

Solution of the Wave Equation Using a Special Transformation of Variables

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ABSTRACT: In this paper we focus our study on the method to solve the wave equation, found by Jean Le Rond D'Alembert in the 18th century. The functions of several variables and having worked through the concept of a partial derivative. Solutions to wave equations will involve functions not just of one variable, but of several variables. We conclude that the solution of these PDEs are second order, and are linear.

KEYWORDS: Wave Equations, Infinite Set, Transformation, Several Variables, Boundary Conditions, Initial Velocity Function.

I. INTRODUCTION

There is important method to solve the wave equation, found by Jean Le Rond D'Alembert in the 18th century. There is solution to the wave equation using the separation of variables technique, in which we made use of the boundary conditions in finding the solutions to the PDE. We glossed over the issue of the initial conditions, until the very end when we claimed that one could make use of something called Fourier Series to build up combinations of solutions.

II. RELATED WORK

2.1 Wave Equation by Separation of Variables : There are several approaches to solving the wave equation. The first one we will work with, using a technique called separation of variables, again, demonstrates one of the most widely used solution techniques for PDEs. The idea behind it is to split up the original PDE into a series of simpler ODEs, each of which we should be able to solve readily using tricks already learned [1]. The second technique, which we will see in the next section, uses a transformation trick that also reduces the complexity of the original PDE, but in a very different manner.

First, note that for a specific wave equation situation, in addition to the actual PDE, we will also have boundary conditions arising from the fact that the endpoints of the string are attached solidly, at the left end of the string, when $x = 0$ and at the other end of the string, which we suppose has overall length l . Let's start the process of solving the PDE by first figuring out what these boundary conditions imply for the solution function, $u(x, t)$.

$$u(0, t) = 0 \quad \text{and} \quad u(l, t) = 0 \quad \text{for all values of } t \quad (2.1, 1)$$

are the boundary conditions for our wave equation. These will be key when we later on need to sort through possible solution functions for functions that satisfy our particular vibrating string set-up.

We might also note that we probably need to specify what the shape of the string is right when time $t = 0$, and you're right - to come up with a particular solution function, we would need to know $u(x, 0)$. In fact we would also need to know the initial velocity of the string, which is just $u_t(x, 0)$.

These two requirements are called the initial conditions for the wave equation, and are also necessary to specify a particular vibrating string solution [4]. For instance, as the simplest example of initial conditions, if no one is plucking the string, and it's perfectly flat to start with, then the initial conditions would just be $u(x, 0) = 0$ (a perfectly

flat string) with initial velocity, $u_t(x,0) = 0$. Here, then, the solution function is pretty unenlightening – it's just $u(x,t) = 0$, i.e. no movement of the string through time.

To start the separation of variables technique we make the key assumption that whatever the solution function is, that it can be written as the product of two independent functions, each one of which depends on just one of the two variables, x or t. Thus, imagine that the solution function, $u(x,t)$ can be written as

$$u(x,t) = F(x)G(t) \quad (2.1, 2)$$

where F, and G, are single variable functions of x and t respectively. Differentiating this equation for $u(x,t)$ twice with respect to each variable yields

$$\frac{\partial^2 u}{\partial x^2} = F''(x)G(t) \quad \text{and} \quad \frac{\partial^2 u}{\partial t^2} = F(x)G''(t) \quad (2.1, 3)$$

Thus when we substitute these two equations back into the original wave equation, which is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (2.1, 4)$$

then we get

$$\frac{\partial^2 u}{\partial t^2} = F(x)G''(t) = c^2 \frac{\partial^2 u}{\partial x^2} = c^2 F''(x)G(t) \quad (2.1, 5)$$

Here's where our separation of variables assumption pays off, because now if we separate the equation above so that the terms involving F and its second derivative are on one side, and likewise the terms involving G and its derivative are on the other, then we get

$$\frac{G''(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} \quad (2.1, 6)$$

Now we have an equality where the left-hand side just depends on the variable t, and the right-hand side just depends on x [3-6]. Here comes the critical observation - how can two functions, one just depending on t, and one just on x, be equal for all possible values of t and x? The answer is that they must each be constant, for otherwise the equality could not possibly hold for all possible combinations of t and x. Aha! Thus we have

$$\frac{G''(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} = k \quad \text{where } k \text{ is a constant.} \quad (2.1, 7)$$

III. METHODOLOGY

3.1 D'Alembert Solution of the Wave Equation : Separately, we look at the existence of solution the initial value problem for the wave equation. In one dimension this is called D'Alembert's Solution. We recall, being given specific initial conditions meant being given both the shape of the string at time $t = 0$, i.e. the function $u(x,0) = f(x)$, as well as the initial velocity, $u_t(x,0) = g(x)$ (note that these two initial condition functions are functions of x alone, as t is set equal to 0) [4]. In the separation of variables solution, we ended up with an infinite set, or family, of solutions, $u_n(x,t)$ that we said could be combined in such a way as to satisfy any reasonable initial conditions.



In using D'Alembert's approach to solving the wave equation, we don't need to use Fourier series to build up the solution from the initial conditions [9-10]. Instead, we are able to explicitly construct solutions to the wave equation for any (reasonable) given initial condition functions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$.

IV RESULT AND DISCUSSION

4.1 Solution Of The Wave Equation Using A Special Transformation Of Variables : The technique involves changing the original PDE into one that can be solved by a series of two simple single variable integrations by using a special transformation of variables [1]. Suppose that instead of thinking of the original PDE

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{4.1, 1}$$

in terms of the variables x , and t , we rewrite it to reflect two new variables

$$v = x + ct \text{ and } z = x - ct \tag{4.1, 2}$$

This then means that u , originally a function of x , and t , now becomes a function of v and z , instead. How does this work? Note that we can solve for x and t in (2), so that

$$x = \frac{1}{2}(v + z) \text{ and } t = \frac{1}{2c}(v - z) \tag{4.1, 3}$$

Now using the chain rule for multivariable functions, you know that

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial v} \frac{\partial v}{\partial t} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial t} = c \frac{\partial u}{\partial v} - c \frac{\partial u}{\partial z} \tag{4.1, 4}$$

since $\frac{\partial v}{\partial t} = c$ and $\frac{\partial z}{\partial t} = -c$, and that similarly

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial x} = \frac{\partial u}{\partial v} + \frac{\partial u}{\partial z} \tag{4.1, 5}$$

since $\frac{\partial v}{\partial x} = 1$ and $\frac{\partial z}{\partial x} = 1$. Working up to second derivatives, another, more involved application of the chain rule yields that

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial t} \left(c \frac{\partial u}{\partial v} - c \frac{\partial u}{\partial z} \right) = c \left(\frac{\partial^2 u}{\partial v^2} \frac{\partial v}{\partial t} + \frac{\partial^2 u}{\partial z \partial v} \frac{\partial z}{\partial t} \right) - c \left(\frac{\partial^2 u}{\partial z^2} \frac{\partial z}{\partial t} + \frac{\partial^2 u}{\partial v \partial z} \frac{\partial v}{\partial t} \right) \\ &= c^2 \left(\frac{\partial^2 u}{\partial v^2} - \frac{\partial^2 u}{\partial z \partial v} \right) + c^2 \left(\frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 u}{\partial v \partial z} \right) = c^2 \left(\frac{\partial^2 u}{\partial v^2} - 2 \frac{\partial^2 u}{\partial z \partial v} + \frac{\partial^2 u}{\partial z^2} \right) \end{aligned} \tag{4.1, 6}$$

Another almost identical computation using the chain rule results in the fact that

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial v} + \frac{\partial u}{\partial z} \right) = \left(\frac{\partial^2 u}{\partial v^2} \frac{\partial v}{\partial x} + \frac{\partial^2 u}{\partial z \partial v} \frac{\partial z}{\partial x} \right) + \left(\frac{\partial^2 u}{\partial z^2} \frac{\partial z}{\partial x} + \frac{\partial^2 u}{\partial v \partial z} \frac{\partial v}{\partial x} \right)$$



$$= \frac{\partial^2 u}{\partial v^2} + 2 \frac{\partial^2 u}{\partial z \partial v} + \frac{\partial^2 u}{\partial z^2} \tag{4.1, 7}$$

Now we revisit the original wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{4.1, 8}$$

and substitute in what we have calculated for $\frac{\partial^2 u}{\partial t^2}$ and $\frac{\partial^2 u}{\partial x^2}$ in terms of $\frac{\partial^2 u}{\partial v^2}$, $\frac{\partial^2 u}{\partial z^2}$ and $\frac{\partial^2 u}{\partial z \partial v}$. Doing this gives the following equation, ripe with cancellations:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial v^2} - 2 \frac{\partial^2 u}{\partial z \partial v} + \frac{\partial^2 u}{\partial z^2} \right) = c^2 \frac{\partial^2 u}{\partial x^2} = c^2 \left(\frac{\partial^2 u}{\partial v^2} + 2 \frac{\partial^2 u}{\partial z \partial v} + \frac{\partial^2 u}{\partial z^2} \right) \tag{4.1, 9}$$

Dividing by c^2 and canceling the terms involving $\frac{\partial^2 u}{\partial v^2}$ and $\frac{\partial^2 u}{\partial z^2}$ reduces this series of equations to

$$-2 \frac{\partial^2 u}{\partial z \partial v} = +2 \frac{\partial^2 u}{\partial z \partial v} \tag{4.1, 10}$$

which means that $\frac{\partial^2 u}{\partial z \partial v} = 0$ (4.1, 11)

So what, we ask, after all, we still have a second order PDE, and there are still several variables involved [4-9]. But wait, think about what (4.1,11) implies. Picture (4.1,11) as it gives you information about the partial derivative of a partial derivative:

$$\frac{\partial}{\partial z} \left(\frac{\partial u}{\partial v} \right) = 0 \tag{4.1, 12}$$

In this form, this implies that $\frac{\partial u}{\partial v}$ considered as a function of z and v is a constant in terms of the variable z , so that $\frac{\partial u}{\partial v}$ can only depend on v , i.e.

$$\frac{\partial u}{\partial v} = M(v) \tag{4.1, 13}$$

Now, integrating this equation with respect to v yields that

$$u(v, z) = \int M(v) dv \tag{4.1, 14}$$

This, as an indefinite integral, results in a constant of integration, which in this case is just constant from the standpoint of the variable v . Thus, it can be any arbitrary function of z alone, so that actually

$$u(v, z) = \int M(v) dv + N(z) = P(v) + N(z) \tag{4.1, 15}$$



where $P(v)$ is a function of v alone, and $N(z)$ is a function of z alone, as the notation indicates [2-3].

Substituting back the original change of variable equations for v and z in (2) yields that

$$u(x, t) = P(x + ct) + N(x - ct) \tag{4.1, 16}$$

where P and N are arbitrary single variable functions. This is called D’Alembert’s solution to the wave equation. Except for the somewhat annoying but easy enough chain rule computations, this was a pretty straightforward solution technique [11]. The reason it worked so well in this case was the fact that the change of variables used in (4.1,2) were carefully selected so as to turn the original PDE into one in which the variables basically had no interaction, so that the original second order PDE could be solved by a series of two single variable integrations, which was easy to do.

According to this solution, you can pick any functions for P and N such as $P(v) = v^2$ and $N(v) = v + 2$.

Then
$$u(x, t) = (x + ct)^2 + (x - ct) + 2 = x^2 + x + ct + c^2t^2 + 2 \tag{4.1, 17}$$

$$\frac{\partial^2 u}{\partial t^2} = 2c^2 \tag{4.1, 18}$$

and that
$$\frac{\partial^2 u}{\partial x^2} = 2 \tag{4.1, 19}$$

so that indeed
$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{4.1, 20}$$

and so this is in fact a solution of the original wave equation.

This same transformation trick can be used to solve a fairly wide range of PDEs. For instance one can solve

the equation
$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y^2} \tag{4.1, 21}$$

by using the transformation of variables

$$v = x \text{ and } z = x + y \tag{4.1, 22}$$

Note that in our solution (4.1,16) to the wave equation, nothing has been specified about the initial and boundary conditions yet, and we said we would take care of this time around. So now we take a look at what these conditions imply for our choices for the two functions P and N .

If we were given an initial function $u(x,0) = f(x)$ along with initial velocity function $u_t(x,0) = g(x)$ then we can match up these conditions with our solution by simply substituting in $t = 0$ into (16) and follow along. We start first with a simplified set-up, where we assume that we are given the initial displacement function $u(x,0) = f(x)$, and that the initial velocity function $g(x)$ is equal to 0 (i.e. as if someone stretched the string and simply released it without imparting any extra velocity over the string tension alone).

Now the first initial condition implies that

$$u(x,0) = P(x + c \cdot 0) + N(x - c \cdot 0) = P(x) + N(x) = f(x) \tag{4.1, 23}$$



We next figure out what choosing the second initial condition implies. By working with an initial condition that $u_t(x,0) = g(x) = 0$, we see that by using the chain rule again on the functions P and N

$$u_t(x,0) = \frac{\partial}{\partial t} (P(x+ct) + N(x-ct)) = cP'(x+ct) - cN'(x-ct) \quad (4.1, 24)$$

(remember that P and N are just single variable functions, so the derivative indicated is just a simple single variable derivative with respect to their input). Thus in the case where $u_t(x,0) = g(x) = 0$, then

$$cP'(x+ct) - cN'(x-ct) = 0 \quad (4.1, 25)$$

Dividing out the constant factor c and substituting in $t = 0$

$$P'(x) = N'(x) \quad (4.1, 26)$$

and so $P(x) + k = N(x)$ for some constant k. Combining this with the fact that $P(x) + N(x) = f(x)$, means that $2P(x) + k = f(x)$, so that $P(x) = (f(x) - k)/2$ and likewise $N(x) = (f(x) + k)/2$. Combining these leads to the solution

$$u(x,t) = P(x+ct) + N(x-ct) = \frac{1}{2} (f(x+ct) + f(x-ct)) \quad (4.1, 27)$$

To make sure that the boundary conditions are met, we need

$$u(0,t) = 0 \text{ and } u(l,t) = 0 \quad \text{for all values of } t \quad (4.1, 28)$$

The first boundary condition implies that

$$u(0,t) = \frac{1}{2} (f(ct) + f(-ct)) = 0 \quad (4.1, 29)$$

or $f(-ct) = -f(ct) \quad (4.1, 30)$

so that to meet this condition, then the initial condition function f must be selected to be an odd function. The second boundary condition that $u(l,t) = 0$ implies

$$u(l,t) = \frac{1}{2} (f(l+ct) + f(l-ct)) = 0$$

so that $f(l+ct) = -f(l-ct)$. Next, since we've seen that f has to be an odd function, then $-f(l-ct) = f(-l+ct)$. Putting this all together this means that

$$f(l+ct) = f(-l+ct) \text{ for all values of } t$$

which means that f must have period 2l, since the inputs vary by that amount. Remember that this just means the function repeats itself every time 2l is added to the input, the same way that the sine and cosine functions have period 2π .

What happens if the initial velocity isn't equal to 0? Thus suppose $u_t(x,0) = g(x) \neq 0$. Tracing through the same types of arguments as the above leads to the solution function

$$u(x, t) = \frac{1}{2} (f(x + ct) + f(x - ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds$$

V CONCLUSION

However, with more variables involved, one typically has to invoke the technique several times in a row, splitting PDEs involving functions of more than two variables into a sequence of PDEs each involving just one-variable functions. Solutions are then found for each of the one-variable differential equations, and combined to yield solutions to the original PDEs, in much the same way that we saw in the two separations of variables solutions.

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