

# A Study on Radio Mean Graceful for Certain Graphs

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**ABSTRACT:** The channel assignment problem in graph theory is a real-life problem to find the maximum number of channels in a specified bandwidth. A *radio labelling mean* of a connected graph  $G$  is an injection  $h$  from the vertices of  $G$  to the natural numbers such that  $d(u, w) + \left\lceil \frac{h(u)+h(w)}{2} \right\rceil \geq \text{diam}(G) + 1$ , for every pair of vertices  $u$  and  $w$  of  $G$ . The *radio mean number of  $h$*  denoted by  $\text{rmn}(h)$ , is the maximum number assigned to any vertex of  $G$ . The *radio mean number of  $G$* , denoted  $\text{rmn}(G)$ , is the minimum value of  $\text{rmn}(h)$ , taken over all labellings  $h$  of  $G$ . A graph  $G$  is said to be a radio mean graceful if  $\text{rmn}(G) = |V(G)|$ . In this paper we have determined certain graphs having fixed diameter which satisfies the radio mean graceful conditions.

**KEYWORDS:** Labelling, Mean labelling, Radio mean labelling, Radio mean graceful, Shell graph, Multiple shell graph,  $WW(n, r)$  graph and  $HW(n, r)$  graph.

## I. INTRODUCTION

In graph theory an assignment problem predominantly represents a channel assignment problem because it is used in the field of communication engineering to maximize the number of channels by minimizing the bandwidth. To study this real life assignment problem in graph theoretical concept, Chartrand et.al [1] introduced the definition of radio labelling. A radio labelling of a connected graph  $G$  is an injection  $h$  from the vertices of  $G$  to the natural numbers such that  $d(u, w) + |h(u) - h(w)| \geq 1 + \text{diam}(G)$  for every pair of vertices  $u$  and  $w$  of  $G$ . The radio number of  $h$  denoted by  $\text{rn}(h)$ , is the maximum number assigned to any vertex of  $G$ . The radio number of  $G$ , denoted  $\text{rn}(G)$ , is the minimum value of  $\text{rn}(h)$ , taken over all labellings  $h$  of  $G$ . Several authors studied the radio number problem for graphs such as paths, trees, cycles, wheel related graphs, hypercubes, Mesh derived architectures [2,5,6,8,16,17]. Motivated by the combined concept of mean labelling [15] and radio labelling [1], Ponraj et al. [11] introduced the concept of radio mean labelling in 2014. Further they determined the radio mean number of graphs like cycle, path, complete bipartite graph, gear graph, wheel graph, sun flower graph, helm graphs, subdivision of wheel, star graph, jelly fish graph and subdivision of jelly fish graph [12,13,14]. Mehmet Ali Balci et.al [10] studied the average edge distance in graphs. Further in 2019, Lavanya et.al [9] introduced the concept of radio mean graceful labelling. A graph  $G$  is said to be a radio mean graceful if  $\text{rmn}(G) = |V(G)|$ . Also, they proved that certain graphs like subdivision of complete graph, Mongolian tent graph and friendship graphs are radio mean graceful. In this paper we have shown that the complete bipartite graph, shell graph, multiple shell graph, the  $WW(n, r)$  graph and  $HW(n, r)$  graphs are radio mean graceful.

## II. PRELIMINARIES

In this section we have given the definitions and basic results of complete bipartite graph, shell graph, multiple shell graph,  $WW(n, r)$  graph and  $HW(n, r)$  graph.

**Definition 2.1:** A *complete bipartite graph* is a graph whose vertices can be partitioned into two disjoint subsets  $V_1$  and  $V_2$  such that no edge has both endpoints in the same subset, and every possible edge that could connect vertices in different subsets is part of the graph. A complete bipartite graph with partitions of cardinality  $|V_1| = m$  and  $|V_2| = n$ , is denoted by  $K_{m,n}$ .

**Definition 2.2:** A shell graph denoted by  $C_{n,n-3}$  and is defined as a cycle  $C_n$  with  $(n - 3)$  chords sharing a common end point called the apex.

**Remark 1:** The number of vertices and edges  $C_{n,n-3}$  are  $n$  and  $2n - 3$  respectively. Also, the diameter of the graph is 2.

**Definition 2.3:** A multiple shell graph is defined to be a collection of edge disjoint shells that have their apex is common. It is denoted by  $MC_{n,n-3}$ .

**Remark 2:** Suppose there are  $r -$  edge disjoint shells, then the number of vertices in  $MC_{n,n-3}$  is  $r(n - 1) + 1$ . Furthermore, the diameter of the graph is same as that of the shell graph  $C_{n,n-3}$ .

**Definition 2.4:** [7] Let  $W_{n+1} = C_n + K_1$  be a wheel with  $V(C_n) = \{w_i : 1 \leq i \leq n\}$  and hub  $w_{n+1}$ . Let  $W_r^i = C_r^i + K_1$  be the wheels with hubs  $w_i, 1 \leq i \leq n$ , respectively. The graph obtained is denoted by  $WW(n, r)$ .

**Remark 3:** The number of vertices in  $WW(n, r)$  is  $n(r + 1) + 1$  and the number of edges is  $2n(r + 1)$ . The diameter of  $WW(n, r)$  is 4.

**Definition 2.5:** [7] Let  $H_{n+1}$  be the helm with the pendant vertices  $v_i, 1 \leq i \leq n$ . Let  $W_r^i = C_r^i + K_1$  be wheels with hubs  $v_i, 1 \leq i \leq n$  respectively. The graph constructed is denoted by  $HW(n, r)$ .

**Remark 4:** The number of vertices and edges of  $HW(n, r)$  are  $n(r + 2) + 1$  and  $2n(r + 1) + n$  respectively. Also, the diameter of  $HW(n, r)$  is 6.

### III. MAIN RESULTS

In this section we have proved that the graphs defined in previous section are radio mean graceful.

**Theorem 3.1:** The complete bipartite graph  $K_{m,n}$  ( $m > n$ ) is radio mean graceful.

**Proof:** First we name the two set of vertices in the complete bipartite graph as  $\{v_1, v_2 \dots v_m\}$  and  $\{w_1, w_2 \dots w_n\}$ .

Define a mapping  $h: V(K_{m,n}) \rightarrow \{1, 2 \dots m + n\}$  as  $h(v_i) = i, i = 1, 2, \dots, m$  and  $h(w_i) = m + i, i = 1, 2, \dots, n$ .

Next, we claim that  $h$  is a valid radio mean labelling.

Since the diameter of the graph is 2, we must verify that  $d(u, w) + \left\lceil \frac{h(u) + h(w)}{2} \right\rceil \geq 3$  for every pair of vertices in  $K_{m,n}$  ( $m > n$ ). Also, since  $m > n$ , there are at least three vertices in the graph.

Let  $u$  and  $w$  be any two vertices in  $K_{m,n}$ .

**Case 1:** Let  $u = v_k$  and  $w = v_l$ , then  $d(u, w) = 2$  and  $h(u) = k, h(w) = l$ , where  $k \neq l$ . Therefore  $d(u, w) + \left\lceil \frac{h(u) + h(w)}{2} \right\rceil \geq 2 + \left\lceil \frac{1+2}{2} \right\rceil > 3$ .

**Case 2:** Suppose  $u = w_k$  and  $w = w_l$ , then  $d(u, w) = 2$  and  $h(u) = m + k, h(w) = m + l, k \neq l$ . Therefore,  $d(u, w) + \left\lceil \frac{h(u) + h(w)}{2} \right\rceil \geq 2 + \left\lceil \frac{2m+1+2}{2} \right\rceil > 3$ .

**Case 3:** If  $u = v_k$  and  $w = w_l$ , then  $d(u, w) = 1$  and  $h(u) = k, h(w) = m + l$ . Hence  $d(u, w) + \left\lceil \frac{h(u) + h(w)}{2} \right\rceil \geq 1 + \left\lceil \frac{m+2}{2} \right\rceil > 1 + 2 = 3$ , since  $m \geq 2$ .

Therefore,  $h$  is a valid radio mean labelling which attains the maximum value  $m + n = |V(K_{m,n})|$ . In addition to this,  $h$  is an injection and the range of  $h$  is  $\{1, 2 \dots m + n\}$  and hence we conclude that the complete bipartite graph  $K_{m,n}$  ( $m > n$ ) is radio mean graceful.

**Theorem 3.2:** The shell graph  $C_{n,n-3}, n > 3$  is radio mean graceful.

**Proof:** First we start naming the vertices of  $C_{n,n-3}$  from the apex vertex to the remaining vertices in clockwise order as  $\{x_1, x_2 \dots x_n\}$ . Define a mapping  $h: V(C_{n,n-3}) \rightarrow \{1, 2 \dots n\}$  as  $h(x_{2i-1}) = i, i = 1, 2, \dots, \left\lceil \frac{n}{2} \right\rceil$  and  $h(x_{2i}) = \left\lceil \frac{n}{2} \right\rceil + i, i =$

$1, 2, \dots, \lfloor \frac{n}{2} \rfloor$ . Since the diameter of the graph is 2, we must verify that  $d(u, w) + \lfloor \frac{h(u)+h(w)}{2} \rfloor \geq 3$  for every pair of vertices in  $C_{n,n-3}, n > 3$ .

**Case 1:** Suppose  $u = x_{2k-1}$  and  $w = x_{2l-1}, k \neq l$  then  $h(u) = k, h(w) = l$  and  $d(u, w) = 2$ . Therefore,  $d(u, w) + \lfloor \frac{h(u)+h(w)}{2} \rfloor \geq 2 + \lfloor \frac{1+2}{2} \rfloor > 3$ .

**Case 2:** Assume  $u$  and  $w$  are of the form  $x_{2k}$  and  $x_{2l}, k \neq l$ , then  $d(u, w) = 2$  and  $h(u) = \lfloor \frac{n}{2} \rfloor + k, h(w) = \lfloor \frac{n}{2} \rfloor + l$ . Therefore,  $d(u, w) + \lfloor \frac{h(u)+h(w)}{2} \rfloor \geq 2 + \lfloor \frac{n+1+2}{2} \rfloor > 3$ .

**Case 3:** If  $u = x_{2k-1}$  and  $w = x_{2l}$ , then  $d(u, w) \geq 1$  and  $h(u) = k, h(w) = \lfloor \frac{n}{2} \rfloor + l$ . Hence the required condition becomes,  $d(u, w) + \lfloor \frac{h(u)+h(w)}{2} \rfloor \geq 1 + \lfloor \frac{\lfloor \frac{n}{2} \rfloor + 2}{2} \rfloor > 1 + 2 = 3$ , since  $n > 3$ .

Therefore, the defined injection  $h$  is a valid radio mean labelling with the maximum value  $|V(C_{n,n-3})| = n$  and hence we agree that the shell graph  $C_{n,n-3}, n > 3$  is radio mean graceful.

**Theorem 3.3:** The multiple shell graph  $MC_{n,n-3}$  is radio mean graceful.

**Proof:** Let us assume that the multiple shell graph contains atleast  $r = 2$  edge disjoint shells. First, we name the  $r(n - 1) + 1$  vertices of the multiple shell graph as  $\{u_1, u_2 \dots u_{r(n-1)+1}\}$  starting from the apex vertex and move towards the other shells in clockwise sense.

Define a mapping  $h: V(MC_{n,n-3}) \rightarrow \{1, 2 \dots r(n - 1) + 1\}$  as  $h(u_{2i}) = i, i = 1, 2, \dots, \lfloor \frac{r(n-1)+1}{2} \rfloor$  and  $h(u_{2i-1}) = \lfloor \frac{r(n-1)+1}{2} \rfloor + i, i = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor$ . See Figure 1. Since the diameter of the graph is 2, the rest of the proof is similar to Theorem 3.2.

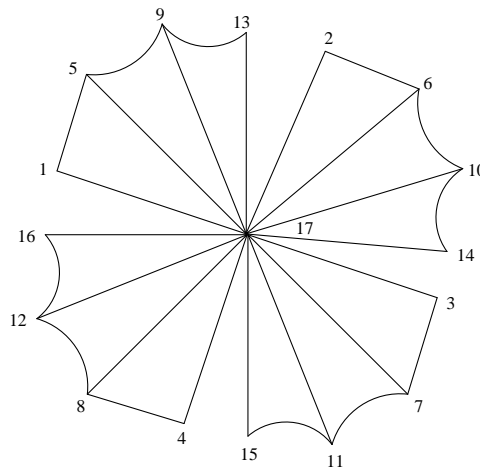


Figure 1: A multiple shell graph with 4- edge disjoint shells and its radio mean graceful labelling

**Theorem 3.4:** For  $n > 4$ , the graph  $WW(n, r)$  is radio mean graceful.

**Proof:** Let  $V(C_r^i) = \{v_{r(i-1)+j} : j = 1, 2 \dots r\}, i = 1, 2 \dots n$ . Also, we name the centre wheel vertices as  $\{u_i : i = 1, 2 \dots n + 1\}$ . Let us assume that the vertices are named in clockwise sense.

Define a mapping  $h: V(WW(n, r)) \rightarrow \{1, 2 \dots n(r + 1) + 1\}$  as follows:

$$h(v_{(i-1)r+j}) = n(j - 1) + i, j = 1, 2 \dots r, i = 1, 2 \dots n.$$

$$h(u_i) = nr + i, i = 1, 2 \dots n + 1.$$

Since the diameter of the graph is 4, we must claim that  $d(u, w) + \left\lceil \frac{h(u)+h(w)}{2} \right\rceil \geq 5$  holds for every pair of vertices  $u$  and  $w$  in  $WW(n, r)$ ,  $n > 4$ .

**Case 1:** Suppose  $u$  and  $w$  are any two vertices in the outer cycle, then  $u = v_{(k-1)r+l}$  and  $w = v_{(s-1)r+t}$ ,  $1 \leq k, s \leq n$ ,  $1 \leq l, t \leq r$ . Therefore,  $h(u) = n(l - 1) + k$  and  $h(w) = n(t - 1) + s$ .

**Case 1.1:** If  $k \neq s$ , then  $d(u, w) = 4$  and  $\left\lceil \frac{h(u)+h(w)}{2} \right\rceil \geq \left\lceil \frac{3}{2} \right\rceil = 2$ . Therefore,  $d(u, w) + \left\lceil \frac{h(u)+h(w)}{2} \right\rceil \geq 5$ .

**Case 1.2:** If  $l \neq t$ , then  $d(u, w) \geq 1$  and  $\left\lceil \frac{h(u)+h(w)}{2} \right\rceil \geq \left\lceil \frac{n(l+t-2)+2}{2} \right\rceil \geq 4$ , since  $n > 4$ . Hence,  $d(u, w) + \left\lceil \frac{h(u)+h(w)}{2} \right\rceil \geq 5$ .

**Case 2:** Suppose  $u$  and  $w$  are any two vertices in the inner wheel, then  $u = u_k$  and  $w = u_l$ ,  $1 \leq k \neq s \leq n + 1$ . Therefore  $h(u) = nr + k$  and  $h(w) = nr + l$ , which trivially verifies the radio mean labelling condition.

**Case 3:** Suppose  $u$  is a vertex in the outer cycle and  $w$  is a vertex in the inner wheel, then  $h(u) = n(l - 1) + k$ ,  $1 \leq k \leq r$ ,  $1 \leq l \leq n$  and  $h(w) = nr + s$ ,  $1 \leq s \leq n + 1$ . Again, the distance between them is atleast 1. Therefore, the radio mean labelling condition becomes,  $d(u, w) + \left\lceil \frac{h(u)+h(w)}{2} \right\rceil \geq 5$ . Thus,  $h$  is a valid radio mean labelling.

Hence, we conclude that the graph  $WW(n, r)$ ,  $n > 4$  is radio mean graceful.

**Theorem 3.5:** For  $n > 9$ , the graph  $HW(n, r)$  is radio mean graceful.

**Proof:** Let the vertices of the  $n$ -cycles be named as  $V(C_r^i) = \{u_{r(i-1)+j} : j = 1, 2 \dots r\}$ ,  $i = 1, 2 \dots n$ . Let  $w_i$ ,  $i = 1, 2 \dots n + 1$  be the vertices inner wheel. Also, we assume that the vertices  $u_i$ ,  $v_i$  and  $w_i$  are appear in the clockwise sense. Define an injection  $h: V(HW(n, r)) \rightarrow \{1, 2 \dots n(r + 2) + 1\}$  as follows:

$$h(u_{(i-1)r+j}) = n(j - 1) + i, j = 1, 2 \dots r, i = 1, 2 \dots n.$$

$$h(v_i) = nr + i, i = 1, 2 \dots n.$$

$$h(w_i) = n(r + 1) + i, i = 1, 2 \dots n + 1.$$

Since, the diameter of  $HW(n, r)$  is 6, we must show that  $d(x, y) + \left\lceil \frac{h(x)+h(y)}{2} \right\rceil \geq 7 \forall x, y \in V(HW(n, r))$ ,  $n > 9$ .

**Case 1:** Suppose  $x$  and  $y$  are any two vertices in the outer cycle, then  $x = u_{(k-1)r+l}$  and  $y = u_{(s-1)r+t}$ ,  $1 \leq k, s \leq n$ ,  $1 \leq l, t \leq r$  and hence  $h(x) = n(l - 1) + k$ ,  $h(y) = n(t - 1) + s$ .

**Case 1.1:** If  $k \neq s$ , then  $d(x, y) = 6$  and  $\left\lceil \frac{h(x)+h(y)}{2} \right\rceil \geq \left\lceil \frac{3}{2} \right\rceil = 2$ . Therefore,  $d(x, y) + \left\lceil \frac{h(x)+h(y)}{2} \right\rceil \geq 7$ .

**Case 1.2:** If  $l \neq t$ , then  $d(x, y) \geq 1$  and  $\left\lceil \frac{h(x)+h(y)}{2} \right\rceil \geq \left\lceil \frac{n(l+t-2)+2}{2} \right\rceil \geq 4$ , since  $n > 4$ . Hence,  $d(x, y) + \left\lceil \frac{h(x)+h(y)}{2} \right\rceil \geq 7$ .

**Case 2:** Suppose  $x$  and  $y$  are any two hub vertices, then  $x = v_k$  and  $w = v_l$ ,  $1 \leq k \neq l \leq n$ . Therefore,  $h(x) = nr + k$ ,  $h(y) = nr + l$  and the distance between them is atleast 3. So,  $d(x, y) + \left\lceil \frac{h(x)+h(y)}{2} \right\rceil \geq 3 + \left\lceil \frac{2nr+2}{2} \right\rceil > 7$ , since  $n > 9$ .

**Case 3:** Suppose  $x$  and  $y$  are any two vertices in the inner wheel, then  $x = w_k$  and  $w = w_l$ ,  $1 \leq k \neq l \leq n + 1$ . Therefore,  $d(x, y) \geq 1$  and  $h(x) = n(r + 1) + k$ ,  $h(y) = n(r + 1) + l$ . Hence,  $d(x, y) + \left\lceil \frac{h(x)+h(y)}{2} \right\rceil > 7$ , since  $n > 9$ .

**Case 4:** Let  $x$  be a vertex in the outer cycle and  $y$  be a hub vertex, then the labellings of  $x$  and  $y$  are  $n(l - 1) + k$ ,  $1 \leq k \leq r$ ,  $1 \leq l \leq n$  and  $nr + s$ ,  $1 \leq s \leq n$  respectively. Also,  $d(x, y) \geq 1$ . Therefore,  $d(x, y) + \left\lceil \frac{h(x)+h(y)}{2} \right\rceil \geq 1 + \left\lceil \frac{n(l-1)+k+nr+s}{2} \right\rceil > 7$ .

**Case5:** Suppose  $x$  lie in the outer cycle and  $y$  lie in the inner wheel, then the labellings of  $x$  and  $y$  are  $h(x) = n(l-1) + k, 1 \leq k \leq r, 1 \leq l \leq n$  and  $h(y) = n(r+1) + t, 1 \leq t \leq n+1$  respectively. Also,  $d(x, y) \geq 2$ . Therefore,  $d(x, y) + \left\lceil \frac{h(x)+h(y)}{2} \right\rceil \geq 2 + \left\lceil \frac{n(l-1)+k+n(r+1)+t}{2} \right\rceil > 7$

**Case 6:** Let  $x$  and  $y$  be a hub vertex and centre wheel vertex respectively, then  $h(x) = nr + s, 1 \leq s \leq n+1$ ,  $h(y) = n(r+1) + k, 1 \leq k \leq n+1$  and  $d(x, y) \geq 1$ . Therefore,  $d(x, y) + \left\lceil \frac{h(x)+h(y)}{2} \right\rceil \geq 7$ . Hence in all the cases  $h$  satisfies the radio mean labelling condition. Also, the range of  $h$  is  $\{1, 2, \dots, n(r+2) + 1\}$ . Therefore, we arrived the result that, the graph  $HW(n, r), n > 9$  is radio mean graceful.

#### IV. CONCLUSION

In this paper we have proved that the complete bipartite graph  $K_{m,n}$  is radio mean graceful for  $m > n$ . Also, we have shown that the shell graph  $C_{n,n-3}, n > 3$  and multiple shell graph  $MC_{n,n-3}$  are radio mean graceful. In addition to this, we have proved the special graphs  $HW(n, r), n > 9$  and  $WW(n, r), n > 4$  are radio mean graceful for  $n > 9$  and  $n > 4$  respectively. Further the study is extended to the varying diameter interconnection networks such as grid networks, honeycomb networks, hexagonal derived networks, silicate and oxide networks.

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