

# A Overview in the Field of Complex Variables

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**ABSTRACT:** Complex-valued functions or simply complex functions are functions which produce complex numbers from complex numbers. Complex analysis or complex function theory is the study of complex analytic functions.

**KEYWORDS:** complex variable, analytic functions, harmonic function, conformal mapping.

## I. INTRODUCTION

This paper deals with brief overview of complex valued function  $w=f(z)$  and its associated topics, analytical functions and conformal mappings. Complex analysis is study of complex analytic functions. It is a powerful method in the study of heat flow, fluid dynamics and electrostatics. Two dimensional potential problem can be solved using analytic functions since the real and imaginary part of an analytic function are solutions of two dimensional Laplace equations. conformal mapping, which preserves angles in magnitude and sense is useful in solving boundary value problems in two dimensional potential theory by transforming a complicated region to a simpler region. i.e conformal mapping preserves solutions of two dimensional Laplace equations.

Definition of analytic function :A function  $f(z)$  is said to be analytic at a point  $z_0$  if  $f$  is differentiable not only at  $z_0$  but at every point of some neighbourhood of  $z_0$ .

A function  $f(z)$  is analytic in a domain if it is analytic at every point of domain.

An analytic function is also known as holomorphic, regular, monogenic.

## II. TOPICS RELATED TO ANALYTIC FUNCTIONS.

- 1.1 If  $f$  is analytic in a domain  $D$  then  $u, v$  satisfy C-R conditions at all points in  $D$ .
- 1.2 C-R conditions are necessary but not sufficient.
- 1.3 C-R conditions are sufficient if the partial derivatives are continuous.
- 1.4 The necessary conditions that a single valued function  $w = f(z) = u(x, y) + iv(x, y)$  may be analytic at any point  $z = x + iy$  is that there exists four continuous first order partial derivatives  $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$  and must satisfy equations  $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$  these are known as Cauchy-Riemann equations or in brief C-R equations.
- 1.5 The necessary conditions that a single valued function  $w = f(z) = u(r, \theta) + iv(r, \theta)$  may be analytic at any point  $z = r + i\theta$  is that there exists four continuous first order partial derivatives  $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}$  and must satisfy equations  $\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$  these are known as Cauchy-Riemann equations or in brief C-R equations.
- 1.6 Harmonic function : A function  $\phi$  is said to be harmonic function if it satisfies Laplace's equation  $\nabla^2 \phi = 0$ .  
In the Cartesian form  $\phi(x, y)$  is harmonic if  $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ .  
In the polar form  $\phi(r, \theta)$  is harmonic if  $\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = 0$ .

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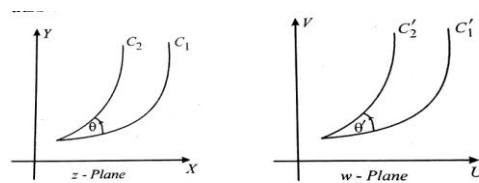
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1.7 Orthogonal system : If  $f(z) = u + iv$  is analytic then the family of curves  $u(x, y) = c_1, v(x, y) = c_2, c_1$  and  $c_2$  being constants, intersect each other orthogonally.

2.1 Conformal mapping : A mapping  $w=f(z)$  is said to be conformal if the angle between any two smooth curves  $c_1, c_2$  in the Z-plane intersecting at the point  $z_0$  is equal in magnitude and sense to the angle between their images  $c_1^*, c_2^*$  in the W-plane at the point  $w_0$ .



Thus conformal mapping preserves angles both in magnitude and sense also known as conformal mapping of the first kind. Conformal mapping of the second kind (isogonal mapping) preserves angles only in magnitude but not in sense which is reversible like  $w = \bar{z}$  where  $\arg \bar{z} = -\arg z$ .

Conformal mapping is used to map complicated region conformally onto simpler, standard regions such as circular disks, half planes and stripes for which the boundary value problems are easier.

Given two mutually orthogonal one parameter families of curves say  $\phi(x, y) = c_1$  and  $\psi(x, y) = c_2$ , their image curves in the w-plane  $\phi(u, v) = c_3$  and  $\psi(u, v) = c_4$  under a conformal mapping are also mutually orthogonal. Thus conformal mapping preserves the property of mutual orthogonality of system of curves in the plane.

2.2 Condition for conformality : A mapping  $w=f(z)$  is conformal at each point  $z_0$  where  $f(z)$  is analytic and  $f'(z_0) \neq 0$

3.1 Conformal mapping by elementary functions : General Linear Transformation or simply linear transformation defined by the function  $w = f(z) = az + b$  ( $a \neq 0$ , and  $b$  are arbitrary complex constants) maps conformally the extended complex z-plane onto the extended w-plane, since this function is analytic  $f'(z) = a \neq 0$  for any  $z$ . If  $a=0$  above transformation reduces to a constant function.

3.2 Special cases of linear transformation are

3.3 Identity Transformation :  $w=z$  -----(1)

For  $a=1, b=0$ , which maps a point  $z$  onto itself.

3.4 Translation:  $w=z+b$ -----(2)

For  $a=1$ , which translates  $z$  through a distance  $|b|$  in the direction of  $b$ .

3.5 Rotation :  $w = e^{i\theta_0} z$ ----- (3)

For  $a = e^{i\theta_0}, b = 0$  which rotates (the radius vector of point)  $z$  through a scalar angle  $\theta_0$ .

3.6 Stretching (scaling) :  $w=az$  -----(4)

For 'a' real stretches if  $a>1$  (contracts if  $0<a<1$ ) the radius vector by a factor 'a'.

Thus the linear transformation  $w=az+b$  consists of rotation through angle  $\arg a$ , scaling by factor  $|a|$ , followed by translation through vector  $b$ . This transformation is used for constructing conformal mappings of similar figures.

4.1 Discussion of standard conformal transformation :

Given the transformation  $w=f(z)$  put  $z = x + iy$  or  $z = re^{i\theta}$  to obtain  $u$  and  $v$  as functions of  $x, y$  or  $r, \theta$  we find the image in w-plane corresponding to the given curve in the z-plane. Sometimes it is required to make some judicious elimination from  $u$  &  $v$  for obtaining the image in the w-plane.

4.2 Discussion of  $w = e^z$  :

Consider  $w = e^z$

i.e.  $u + iv = e^{x+iy} = e^x \cdot e^{iy}$  or  $u + iv = e^x (\cos y + i \sin y)$

$$\therefore u = e^x \cos y, v = e^x \sin y \text{ ----- (i)}$$

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We shall find the image in the w-plane corresponding to the straight lines parallel to the co-ordinate axis in the z-plane .That is  $x=\text{constant}$  and  $y=\text{constant}$  .

Eliminating x and y separately from (1)

Squaring and adding we get  $u^2 + v^2 = e^{2x}$  -----(ii)

Also by dividing we get  $\frac{v}{u} = \frac{e^x \sin y}{e^x \cos y}$

i.e.  $e \frac{v}{u} = \tan y$  ----- (iii)

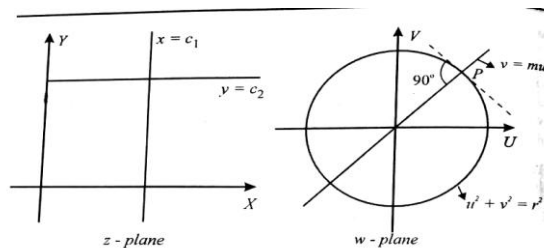
case 1: Let  $x = c_1$  where  $c_1$  is constant .

i.e.  $u^2 + v^2 = r^2$  This represents a circle with centre origin and radius r in the w-plane .

case 2: Let  $y=c_2$  where  $c_2$  is constant .

Equation (iii) becomes  $\frac{v}{u} = \tan c_2 = m(\text{say}) \quad \therefore v = mu$

This represents a straight line passing through the origin in the w-plane



Conclusion : The straight line parallel to the x-axis ( $y = c_2$ ) in the z-plane maps on to a straight line passing through the origin in the w-plane .

The straight line parallel to the y-axis ( $x = c_1$ ) in the z-plane maps on to a circle with centre origin and radius r in the w-plane .

### 4.3 Discussion of $w = z^2$

Consider  $w = z^2$

i.e.  $u + iv = (x + iy)^2$  or  $u + iv = (x^2 - y^2) + i(2xy)$

$$\therefore u = x^2 - y^2 \text{ and } v = 2xy \text{ ----- (i)}$$

Case1: Let us consider  $x = c_1, c_1$  is a constant .

The set of equations(i) become  $u = c_1^2 - y^2 ; v = 2c_1y$

Now  $y = v/2c_1$  and substituting this in u we get

$$u = c_1^2 - \left( \frac{v^2}{4c_1^2} \right) \text{ or } v^2 = -4c_1^2(u - c_1^2)$$

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This is a parabola in the w-plane symmetrical about the real axis with its vertex at  $(c_1^2, 0)$

and focus at the origin. It may be observed that the line  $x = -c_1$  is also transformed into the same parabola.

Case 2 : Let us consider  $y = c_2, c_2$  is a constant .

The set of equations (i) become  $u = x^2 - c_2^2, v = 2xc_2$

Now  $x = v/2c_2$  and substituting this in u ,we get

$$u = \left( \frac{v^2}{4c_2^2} \right) - c_2^2$$

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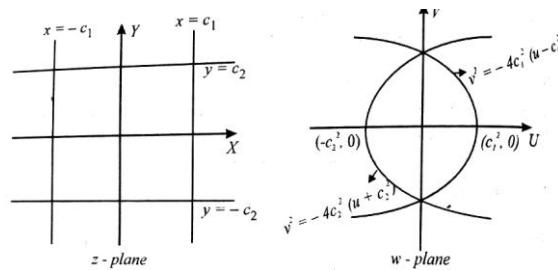
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$$\text{Or } v^2 = 4c_2^2(u + c_2^2)$$

This is also a parabola in the w-plane symmetrical about the real axis with its vertex at  $(-c_2^2, 0)$  and focus at the origin. It may be observed that the line  $y = -c_2$  is also transformed into the same parabola.

Hence from these two cases it is concluded that the straight lines parallel to the co-ordinate axes in the z-plane map onto parabolas in the w-plane.



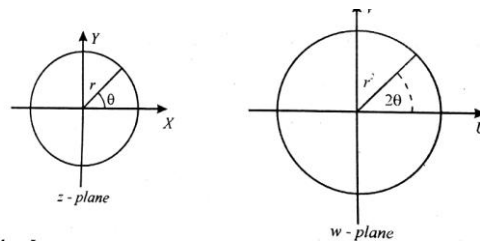
Case3: Let us consider a circle

$$\text{i.e. } |z| = r \therefore z = re^{i\theta} \text{ Hence } w = z^2 = (re^{i\theta})^2$$

$$\text{i.e. } w = r^2 e^{2i\theta} = Re^{i\phi} \text{ so that } R = r^2 \text{ and } \phi = 2\theta.$$

This is also a circle in the w-plane having radius  $r^2$  and subtending an angle  $2\theta$  at the origin.

Hence we conclude that a circle with centre origin and radius  $r$  in the z-plane maps onto a circle with centre origin and radius  $r^2$  in the w-plane.



Case4: Let us consider a circle with centre  $a$  and radius  $r$  in the z-plane whose equation is  $|z - a| = r$

$$\text{i.e. } z - a = re^{i\theta} \text{ or } z = a + re^{i\theta}$$

$$\text{Hence } w = z^2 = (a + re^{i\theta})^2 = a^2 + 2are^{i\theta} + r^2 e^{2i\theta}$$

$$\text{i.e., } w - a^2 = 2are^{i\theta} + r^2 e^{2i\theta}$$

Adding  $r^2$  on both sides we have

$$w - a^2 + r^2 = 2are^{i\theta} + r^2(1 + e^{2i\theta})$$

$$\text{i.e., } w - (a^2 - r^2) = 2are^{i\theta} + r^2(1 + e^{2i\theta})$$

$$\text{i.e., } w - (a^2 - r^2) = 2re^{i\theta} \left[ a + \frac{r}{2}(e^{i\theta} + e^{-i\theta}) \right]$$

$$\text{i.e., } w - (a^2 - r^2) = 2re^{i\theta} [a + r \cos \theta]$$

Suppose  $w - (a^2 - r^2) = Re^{i\phi}$  then this equation becomes  $Re^{i\phi} = 2re^{i\theta} [a + r \cos \theta]$  so that pole in the w-plane is at the point  $a^2 - r^2$

Now  $R(\cos \phi + i \sin \phi) = 2r(a + r \cos \theta)(\cos \theta + i \sin \theta)$

$$R \cos \phi = 2r(a + r \cos \theta) \cos \theta$$

$$R \sin \phi = 2r(a + r \cos \theta) \sin \theta$$

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Squaring and adding these we have

$$R^2(\cos^2\theta + \sin^2\theta) = [2r(a + r \cos \theta)]^2(\cos^2\theta + \sin^2\theta)$$

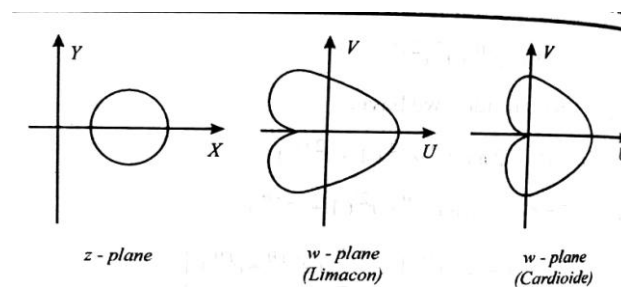
$$\text{i.e. } R^2 = [2r(a + r \cos \theta)]^2 \implies R = 2r(a + r \cos \theta) \text{ ----- (i)}$$

Also  $R \sin \phi / R \cos \phi = \tan \theta$  or  $\tan \phi = \tan \theta \implies \theta = \phi$  ----- (ii)

Using (ii) in (i) we have  $R = 2r(a + r \cos \phi)$  ----- (iii)

The curve given by (iii)(where  $a > r > 0$ ) is called Limacon .(standard form being  $r = a + b \cos \theta$ ).

Hence we conclude that a circle with centre ‘a’ and radius r in the z-plane is mapped on to a Limacon in the w-plane. In particular if  $r=a$ , (iii) becomes  $R = 2a^2(1 + \cos \phi)$  which is a cardioid in the w-plane .[standard form being  $r = a(1 + \cos \theta)$ ].



### III. CONCLUSION

This paper presents a brief overview of complex variable and its mappings . Major properties of complex analysis were discussed in detail. The primary use of conformal mapping is discussed with illustrative examples with clear representation.

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