The Diagonal Equivalence of A Non-Negative Quaternion Matrix to A Doubly Stochastic Matrix

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ABSTRACT: Let A be a positive square quaternion matrix. Then there exists two diagonal matrices $D_1, D_2$ whose diagonal elements are positive such that $D_1 AD_2$ is quaternion doubly stochastic matrices.

Marcus and Newman [6] and maxfield and minc [7] also studied their problem. Perfect and mirsky have shown that given a fully indecomposable matrix $B$, there exists a doubly stochastic non-negative matrix with the same zero pattern.

Morishima [2] and Thompson [3] have studied such operators in extending the theorems of perron frobenius theory.

In this last two sections of the paper the diagonal equivalence of a non-negative quaternion matrix to a doubly stochastic matrix. We consider the diagonal equivalence of a non-negative quaternion matrix to a row doubly stochastic matrix. Given any square non-negative matrix $A$ with atleast one positive element in each row it is clear that there exists a row doubly stochastic matrix $C$ having zeros in exactly the same position as $A$.

Recently, sinkhorn [1] proved the existence of $D$ sure that $DAD$ is row stochastic under the assumption $A$ is a positive matrix.

KEYWORDS: quaternion doubly stochastic matrices, fully indecomposable matrices, Frobenius-Konig theorem, eigen values.

1. INTRODUCTION

FULLY INDECOMPOSABLE MATRICES

1.1 Notation:

Let $A = [a_{st}]$ be an $m \times n$ quaternion doubly stochastic matrix. If $a_{st} > 0$ for each $s$ and $t$ then we write $A >> 0$; if $a_{st} \geq 0$, then we write $A \geq 0$; if $A \geq 0$ but $A \neq 0$, then we write $A > 0$.

1.2 Definition:

Let $A \geq 0$ be an $n \times n$ quaternion doubly stochastic matrix. Then $A$ is called reducible provided there exists a permutation matrix $P$ such that $PAP^T$ has the form
Where \( A_1 \) and \( A_2 \) are non-empty square matrices. If \( A \) is not reducible, it is irreducible. A \( n \times n \) matrix \( A > 0 \) is called fully indecomposable. If there do not exists permutation matrices \( P \) and \( Q \) such that \( PAQ \) has the form\((2.2.1)\). By convention every \( 1 \times 1 \) matrix is fully indecomposable if and only if single entry is positive.

An \( n \times n \) matrix \( A > 0 \) is quaternion column stochastic if all its (column sums) are one. Then \( n \times n \) matrix \( A > 0 \) is quaternion doubly stochastic provided it is both row stochastic & column stochastic.

**Lemma 2.3:**

Let \( A \geq 0 \) be an \( n \times n \) quaternion doubly stochastic matrix. Then \( A \) is fully indecomposable if and only is there are permutation matrices \( P \) and \( Q \) such that \( PAQ \) has a positive main diagonal and is irreducible.

**Proof:**

Let \( A \) be fully indecomposable. Then by Frobenius-Konig theorem \([7,P.97]\) there exists permutation matrices \( P \) and \( Q \) such that \( PAQ \) has a positive main diagonal. But obviously \( PAQ \) is fully indecomposable and thus irreducible.

Conversely, suppose that \( C = PAQ \) has a positive main diagonal (positive main diagonal) and is irreducible. Since \( C \) is fully indecomposable if and only if \( A \) is, we may assume \( C = A \). Suppose \( A \) is not fully indecomposable, and let \( P_1 \) and \( Q_1 \) be two permutation matrices such that \( P_1AQ_1 \) is of the form\((2.2.1)\). Suppose \( A \) has \( r \) columns, and \( A_2 \) has \( n-r \) columns. Then we may write \( P_1AQ_1 = A'Q' \) where \( A' = P_1AP_1^T \) is again a matrix with a positive main diagonal and \( Q' = P_1Q_1 \) is a permutation matrix. But then it follows that \( Q' \) permutes the first \( r \) columns of \( A' \) among themselves and the last \( n-r \) columns of \( A' \) among themselves. Hence \( A' \) is of the form \((2.2.1)\) and \( A \) is reducible, which is contradiction.

**Theorem 2.4:**

If \( D_1 \) and \( D_2 \) are diagonal matrices such that \( D_1AD_2 = S \) is a quaternion doubly stochastic matrices, then for any permutation matrices \( P \) and \( Q \) we have \( D_1'(PAQ)D_2' = S \). Where \( D_1' = PD_1P^T \) and \( D_2' = QD_2Q^T \) are diagonal matrices.

**Proof:**

Let \( A \) is irreducible with positive main diagonal, then by the Frobenius-Konig theorem \([7,P.97]\) there exists permutation matrices \( P \) and \( Q \) such that \( PAQ \) has a positive main diagonal.

But obviously \( PAQ \) is irreducible with positive main diagonal and thus fully indecomposable.
Conversely, suppose that $C = PAQ$ with has a fully indecomposable. Since $C$ is irreducible positive main diagonal, and let $P_1$ and $Q_1$ be two permutation matrices such that $P_1AQ_1$ is of the form (2.2.1). Suppose $A_1$ has $r$ rows and $A_2$ has $n-r$ rows. Then we may write $P_1AQ_1 = A'Q'$ where $A' = P_1AP_1^T$ is again a matrix with a positive main diagonal and $Q' = P_1Q_1$ is a permutation matrix. But then it follows that $Q'$ permutes the first $r$ rows of $A'$ among themselves and the last $n-r$ rows of $A'$ of among themselves. Hence $A'$ is of the form (2.2.1) and $D$ is fully indecomposable which is contradiction. This proves the above lemma.

**Example:**

$$A \geq 0,$$ Since $A$ fully indecomposable.

$$A = \begin{bmatrix} 2 + i + 2j + k & 3 - i + j + k & -4 - 3j - 2k \\ 3 - i + j + k & 2 + i - j + 2k & -4 - 3k \\ -4 - 3j - 2k & -4 - 3k & 9 + 3j + 5k \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$PAQ = C, \quad C \geq 0.$$ 

$$PAQ = \begin{bmatrix} 2 + i + 2j + k & 3 - i + j + k & -4 - 3j - 2k \\ 3 - i + j + k & 2 + i - j + 2k & -4 - 3k \\ -4 - 3j - 2k & -4 - 3k & 9 + 3j + 5k \end{bmatrix}$$

The $D_1'AD_2'$ Theorems

**Theorems 3.1:**

Let $A > 0$ be an $n \times n$ fully indecomposable quaternion matrix. Then there exist diagonal matrices $D_1$ and $D_2$ with positive main diagonals such that $D_1'AD_2'$ is quaternion doubly stochastic. Moreover $D_1'$ and $D_2'$ are uniquely determine up to scalar multiples.

**Proof:**

By(2.4), $E$ is enough to prove this theorem for an irreducible matrix $A$ with a positive main diagonal. Let $x >> 0$ and set $D_2' = \text{diag}(x_1', x_2', ..., x_n')$ and $D_1' = \text{diag}((\alpha x)_1, ..., (\alpha x)_n)$. Then $D_1'AD_2'$ is quaternion doubly stochastic if and only if $Tx = x$, where $T$ is the operator associated with $A$. Hence the results from
Let $A \succ 0$ be an $n \times n$ irreducible matrix with a positive main diagonal. Let $T$ be the operator associated with the matrix $A$. Then 1 is an eigen value of $T$ with a unique eigen vector $u$ in $P$. Further more $u \gg 0$.

We now show that the condition that $A \succ 0$ fully indecomposable is essential necessary for (3.1) to hold.

Theorem 3.2:

Let $A \succ 0$ be an $n \times n$ quaternion matrix. Then there exist diagonal matrices $D_1$ and $D_2$ such that $D_1'AD_2'$ is quaternion doubly stochastic matrices if and only if after independent permutations of rows and columns $A$ is the direct sum of fully indecomposable matrices.

Proof:

Suppose the matrix $A'$ obtained from $A$ by independent permutations of rows and columns is equal to $A^{(1)} + ... + A^{(K)}$ where each $A^{(i)}$ is fully indecomposable. By there exists diagonal matrices $D_1^{(1)}...D_1^{(K)}$ and $D_1^{(1)}...D_2^{(K)}$ such that $D_1^{(i)}A^{(i)}D_2^{(i)}$ is quaternion doubly stochastic. If

$$D_1' = D_1^{(1)} + ... + D_1^{(K)}; \quad D_2' = D_1^{(1)} + ... + D_2^{(K)}$$

Then $D_1'AD_2'$ is quaternion doubly stochastic. Hence (2.4 ) we can find $D_1$ and $D_2$ such that $D_1AD_2$ is quaternion doubly stochastic.

Conversely, suppose that there exists $D_1$ and $D_2$ such that $D_1AD_2 = S$ is quaternion doubly stochastic

$$S = \begin{bmatrix} S_1' & 0 & \cdots & 0 \\ S_{12}' & S_2' & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ S_{K1}' & S_{K2}' & \cdots & S_K' \end{bmatrix}$$

Where each $S_i$ is either fully indecomposable or a 1x1 zero matrix. But since $S'$ is quaternion doubly stochastic, the row sums of $S_i'$ are all one and hence the column sums of $S_i'$ being less than or equal to 1 must in fact be equal to 1. Hence $S_{12}' = 0, ..., S_K' = 0$. Repeating this argument we obtain $S' = S_1' + S_2' + ... + S_K'$, where each $S_i'$ is a quaternion doubly stochastic and hence fully indecomposable.

Theorem 3.3:

The Kronecker product of two quaternion fully indecomposable double stochastic matrices if and only if both are fully indecomposable doubly stochastic matrices.
Proof:

Suppose that $A \otimes B$ is fully indecomposable, but $A$ is not fully indecomposable quaternion doubly stochastic matrices, then there exist permutation matrices $P$ and $Q$ such that

$$PAQ = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$ ---- (3.3.1)

Where $A_{11}$ and $A_{22}$ are as in (1). Now

$$(P \otimes I)(A \otimes B)(Q \otimes I) = PAQ \otimes B = \begin{bmatrix} A_{11} \otimes B & A_{12} \otimes B \\ 0 & A_{22} \otimes B \end{bmatrix}$$

Where the Kronecker product of non-square matrices is defined in the obvious way since $(P \otimes I)$ and $(Q \otimes I)$ are permutation matrices this gives a contradiction quaternion doubly stochastic. Similarly, if $B$ is not fully indecomposable then there permutation matrix $R$, where $[R]_{i,j,k,l} = S_{i,j}S_{k,l}$, satisfies $B \otimes A = R(A \otimes B)RT$, from which the result follows as above.

II. CONCLUSION

In this paper, we can discuss fully indecomposable doubly stochastic matrices to a diagonal equivalence of a non-negative quaternion matrix.

REFERENCES