)

Problem P1:

$$\begin{aligned} u''(t) + q(t)u(t) &= 0, a < t < b \\ u(a) = 0, u(b) &= 0 \end{aligned} \tag{1.1}$$

to have nontrivial solutions is that

$$\int_a^b |q(s)| ds > 4/(b-a), \tag{1.2}$$

where  $q$  is a real and continuous function. The constant 4 in equation (1.2) is sharp so that it cannot be replaced by a larger number. There are several generalizations and extensions of Theorem 1.1. Hartman and Wintner [4] proved that if  $u$  is a nontrivial solution to equations (1.1)-(1.2) then

$$\int_a^b (b-s)(s-a)q^+(s)ds > b-a$$

where  $q^+(s)$  is the positive part of  $q(s)$  defined as  $q^+(s) = \max\{q(s), 0\}$ . We call the above inequality as Hartman and Wintner inequality.

Recently, research on Lyapunov Type Inequalities (LTIs) for Fractional Boundary Value Problems (FBVPs) has begun. Ferreira [5], [6], Jleli and Samet [7], Rong and Bai [8], Pathak [9], [10] have established Lyapunov-type inequalities for FBVPs of order  $\alpha$ ,  $\alpha \in (1, 2]$  and different boundary conditions. Motivated by the above work, in this paper we consider a FBVP involving a Riemann-Liouville derivative of order  $\alpha$ ,  $\alpha \in (2, 3]$  and obtain a Lyapunov type inequality for it. Specifically, we consider the following problem:

$$\begin{aligned} (D_{a+}^{\alpha} u)(t) + q(t)u(t) &= 0, a < t < b \\ u(a) = 0, u'(a) = 0, u'(b) &= 0, \end{aligned} \tag{1.3}$$

# International Journal of Innovative Research in Science, Engineering and Technology

(A High Impact Factor, Monthly, Peer Reviewed Journal)

Visit: [www.ijirset.com](http://www.ijirset.com)

Vol. 8, Issue 2, February 2019

where  $q: [a, b] \rightarrow \mathbb{R}$  is a continuous function,  $D_{a+}^{\alpha}$  is the standard Riemann-Liouville differential operator defined later. We also find lowerbound estimates for the smallest eigenvalue of a fractional eigenvalue problem obtained from problem P3.

The outline of the paper is as follows. First, for the FBVP P2 we derive the Green's function and use it a) to reduce problem P2 to an equivalent Fredholm integral equation of the second kind and b) to establish a Lyapunov-type inequality. Second, we consider a Fractional Eigenvalue Problem (FEP) and determine a lowerbound for the first eigenvalue from Lyapunov-type inequality. Third, we improve the lower bound for the first eigenvalue using a Semi Maximum Norm (SMN) and a Cauchy-Schwarz Inequality (CSI). We compare the smallest eigenvalues and their lower bounds obtained from the semi maximum norm, Lyapunov-type and Cauchy-Schwarz inequalities for fractional as well as integer order systems. Finally, to demonstrate an application of the inequalities developed in the paper, we apply the improved bounds for the smallest eigenvalues to obtain intervals where a certain Mittag-Leffler function has no real zero.

## II. BACKGROUND MATERIALS AND PRELIMINARIES

Considerable work has been done on fractional calculus in recent years. Much of the tools from fractional calculus necessary for this paper could be found, among others, in [11]-[14]. To make the notations uniform and this paper self-contained, we briefly review here the definitions of Riemann-Liouville fractional operators and some preliminary materials.

**Definition 2.1** The Riemann-Liouville fractional derivative operator ( $D_{a+}^{\alpha}u$ ) of order  $\alpha > 0$  of a continuous function  $f: (a, \infty) \rightarrow \mathbb{R}$ , is given by

$$I_{a+}^{\alpha} = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds \tag{2.1}$$

provided that the right side is pointwise defined on  $(a, \infty)$ . Here  $\Gamma(\alpha)$  denotes the Euler's gamma function.

**Definition 2.2** The Riemann-Liouville fractional derivative operator ( $D_{a+}^{\alpha}$ ) of order  $\alpha > 0$  of a continuous function  $f: (a, \infty) \rightarrow \mathbb{R}$ , is given by

$$D_{a+}^{\alpha} f(t) = D^n I_{a+}^{n-\alpha} f(t), D \equiv \frac{d}{dt}, \tag{2.2}$$

where  $n = [\alpha] + 1$ , provided that the right side is pointwise defined on  $(a, \infty)$ .

**Definition 2.3** The one and two-parameter Mittag-Leffler functions are defined respectively by

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \alpha, z \in \mathbb{R}, \alpha > 0 \tag{2.3}$$

and

$$E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \alpha, \beta, z \in \mathbb{R}, \alpha, \beta > 0 \tag{2.4}$$

Note that  $E_{\alpha, 1}(z) = E_{\alpha}(z)$  and  $E_{1, 1}(z) = E_1(z) = \exp(z)$ .

**Definition 2.4** The Laplace transform of the function  $f(t), t \in (0, \infty)$  is defined by  $(\mathcal{L}f)(s) = \int_a^{\infty} e^{-st} f(t) dt, s \in \mathbb{C}$ .

If the above integral is convergent at a point  $s_0 \in \mathbb{C}$ , then it converges absolutely for  $s \in \mathbb{C}, \mathbb{R}(s) > \mathbb{R}(s_0)$ . Here  $\mathcal{L}$  denotes the Laplace transform operator.

Using definition 2.4, the Laplace transform of the function  $\phi(t) = t^{\beta-1} E_{\alpha, \beta}(\pm \lambda t^{\alpha})$  is given as

$$(\mathcal{L}\phi)(s) = \frac{s^{-\alpha-\beta}}{s^{\alpha} \mp \lambda}, \mathbb{R}(s) > 0, \lambda \in \mathbb{C}, |\lambda s^{-\alpha}| < 1,$$

and its inverse relationship is given as

$$\mathcal{L}^{-1} \left[ \frac{s^{-\alpha-\beta}}{s^{\alpha} \mp \lambda} \right] = t^{\beta-1} E_{\alpha, \beta}(\pm \lambda t^{\alpha}), \mathbb{R}(s) > 0, \lambda \in \mathbb{C}, |\lambda s^{-\alpha}| < 1, \tag{2.5}$$

where  $\mathcal{L}^{-1}$  is the inverse Laplace transform operator. Further, function  $t^{\beta-1} E_{\alpha, \beta}(\pm \lambda t^{\alpha})$

satisfies the following property [11],

$$\frac{d^n}{dt^n} [t^{\beta-1} E_{\alpha, \beta}(\pm \lambda t^{\alpha})] = t^{\beta-n-1} E_{\alpha, \beta-n}(\pm \lambda t^{\alpha}). \tag{2.6}$$

# International Journal of Innovative Research in Science, Engineering and Technology

(A High Impact Factor, Monthly, Peer Reviewed Journal)

Visit: [www.ijirset.com](http://www.ijirset.com)

Vol. 8, Issue 2, February 2019

The results given below in Lemmas 2.1 and 2.2 are standard within the fractional calculus theory involving Riemann-Liouville differential operator (see [6], [14], [15]).

**Lemma 2.1.** Let  $\alpha > 0$ . If  $u \in C(a, b) \cap L(a, b)$ , then the differential equation

$$D_{a+}^{\alpha} u(t) = 0 \tag{2.7}$$

has a general solution

$$u(t) = c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2} + \dots + c_n(t-a)^{\alpha-n}, c_i \in \mathbb{R}, i = 1, 2, \dots, n, \tag{2.8}$$

where  $n$  is the smallest integer greater than or equal to  $\alpha$ , and  $L(a, b)$  and  $C(a, b)$  are spaces of all Lebesgue integrable and continuously differentiable functions defined on  $(a, b)$ .

From Lemma 2.1, we deduce the following statement:

**Lemma 2.2.** Let  $\alpha > 0$ . Assume that  $u \in C(a, b) \cap L(a, b)$ , then

$$I_{a+}^{\alpha} D_{a+}^{\alpha} u(t) = u(t) - c_1(t-a)^{\alpha-1} - c_2(t-a)^{\alpha-2} - \dots - c_n(t-a)^{\alpha-n}, \tag{2.9}$$

for some  $c_i \in \mathbb{R}, i = 1, 2, \dots, n$ .

### III. MAIN RESULTS

The main results of this paper are given in Theorems 3.1, 3.2 and Corollary 3.1. To prove these results, we begin by writing problem P2 in an equivalent Fredholm integrodifferential equation of second kind.

**Lemma 3.1.** If  $q \in C[a, b]$  and  $2 < \alpha \leq 3$ , then the FBVP P2 can be written as

$$u(t) = \int_a^b G(t, s)q(s)u(s)ds, \tag{3.1}$$

where

$$G(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-2}}{(b-a)^{\alpha-2}} - (t-s)^{\alpha-1}, & a \leq s \leq t \leq b, \\ \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-2}}{(b-a)^{\alpha-2}}, & a \leq t \leq s \leq b, \end{cases} \tag{3.2}$$

is the Green's function for the problem.

**Proof.** From Lemma 2.2, we write equation (1.3) in an equivalent integral form:

$$u(t) = c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2} + c_3(t-a)^{\alpha-3} - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} q(s)u(s)ds, \tag{3.3}$$

where  $c_1, c_2$  and  $c_3$  are some real constants. Since  $u(a) = 0$ , we get  $c_3 = 0$ . Taking time derivative of equation (3.3), and setting  $u'(a) = u'(b) = 0$ , we obtain  $c_2 = 0$  and  $c_1 = \frac{1}{\Gamma(\alpha)(b-a)^{\alpha-2}} \int_a^b (b-s)^{\alpha-2} q(s)u(s)ds$ , and equation (3.3) becomes

$$u(t) = \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)(b-a)^{\alpha-2}} \int_a^b (b-s)^{\alpha-2} q(s)u(s)ds - \frac{1}{\Gamma(\alpha)} \int_a^t (b-s)^{\alpha-1} q(s)u(s)ds,$$

which can be written as equation (3.1) with  $G(t, s)$  given by equation (3.2). This concludes the proof.

Next, we give three properties of the Green's function  $G(t, s)$  defined in equation (3.2).

**Lemma 3.2.** The Green's function  $G$  defined by equation (3.2) satisfies the following properties:

1.  $G(t, s)$  is a continuous function on  $[a, b] \times [a, b]$ .
2.  $G(t, s) \geq 0$  for all  $a \leq t \leq s \leq b$ .
3.  $\max_{t \in [a, b]} G(t, s) = G(b, s) = \frac{(b-s)^{\alpha-2}(s-a)}{\Gamma(\alpha)}$  for  $s \in [a, b]$ .

**Proof.** Let us define two functions

$$G_1(t, s) = \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-2}}{(b-a)^{\alpha-2}} - (t-s)^{\alpha-1}, a \leq s \leq t \leq b \text{ and } G_2(t, s) = \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-2}}{(b-a)^{\alpha-2}}, a \leq t \leq s \leq b.$$

Property (1) is trivial. Indeed, it is clear that both  $G_1$  and  $G_2$  are continuous on their domains, and that  $G_1(s, s) = G_2(s, s)$ , whence (1) follows. To prove (2), note that  $G_2(t, s) \geq 0$ . Further,  $0 < \alpha - 2 \leq 1, \frac{t-s}{t-a} \leq 1$  and  $(b-s)(t-a) - (t-s)(b-a) = (b-t)(s-t) \geq 0$ , and hence,

$$G_1(t, s) = (t-a)^{\alpha-1} \left[ \left( \frac{b-s}{b-a} \right)^{\alpha-2} - \left( \frac{t-s}{t-a} \right)^{\alpha-1} \right] \geq 0, a \leq s \leq t \leq b.$$

# International Journal of Innovative Research in Science, Engineering and Technology

(A High Impact Factor, Monthly, Peer Reviewed Journal)

Visit: [www.ijirset.com](http://www.ijirset.com)

Vol. 8, Issue 2, February 2019

Thus, (2) is true. To prove (3), note that  $G_2(t, s)$  is an increasing function of  $t$ . Using  $(b - s)(t - a) - (t - s)(b - a) = (b - t)(s - t) \geq 0$  for  $a \leq s < t \leq b$ , we get

$$\frac{\partial G_1}{\partial t} = (\alpha - 1)(t - a)^{\alpha-2} \left[ \left( \frac{b-s}{b-a} \right)^{\alpha-2} - \left( \frac{t-s}{t-a} \right)^{\alpha-2} \right] \geq 0$$

Hence, for a given  $s$ ,  $G_1(t, s)$  is an increasing function of  $t \in [s, b]$ . This proves the first part of property (3). The second part of property (3) follows by setting  $t = b$  in equation (3.2). Hence, the proof is complete.

**Theorem 3.1.** If a nontrivial continuous solution of the FBVP P2 exists, then

$$\int_a^b (s - a)(b - s)^{\alpha-2} |q(s)| ds \geq \Gamma(\alpha), \tag{3.4}$$

and more specifically,

$$\int_a^b |q(s)| ds \geq \frac{\Gamma(\alpha)(\alpha-1)^{\alpha-1}}{(b-a)^{\alpha-1}(a-2)^{\alpha-2}}, \tag{3.5}$$

and for integer order case  $\alpha = 3$ ,

$$\int_a^b |q(s)| ds \geq \frac{8}{(b-a)^2}. \tag{3.6}$$

The inequality in equation (3.4) is called a Hartman-Wintner type inequality, and the inequalities in equations (3.5) and (3.6) are called the Lyapunov-type inequalities for problem P2 of fractional and integer orders, respectively.

**Proof.** Let  $B = C[a, b]$  be a Banach space of all continuous functions  $u : [a, b] \rightarrow \mathbb{R}$  endowed with a norm

$$\|u\|_\infty = \max_{a \leq t \leq b} |u(t)|, u \in B.$$

From Lemma 3.1, for all  $t \in [a, b]$  we have  $u(t) = \int_a^b G(t, s)q(s)u(s)ds$ , where  $G(t, s)$  for problem P2 is defined in equation (3.2). Hence  $\|u\|_\infty \leq \max_{a \leq t \leq b} \int_a^b |G(t, s)q(s)| ds \|u\|_\infty$ , or equivalently,

$$1 \leq \max_{a \leq t \leq b} \int_a^b |G(t, s)q(s)| ds. \tag{3.7}$$

Finally, using property (3) of Lemma 3.2, we get the inequality in equation (3.4). To prove equation (3.5), let  $\phi(s) = (s - a)(b - s)^{\alpha-2}$ ,  $s \in [a, b]$ . Differentiating  $\phi(s)$  with respect to  $s$ , and setting  $\phi'(s)$  to 0, we obtain that  $\phi(s)$  has an extremum at  $s = s^* = \frac{b+a(\alpha-2)}{\alpha-1}$ ,  $s^* \in (a, b)$ . The second derivative of  $\phi(s)$  with respect to  $s$  at  $s = s^*$  is  $\phi''(s^*) = -(\alpha - 1)^{(4-\alpha)} / [(\alpha - 2)(b - a)]^{(3-\alpha)}$  which indicates that  $\phi(s)$  is maximum at  $s = s^*$ , and  $\max_{a \leq t \leq b} \phi(s) = \phi(s^*) = (\alpha - 2)^{(\alpha-2)} \left( \frac{b-a}{\alpha-1} \right)^{\alpha-1}$ .

Now, replacing  $(s - a)(b - s)^{\alpha-2} = \phi(s)$  in equation (3.4) with the expression for  $\phi(s^*)$ , we get equation (3.5), and setting  $\alpha = 3$  in equation (3.5), we get equation (3.6). This completes the proof of Theorem 3.1.

Now, consider the following linear Fractional Differential Equation (FDE) and the boundary conditions,

Problem P3:

$$(D_{a+}^\alpha u)(t) + \lambda u(t) = 0, a < t < b, \tag{3.8}$$

$$u(a) = 0, u'(a) = 0, u'(b) = 0, \tag{3.9}$$

where the function  $u(t)$  and the number  $\lambda$  are unknown. A function  $u(t)$  that satisfies equations (3.8) and (3.9) is known as an eigenfunction, the corresponding  $\lambda$  the eigenvalue associated with  $u(t)$ , and the problem a Fractional Eigenvalue Problem (FEP). By setting  $\alpha = 3$  in equation (3.8), we obtain an Integer Order Eigenvalue Problem (IOEP). Note that problem P2 and P3 are the same except that  $q(t)$  in equation (1.3) has been replaced with  $\lambda t$  to obtain equation (3.8). Thus, equations (3.5) and (3.6) for problem P2 can be used to find a lower bound for the smallest eigenvalue of problem P3. We note that the eigenvalues of FEP P3 are known to be real. In the discussion to follow, we will use the following definition for a Lyapunov inequality lower bound.

**Definition 3.1.** A Lyapunov Inequality Lower Bound (LILB) is defined as a lower estimate for the smallest eigenvalue obtained from Lyapunov type inequalities given in equations (1.2), (3.5) and (3.6).

**Corollary 3.1.** The LILB for the smallest eigenvalue of FEP P3 for  $\alpha \in (2, 3]$  is given by

$$\lambda \geq \frac{\Gamma(\alpha)(\alpha-1)^{\alpha-1}}{(\alpha-2)^{\alpha-2}(b-a)^\alpha} \tag{3.10}$$

and in particular, for IOEP P3, i.e.  $\alpha = 3$ , this bound is

$$\lambda \geq \frac{8}{(b-a)^3} \tag{3.11}$$

# International Journal of Innovative Research in Science, Engineering and Technology

(A High Impact Factor, Monthly, Peer Reviewed Journal)

Visit: [www.ijirset.com](http://www.ijirset.com)

Vol. 8, Issue 2, February 2019

**Proof.** Setting  $q(t) = \lambda$  in equations (3.5) and (3.6), the inequalities in equations (3.10) and (3.11) follow. In [5], [6], [7] and [8], the authors have used Lyapunov-type inequalities for FBVPs for  $\alpha \in (1, 2]$  to obtain the lower bound estimates for the smallest eigenvalues for several FEPs. The method involves replacing the Green's functions in some integral equations with their maximum norms. As a result, Lyapunov-type inequalities highly underestimate the lower bounds for smallest eigenvalues. In this paper, these estimates are improved by using a Semi Maximum Norm (SMN) and Cauchy-Schwarz Inequality (CSI). The SMN and the CSI methods and the numerical results obtained from them for some FEPs of order  $\alpha \in (2, 3]$  are presented below. Similar results for some fractional EPs of order  $\alpha \in (1, 2]$  are presented in [9] and [10].

To describe the Semi Maximum Norm method, note that a linear FBVP of type P2

reduces to  $1 \leq \max_{a \leq t \leq b} \int_a^b |G(t, s)q(s)| ds$ .

(see (3.7)), and for a FEP of type P3,  $q(s)$  in the above equation is replaced with  $\lambda$  to obtain

$$\lambda \geq \frac{1}{\max_{a \leq t \leq b} \int_a^b |G(t, s)q(s)| ds}. \tag{3.12}$$

The above inequality gives a lower bound estimate for the smallest eigenvalue. In this case, we do not take the maximum norm of  $|G(t, s)|$  but only the maximum norm of the integral  $\int_a^b |G(t, s)| ds$  over  $[a, b]$ , and for this reason, we call this method of obtaining a lower bound for  $\lambda$  the Semi Maximum Norm method. Also note that

$$\max_{a \leq t \leq b} \int_a^b |G(t, s)q(s)| ds \leq (b - a) \max_{[a, b] \times [a, b]} |G(t, s)|$$

and therefore, the Semi Maximum Norm method provides a better estimate for the smallest eigenvalue than that provided by the Lyapunov type inequalities. In the sequel we defined a Semi Maximum Norm Lower Bound as follows.

**Definition 3.2.** A Semi Maximum Norm Lower Bound (SMNLB) is defined as the lower estimate for the smallest eigenvalue obtained from the Semi Maximum Norm inequality of type given in (3.12).

**Corollary 3.2.** Let  $\lambda$  be the smallest eigenvalue of the FEP P3, then the SMNLB for the smallest eigenvalue is given by

$$\lambda \geq \frac{(\alpha - 1)\Gamma(\alpha + 1)}{(b - a)^\alpha}, \tag{3.13}$$

and in particular, for IOEP P3, i.e.  $\alpha = 3$ , this bound is

$$\lambda \geq \frac{12}{(b - a)^3}, \tag{3.14}$$

**Proof.** For the FEP P3, the Green's function is given by equation (3.2), and using this Green's function, we obtain

$$\int_a^b |G(t, s)| ds = \frac{(t - a)^{\alpha - 1}}{(\alpha - 1)\Gamma(\alpha + 1)} [\alpha(b - t) + (t - a)].$$

The right-hand side of the above equation is a positive monotonically increasing function, and therefore its maximum occurs at  $t = b$ , and this maximum is

$$\max_{a \leq t \leq b} \int_a^b |G(t, s)| ds = \frac{(b - a)^\alpha}{(\alpha - 1)\Gamma(\alpha + 1)}. \tag{3.15}$$

Substitution of equation (3.15) in equation (3.12) proves equation (3.13). Setting  $\alpha = 3$ , we obtain equation (3.14).

We now consider the CSI method. The following lemma will prove useful in describing the method.

**Lemma 3.3.** If  $\lambda$  is the lowest eigenvalue and  $G(t, s)$  is the Green's function of the FEP P3 then

$$\lambda \geq \left[ \int_a^b \int_a^b G(t, s) ds dt \right]^{-\frac{1}{2}}. \tag{3.16}$$

**Proof.** Using equation (3.1) with  $q(t) = \lambda$  and the Cauchy-Schwarz inequality we obtain

$$|u(t)|^2 \leq \lambda^2 \left[ \int_a^b |G(t, s)|^2 ds \right] \left[ \int_a^b u^2(s) ds \right]. \tag{3.17}$$

Integrating this inequality with respect to  $t$  over  $[a, b]$  and then dividing the result by  $\|u\|_2$ , we get equation (3.16).

Although, Lemma 3.3 emphasizes the relationship between the lowest eigenvalue and the Green's function for FEP P3, equation (3.16) is applicable to any eigenvalue problem as long as its solution has a representation of type given by equation (3.1). We call equation (3.16) a Cauchy-Schwarz Inequality; it can be used to find a lower bound for



## International Journal of Innovative Research in Science, Engineering and Technology

(A High Impact Factor, Monthly, Peer Reviewed Journal)

Visit: [www.ijirset.com](http://www.ijirset.com)

Vol. 8, Issue 2, February 2019

the smallest eigenvalue of FEP P3. In the discussion to follow, we define a Cauchy-Schwarz Inequality Lower Bound as follows:

**Definition 3.3.** A Cauchy-Schwarz Inequality Lower Bound (CSILB) is defined as an estimate of the lower bound for the smallest eigenvalue obtained from the Cauchy-Schwarz inequality of type given in equation (3.16).

**Theorem 3.2.** If  $\lambda$  is the smallest eigenvalue of the problem FEP P3, then its CSILB is given by

$$\lambda \geq \frac{\Gamma(\alpha)}{(b-a)^\alpha} \left[ \frac{(4\alpha-3)-4(2\alpha-1)(2\alpha-3)c_1(\alpha)}{2\alpha(2\alpha-1)(2\alpha-3)} \right]^{\frac{1}{2}}, \quad (3.18)$$

where  $c_1(\alpha) = \int_0^1 t^{2\alpha-1} {}_2F_1(2-\alpha, 1; \alpha+1; t) dt$  and  ${}_2F_1(a, b; c; t)$  is a hypergeometric function. In particular, for IOEP P3, i.e. for  $\alpha = 3$ , this bound is

$$\lambda \geq \frac{\sqrt{315}}{(b-a)^3}. \quad (3.19)$$

**Proof.** Substituting the Green's function from equation (3.2), in (3.16) and simplifying the results, we obtain equation (3.18), and setting  $\alpha = 3$  in equation (3.18), we get equation (3.19). To better understand the estimates of the lower bound for the smallest eigenvalue of the FEP P3, obtained from LILB, SMNLB and CSILB methods, we reconsider FEP P3. Scaling the time using the transformation  $t = a + (b-a)\tau$ , equations (3.8) and (3.9) reduce to

$$(D_{a+}^\alpha U)(\tau) + \lambda(b-a)^\alpha U(\tau) = 0, 0 < \tau < 1 \quad (3.20)$$

$$U(0) = 0, U'(0) = 0, U'(1) = 0, \quad (3.21)$$

Where  $U(\tau) = u(t) = u(a + (b-a)\tau)$ ,  $D = d/d\tau$ ,  $D_{0+}^\alpha = D^3 I_{0+}^{3-\alpha}$  and  $U' = dU/d\tau$ . Although, equation (3.20) has been given for  $\alpha \in (2, 3]$  only, the transformation of equation (3.8) to equation (3.20) under the change of variable

$t = a + (b-a)\tau$  remains true for arbitrary  $\alpha$ . Thus, to solve the FEP P3, it is only necessary to solve the FEP P3 with  $a = 0$  and  $b = 1$ . Once the transformed problem has been solved, one can obtain the solution to the original problem by replacing  $\tau$  with  $(t-a)/(b-a)$  and  $\lambda$  with  $\lambda/(b-a)^\alpha$ . For this reason, to examine the lower bound estimates for the smallest eigenvalue, we take  $a = 0$  and  $b = 1$  in FEP P3. This will be true for the rest of the paper.

We first consider the integer order case, i.e.  $\alpha = 3$ . For this case, the LILB, SMNLB and CSILB for the smallest  $\lambda$  of FEP P3 are given as 8, 12 and  $\sqrt{315} \approx 27.4545$ , respectively. (see equations (3.11), (3.14) and (3.19)). For  $\alpha = 3$ , the FEP P3 can be solved in closed form using the tools from integer order calculus. Results show, that the smallest eigenvalue of FEP P3 for  $\alpha = 3$  is the root of  $\exp(-3\lambda^{1/3}/2) - 2 \sin(\sqrt{3}\lambda^{1/3}/2 + \pi/6) = 0$  which gives the smallest eigenvalue as  $\lambda \approx 27.4545$ . Comparing this  $\lambda$  with its estimate above, it is clear that among LILB, SMNLB and CSILB for integer  $\alpha$  the CSILB provides the best estimate for the smallest eigenvalue. The FEP P3 can also be solved and its eigenvalues can be determined for arbitrary  $\alpha, \alpha \in (2, 3]$  as a root of the Mittag-Leffler function  $E_{\alpha, \alpha-1}(-\lambda)$ . This is discussed in the following theorem and its proof.

**Theorem 3.3.** The FEP P3 for  $2 < \alpha \leq 3, a = 0$  and  $b = 1$  has infinite eigenvalues, and they are the roots of the Mittag-Leffler function  $E_{\alpha, \alpha-1}(z)$ , i.e. the eigenvalues satisfy

$$E_{\alpha, \alpha-1}(-\lambda) = 0. \quad (3.22)$$

**Proof.** To prove this, we take Laplace transform of equation (3.8), which after some manipulations lead to

$$U(s) = \frac{a_0}{s^{\alpha+\lambda}} + \frac{a_1 s}{s^{\alpha+\lambda}} + \frac{a_2 s^2}{s^{\alpha+\lambda}}, \quad (3.23)$$

where  $U(s)$  is the Laplace transform of  $u(t)$  and  $a_i = (D_{0+}^{\alpha-i-1} u)(0), i = 0, 1, 2$  [12]. Taking inverse Laplace transform of equation (3.23) and using equation (2.5), we obtain

$$u(t) = a_0 t^{\alpha-1} E_{\alpha, \alpha}(-\lambda t^\alpha) + a_1 t^{\alpha-2} E_{\alpha, \alpha-1}(-\lambda t^\alpha) + a_2 t^{\alpha-3} E_{\alpha, \alpha-2}(-\lambda t^\alpha). \quad (3.24)$$

Finally, using equation (2.6) and the boundary conditions given by equation (3.9), we obtain that the eigenvalues of the FEP P3 are the roots of equation (3.22). In [16], the authors have shown that the Mittag-Leffler function  $E_{\alpha, \beta}(z)$  where  $2 \leq \alpha \leq 3, 0 < \beta \leq 6.669$  has infinite number of positive real zeros and no complex zeros. Thus,  $E_{\alpha, \alpha-1}(\lambda), 2 \leq \alpha \leq 3$  has infinite positive real roots, i.e. the FEP P3 has infinite eigenvalues. This completes the proof of Theorem 3.3.

For comparison purpose, we compute the smallest eigenvalues for FEP P3 from equation (3.22) and its LILB, SMNLB and CSILB for different  $\alpha, \alpha \in (2, 3]$  from equations (3.10), (3.13) and (3.18). The results are shown in Figure 3.1.

# International Journal of Innovative Research in Science, Engineering and Technology

(A High Impact Factor, Monthly, Peer Reviewed Journal)

Visit: [www.ijirset.com](http://www.ijirset.com)

Vol. 8, Issue 2, February 2019

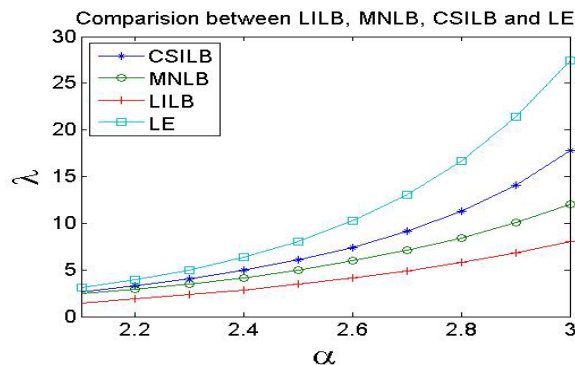


Figure 3.1: Comparison of the lower bounds for  $\lambda$  obtained from maximum norm, Lyapunov-type and Cauchy-Schwarz inequalities with the lowest eigenvalue. (- + -: LILB; -o-: MNLB; -\* -: CSILB; -□ -: LE - the Lowest Eigenvalue  $\lambda$ )

This figure clearly demonstrates that among the three estimates considered here, the LILB provides the worst estimate and the CSILB provides the best estimate for the smallest eigenvalues of FEP P3. We now consider an application of the lower bounds for the smallest eigenvalues of FEPP3 found in Corollaries 3.1 and 3.2 and Theorem 3.2. In [5], [6], [7] and [8], the authors have applied the LILB to the FEPs for  $\alpha \in (1, 2]$  to find the interval in which certain Mittag-Leffler functions have no real zeros. Similar results for  $\alpha \in (2, 3]$  is given in the following theorem.

**Theorem 3.4.** Let  $2 < \alpha \leq 3$ , then based on the LILB, SMNLB and CSILB inequalities, the Mittag-Leffler function  $E_{\alpha, \alpha-1}(z)$  has no real zeros in the following domains:

LILB inequality:

$$z \in \left(-\frac{\Gamma(\alpha)(\alpha-1)^{\alpha-1}}{(\alpha-2)^{\alpha-2}}, 0\right], \tag{3.25}$$

SMNLB inequality:

$$z \in \left(-(\alpha-1)\Gamma(\alpha+1), 0\right], \tag{3.26}$$

CSILB inequality:

$$z \in \left(\Gamma(\alpha) \left[\frac{(4\alpha-3)-4(2\alpha-1)(2\alpha-3)c_1(\alpha)}{2\alpha(2\alpha-1)(2\alpha-3)}\right]^{-\frac{1}{2}}, 0\right]. \tag{3.27}$$

**Proof.** Let  $\lambda$  be the smallest eigenvalue of the FEP P3, then  $z = \lambda$  is the smallest value of  $z$  for which  $E_{\alpha, \alpha-1}(-z) = 0$ . If there is another  $z$  smaller than  $\lambda$  for which  $E_{\alpha, \alpha-1}(-z) = 0$ , then it will contradict that  $\lambda$  is the smallest eigenvalue.

Therefore,  $E_{\alpha, \alpha-1}(z)$  has no real zero for  $z \in (-\lambda, 0]$ . Now, according to LILB,  $\lambda \geq \frac{\Gamma(\alpha)(\alpha-1)^{\alpha-1}}{(\alpha-2)^{\alpha-2}}$  (see equation (3.10)).

Thus,  $E_{\alpha, \alpha-1}(z)$  has no real zero for  $z \in \left(-\frac{\Gamma(\alpha)(\alpha-1)^{\alpha-1}}{(\alpha-2)^{\alpha-2}}, 0\right]$ . This proves equation (3.25). Equations (3.26) and (3.27) are proved in a similar fashion.

From Figure 3.1, it is clear that among the three inequalities discussed in the paper, LILB provides the smallest interval and CSILB provide the largest interval in which the Mittag-Leffler function  $E_{\alpha, \alpha-1}(z)$  has no real zero.

## IV. CONCLUSION

In this paper, we established a Lyapunov-type inequality for a fractional boundary value problem of order  $\alpha$  for  $\alpha \in (2, 3]$ . For the integer and fractional eigenvalue problems, we determined lower bounds for the first eigenvalue from Semi Maximum Norm Lyapunov-type and Cauchy-Schwarz type inequalities. We showed that the Cauchy-Schwarz type inequality provides the best and the Lyapunov type inequality provides the worst estimate for the smallest eigenvalue of the FEP P3. We used these bounds for the smallest eigenvalue to find the domain in which the Mittag-Leffler function  $E_{\alpha, \alpha-1}(z)$ ,  $\alpha \in (2, 3]$  has no zero. Results showed that Cauchy-Schwarz type inequality provides the largest domain in which  $E_{\alpha, \alpha-1}(z)$  has no zero.

# International Journal of Innovative Research in Science, Engineering and Technology

(A High Impact Factor, Monthly, Peer Reviewed Journal)

Visit: [www.ijirset.com](http://www.ijirset.com)

Vol. 8, Issue 2, February 2019

## ACKNOWLEDGMENT

The author would like to thank Prof. Salah Mohammed of Department of Mathematics at Southern Illinois University, Carbondale for proofreading the paper and making some valuable remarks.

## REFERENCES

- [1] A. M. Lyapunov, "Probleme général de la stabilité du mouvement", Ann. Fac. Sci. Univ. Toulouse., 2, 27-247, 1907.
- [2] R. C. Brown, D. B. Hinton, "Lyapunov inequalities and their applications", in Survey on classical inequalities, 1-25, Math. Appl., 517 Kluwer Acad. Publ., Dordrecht.
- [3] B. G. Pachpatte, "Mathematical Inequalities", North Holland Mathematical Library, 2005.
- [4] P. Hartman, A. Wintner, "On an oscillation criterion of Liapunov", Am. J. Math., 73, 885-890, 1951.
- [5] R. A. C. Ferreira, "A Lyapunov-type inequality for a fractional boundary value problem", Fract. Calc., Appl. Anal., 16, 4, 978-984, 2013.
- [6] R. A. C. Ferreira, "On a Lyapunov-type inequality and the zeros of a certain Mittag-Leffler function", J. Math. Anal. Appl., 412, 2, 1058-1063, 2014.
- [7] M. Jleli, B. Samet, "Lyapunov-type inequalities for a fractional differential equation with mixed boundary conditions", Mathematical Inequalities and applications, vol. 18, No. 2, 443-451, 2015.
- [8] Ji. Rong, C. Bai, "Lyapunov-type inequality for a fractional differential equation with fractional boundary conditions", Advances in Difference equations, 2015.
- [9] N. Pathak, "Lyapunov-type inequality and eigenvalue estimates for fractional problems", Dissertation Doctor of Philosophy, Southern Illinois University USA (2016).
- [10] N. Pathak, "Lyapunov-type inequality for fractional boundary value problems with Hilfer derivative", Mathematical Inequalities and Applications, 21(1) (2018) 179-200.
- [11] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, "Theory and applications of fractional differential equations", Elsevier, Amsterdam, 2006.
- [12] I. Podlubny, "Fractional differential equations", Academic Press, San Diego, CA, 1999.
- [13] S. G. Samko, A. A. Kilbas, O. I. Marichev, "Fractional integrals and derivatives", translated from the 1987 Russian original, Gordon and Breach, Yverdon, 1993.
- [14] Z. Bai, H. Lü, "Positive solutions for boundary value problem of nonlinear fractional differential equation", J. of Math. Anal. Appl. Volume 311, 495-505, 2005.
- [15] S. Zhang, "Positive solutions for boundary-value problems of nonlinear fractional differential equations", Electron. J. Differential Equations., Volume 36, 12 pages, 2006.
- [16] John W. Hanneken, B. N. Narahari Achar, David M. Vaught, "An Alpha-Beta Phase Diagram Representation of the Zeros and Properties of the Mittag-Leffler Function", HindaviPubll. corporation, 2013.