

# A Note on Lyapunov-type Inequalities for Class of fractional Differential Equations

Nimisha Pathak

Assistant Professor, Department of Applied Mathematics, The M. S. University of Baroda, Vadodara, Gujarat, India

**ABSTRACT:** In the research article "Lyapunov-type inequalities for a class of fractional differential equations", the authors have established the Lyapunov-type inequality for fractional boundary value problem of order  $\alpha, 3 < \alpha \leq 4$ . They used this inequality to obtain the lower bound for the corresponding eigenvalue problem. In this note, we obtain the lower bound of the eigenvalue by using the Cauchy Schwarz-type inequality bound. Results show that the Lyapunov-type inequality gives the worse and the Cauchy-Schwarz-type inequality gives the best lower bound estimates for the smallest eigenvalues. Furthermore, as an application of Lyapunov-type and Cauchy Schwarz-type inequality bound we obtain an interval in which certain Mittag-Leffler function has no real zeros. Results show that Cauchy Schwarz-type inequality provides the larger interval than Lyapunov-type inequality in which certain Mittag-Leffler function has no real zero.

**KEYWORDS:** Lyapunov inequality, Hartman-Wintner-type inequality, fractional derivative, Mittag-Leffler function.

## I. INTRODUCTION

A classical result of Lyapunov [1] states that if  $u(t)$  is a nontrivial solution of the differential equation

$$\begin{aligned} u''(t) + q(t)u(t) &= 0, a < t < b \\ u(a) = 0, u(b) &= 0 \end{aligned} \tag{1.1}$$

where  $q(t)$  is a continuous and nonnegative function defined in  $[a, b]$ , then

$$\int_a^b |q(s)| ds > 4/(b-a), \tag{1.2}$$

the constant 4 in equation (1.2) is sharp so that it cannot be replaced by a larger number. The Lyapunov inequality has proved useful in the study of various properties of ordinary differential equations. Typical applications include bounds for eigenvalues, oscillation theory, stability criteria for periodic differential equations, and estimates for intervals of disconjugacy [2],[3].

The research on Lyapunov-Type Inequalities (LTIs) for Fractional Boundary Value Problems (FBVPs) has begun since 2013. Ferreira [4], [5], Jleli and Samet [6], Rong and Bai [7], Pathak [8], [9] have established Lyapunov-type inequalities for FBVPs of order  $\alpha, \alpha \in (1, 2]$  and different boundary conditions. In [10], we obtained LTI for FBVP of order  $\alpha, \alpha \in (2, 3]$ . Recently, in [11] the authors considered FBVP having the Riemann-Liouville derivative given by

$$(D_a^\alpha u)(t) + q(t)u(t) = 0, a < t < b, \alpha \in (3, 4] \tag{1.3}$$

with the boundary conditions

$$u(a) = 0, u'(a) = 0, u''(a) = u''(b) = 0, \tag{1.4}$$

where  $q: [a, b] \rightarrow \mathcal{R}$  is a continuous function. They obtained the following results.

**Lemma 1.1.**  $u \in C[a, b]$  is a solution to the boundary value problem (1.3) -(1.4) if and only if  $u$  satisfies the integral equation

$$u(t) = \int_a^b G(t, s)q(s)u(s)ds, \tag{1.5}$$

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where the Green's function is given by

$$G(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-3}}{(b-a)^{\alpha-3}} - (t-s)^{\alpha-1}, & a \leq s \leq t \leq b, \\ \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-3}}{(b-a)^{\alpha-3}}, & a \leq t \leq s \leq b, \end{cases} \quad (1.6)$$

**Lemma 1.2.** If a nontrivial continuous to the boundary value problem (1.3) -(1.4) exists, then the Hartman-Wintner-type inequality [12] is given by

$$\int_a^b (b-s)^{\alpha-3}(s-a)(2b-a-s)|q(s)|ds, \geq \Gamma(\alpha). \quad (1.7)$$

Proof: Let  $B = C[a, b]$  be a Banach space endowed a norm

$$\|u\|_{\infty} \leq \max_{a \leq t \leq b} |u(t)|, u \in B.$$

Hence, from equation (1.5) we get

$$\|u\|_{\infty} \leq \int_a^b |G(t, s)q(s)|ds \|u\|_{\infty},$$

or equivalently,

$$1 \leq \sup_{a \leq t \leq b} \int_a^b |G(t, s)q(s)|ds. \quad (1.8)$$

Hence, equation (1.7) follows since,  $|G(t, s)| < |G(b, s)| = \frac{(b-s)^{\alpha-3}(s-a)(2b-a-s)}{\Gamma(\alpha)}$  for  $[a, b] \times [a, b]$ .

**Lemma 1.3.** If a nontrivial continuous solution to the boundary value problem (1.3) -(1.4) exists, where  $q$  is a real and continuous function in  $[a, b]$  then the Lyapunov-type inequality is given by

$$\int_a^b |q(s)|ds \geq \frac{\Gamma(\alpha)(\alpha-2)^{\alpha-2}}{2(\alpha-3)^{\alpha-3}(b-a)^{\alpha-1}}. \quad (1.9)$$

They applied the Hartmann-Wintner-type inequality (1.7) to the eigenvalue problem defined in the following theorem.

**Theorem 1.1.** If  $\lambda$  is an eigenvalue to the Fractional Eigenvalue Problem (FEP)

$$(D_a^{\alpha}u)(t) + \lambda u(t) = 0, a < t < b, \quad (1.10)$$

$$u(a) = 0, u'(a) = 0, u''(a) = u''(b) = 0, \quad (1.11)$$

then

$$|\lambda| \geq \frac{\Gamma(\alpha)}{2B(2, \alpha-2)}, \quad (1.12)$$

where  $B$  is the beta function defined by

$$B(x, y) = \int_0^1 s^{x-1}(1-s)^{y-1} ds, x, y > 0. \quad (1.13)$$

## II. MAIN RESULTS

The main results of this paper are given in Theorems 2.1, 2.2 and 2.3. To prove these results, we begin by defining the following definitions which will be useful for lower bound estimate of the eigenvalue.

**Definition 2.1** The Hartman-Wintner-type Inequality Lower Bound (HWILB) is defined as the estimate of lower bound for the smallest eigenvalue obtained from the Hartman-Wintner-type inequality (1.7).

Theorem 1.1 gives the HWILB for  $\lambda$  for the eigenvalue problem (1.10) -(1.11). Setting  $\alpha = 4$  in equation (1.10), we obtain an Integer Order Eigenvalue Problem (IOEP). From equation (1.12) the HWILB for IOEP is  $\lambda \geq 18$ .

**Definition 2.2.** A Lyapunov Inequality Lower Bound (LILB) is defined as a lower estimate for the smallest eigenvalue obtained from Lyapunov type inequalities given in equation (2.1).

**Corollary 2.1.** The LILB for the smallest eigenvalue of FEP (1.10) -(1.11) for  $\alpha \in (3, 4]$  is given by

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$$\lambda \geq \frac{\Gamma(\alpha)(\alpha-2)^{\alpha-2}}{2(\alpha-3)^{\alpha-3}(b-a)^{\alpha-2}}, \quad (2.1)$$

for IOEP this bound is  $\lambda \geq 12$ .

**Proof.** Using equation (1.3) and (1.9) with  $q(t) = \lambda$  we obtain the inequality (2.1). In equation (2.1) taking  $\alpha = 4$  gives the lower bound for IOEP.

In the discussion to follow, we will use the following result to improve the lower bound estimate for the smallest eigenvalue of FEP (1.10) - (1.11).

**Lemma 2.1.** If  $\lambda$  is the lowest eigenvalue and  $G(t, s)$  is the Green's function of the FEP (1.10) - (1.11) then

$$\lambda \geq \left[ \int_a^b \int_a^b G(t, s) ds dt \right]^{-\frac{1}{2}}. \quad (2.2)$$

**Proof.** Using equation (1.5) with  $q(t) = \lambda$  and the Cauchy-Schwarz inequality we obtain

$$|u(t)|^2 \leq \lambda^2 \left[ \int_a^b |G(t, s)|^2 ds \right] \left[ \int_a^b u^2(s) ds \right]. \quad (2.3)$$

Integrating this inequality with respect to  $t$  over  $[a, b]$  and then dividing the result by  $\|u\|_2$ , we get equation (2.2).

Although, Lemma 2.1 emphasizes the relationship between the lowest eigenvalue and the Green's function for FEP (1.10) - (1.11), equation (2.2) is applicable to any eigenvalue problem as long as its solution has a representation of type given by equation (1.5). We call equation (2.2) a Cauchy-Schwarz-type Inequality; it can be used to find a lower bound for the smallest eigenvalue of FEP (1.10) - (1.11). We define a Cauchy-Schwarz Inequality Lower Bound as follows [8]-[10]:

**Definition 2.2.** A Cauchy-Schwarz Inequality Lower Bound (CSILB) is defined as an estimate of the lower bound for the smallest eigenvalue obtained from the Cauchy-Schwarz inequality of type given in equation (2.2).

**Theorem 2.1.** If a nontrivial continuous solution of the eigenvalue problem (1.10) - (1.11) with  $a = 0$  and  $b = 1$  exists, then CSILB is given by

$$|\lambda| \geq \Gamma(\alpha) \left[ \frac{2\alpha - 4\alpha(2\alpha - 1)(2\alpha - 5)B(1, \alpha)C_1(\alpha) + (2\alpha - 5)}{2\alpha(2\alpha - 1)(2\alpha - 5)} \right]^{-\frac{1}{2}}, \quad (2.4)$$

where  $C_1(\alpha) = \int_0^1 t^{2\alpha-1} {}_2F_1(3 - \alpha, 1; \alpha + 1; t) dt$  and  ${}_2F_1(a, b; c; t)$  is a hypergeometric function.

For IOEP this bound is  $\lambda \geq 50.5519$ .

**Proof.** Using the Green's function from Lemma 1.1 in equation (2.1) for the Cauchy-Schwarz inequality and taking  $a = 0$  and  $b = 1$ , proves the inequality (2.3). In equation (2.4) taking  $\alpha = 4$  gives the lower bound for IOEP.

**Theorem 2.2.** The FEP (1.10) - (1.11) for  $3 < \alpha \leq 4$ ,  $a = 0$  and  $b = 1$  has infinite eigenvalues, and they are the roots of the Mittag-Leffler function  $E_{\alpha, \alpha-2}(z)$ , i.e. the eigenvalues satisfy

$$E_{\alpha, \alpha-2}(-\lambda) = 0. \quad (2.5)$$

**Proof.** To prove this, we take Laplace transform of equation (1.10) with  $a = 0$  and  $b = 1$ , which after some manipulations lead to

$$U(s) = \frac{a_3}{s^{\alpha+\lambda}} + \frac{a_2 s}{s^{\alpha+\lambda}} + \frac{a_1 s^2}{s^{\alpha+\lambda}} + \frac{a_0 s^3}{s^{\alpha+\lambda}}, \quad (2.6)$$

where  $U(s)$  is the Laplace transform [13] of  $u(t)$  and  $a_i = (D_{0+}^{\alpha-i-1} u)(0)$ ,  $i = 0, 1, 2, 3$  [14]. Taking inverse Laplace transform of equation (2.6), we obtain

$$u(t) = a_0 t^{\alpha-1} E_{\alpha, \alpha}(-\lambda t^\alpha) + a_1 t^{\alpha-2} E_{\alpha, \alpha-1}(-\lambda t^\alpha) + a_2 t^{\alpha-3} E_{\alpha, \alpha-2}(-\lambda t^\alpha) + a_3 t^{\alpha-4} E_{\alpha, \alpha-3}(-\lambda t^\alpha). \quad (2.7)$$

Using the boundary conditions given by equation (1.11), we obtain that the eigenvalues of the FEP (1.10) - (1.11) are the roots of equation (2.5). It follows from Hadamard's factorization theory that an entire function of fractional order has infinitely many zeros (Ahlfors [15]). Consequently, all Mittag-Leffler functions  $E_{\alpha, \beta}(z)$  will have an infinite number of

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zeros. Thus,  $E_{\alpha, \alpha-2}(\lambda)$ ,  $3 < \alpha \leq 4$  has infinite number of roots, i.e. the FEP (1.10) -(1.11) has infinite eigenvalues. This completes the proof of Theorem 2.2.

Note that, for the FEP (1.10) – (1.11) with  $\alpha = 4$ , i.e for IOEP the lower bound for the smallest eigenvalue is  $\lambda \geq 125.1403939$ .

For comparison purpose, we compute the smallest eigenvalues for FEP (1.10) -(1.11) from equation (2.5) and its HWILB, LILB, and CSILB for different  $\alpha$ ,  $\alpha \in (3, 4]$  from equations (1.12), (2.1) and (2.4). The results are shown in Figure 2.1.

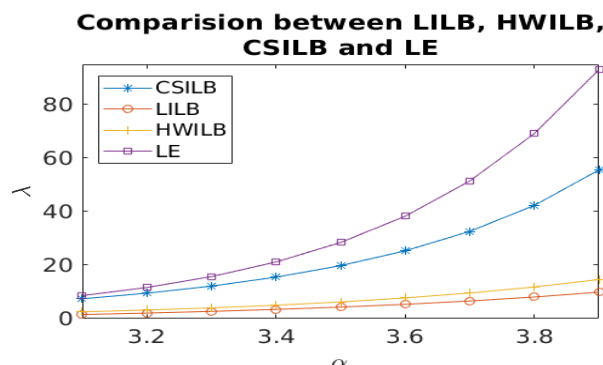


Figure 2.1: Comparison of the lower bounds for  $\lambda$  obtained from maximum norm, Lyapunov-type and Cauchy-Schwarz inequalities with the lowest eigenvalue. (-+ - HWILB; -o- LILB; -\* - CSILB; -□- LE - the Lowest Eigenvalue  $\lambda$ )

This figure clearly demonstrates that among the three estimates considered here, the LILB provides the worse estimate and the CSILB provides the best estimate for the smallest eigenvalues of FEP (1.10) -(1.11). We now consider an application of the lower bounds for the smallest eigenvalues of FEP (1.10) -(1.11) found in Corollary 2.1 and Theorem 1.1 and 2.2. In [5], [6], [7] and [8], the authors have applied the LILB to the FEPs for  $\alpha \in (1, 2]$  to find the interval in which certain Mittag-Leffler functions have no real zeros. Similar results for  $\alpha \in (3, 4]$  are given in the following theorem.

**Theorem 2.3.** Let  $3 < \alpha \leq 4$  then based on the HWILB, LILB, and CSILB inequalities, the Mittag-Leffler function  $E_{\alpha, \alpha-2}(z)$  has no real zeros in the following domains:

HWILB inequality:

$$z \in \left(-\frac{\Gamma(\alpha)}{2B(2, \alpha-2)}, 0\right], \quad (2.8)$$

LILB inequality:

$$z \in \left(-\frac{\Gamma(\alpha)(\alpha-2)^{\alpha-2}}{2(\alpha-3)^{\alpha-3}}, 0\right], \quad (2.9)$$

CSILB inequality:

$$z \in \left(-\Gamma(\alpha) \left[ \frac{2\alpha-4\alpha(2\alpha-1)(2\alpha-5)B(1, \alpha)C_1(\alpha)+(2\alpha-5)}{2\alpha(2\alpha-1)(2\alpha-5)} \right]^{-\frac{1}{2}}, 0\right]. \quad (2.10)$$

**Proof.** Let  $\lambda$  be the smallest eigenvalue of the FEP (1.10) -(1.11), then  $z = \lambda$  is the smallest value of  $z$  for which  $E_{\alpha, \alpha-2}(-z) = 0$ . If there is another  $z$  smaller than  $\lambda$  for which  $E_{\alpha, \alpha-2}(-z) = 0$ , then it will contradict that  $\lambda$  is the smallest eigenvalue. Therefore,  $E_{\alpha, \alpha-2}(z)$  has no real zero for  $z \in (-\lambda, 0]$ . Now, according to HWILB, we have

$$\lambda \geq \frac{\Gamma(\alpha)}{2B(2, \alpha-2)} \text{ (see equation (1.12)).}$$

Thus,  $E_{\alpha, \alpha-2}(z)$  has no real zero for  $z \in \left(-\frac{\Gamma(\alpha)}{2B(2, \alpha-2)}, 0\right]$ . This proves equation (2.8). Equations (2.9) and (2.10) are proved in a similar fashion.

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From Figure 2.1, it is clear that among the three inequalities discussed in the paper, LILB provides the smallest interval and CSILB provide the largest interval in which the Mittag-Leffler function  $E_{\alpha, \alpha-2}(z)$  has no real zero.

### III. CONCLUSION

In this note, we established a Cauchy-Schwarz-type inequality for a fractional boundary value problem of order  $\alpha$  for  $\alpha \in (3, 4]$ . For the integer and fractional eigenvalue problems, we determined lower bounds for the first eigenvalue from Hartman-Wintner-type, Lyapunov-type and Cauchy-Schwarz-type inequalities. We showed that the Cauchy-Schwarz-type inequality provides the best and the Lyapunov-type inequality provides the worse estimate for the smallest eigenvalue of the FEP (1.10) – (1.11). We used these bounds for the smallest eigenvalue to find the domain in which the Mittag-Leffler function  $E_{\alpha, \alpha-2}(z)$ ,  $\alpha \in (3, 4]$  has no zero. Results showed that Cauchy-Schwarz-type inequality provides the largest domain in which  $E_{\alpha, \alpha-2}(z)$  has no zero.

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