

# A SYSTEM OF NONLINEAR EQUATIONS WITH SINGULAR JACOBIAN

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**Abstract:** In this paper, I extend the idea of Otolorin's [1] for solving a single variable nonlinear equation to a system of non linear equations, Permitting Singular Jacobian at some points in the vicinity of the required root. It is proved that this modification has quadratic convergence. Several numerical examples are also given to illustrate the efficiency and the performance of the presented method. Therefore, this method may be viewed as an extension and generalization of the existing methods.

**Keywords:** Nonlinear equations, Iterative's methods, Taylor series expansion, Singular Jacobian.

## I. INTRODUCTION

Due to the fact that system of nonlinear equations arises frequently in science and engineering they have attracted researcher's interest. For example, nonlinear system of equations, after the necessary processing step of implicit discretization, is solved by finding the solutions of system of equations. We consider here the problem of finding a real zero, say  $x^* = (x^*_1, x^*_2, \dots, x^*_n)^T$ , of a system of non linear equations

$$\begin{aligned} f_1(x_1, x_2, \dots, x_n) &= 0; \\ f_2(x_1, x_2, \dots, x_n) &= 0; \\ &\vdots \\ &\vdots \\ f_n(x_1, x_2, \dots, x_n) &= 0; \end{aligned}$$

This system can referred in vector form by

$$F(X) = 0 \quad (I.1)$$

$$\text{where } F = (f_1, f_2, \dots, f_n)^T \quad \text{and} \quad X = (x_1, x_2, \dots, x_n)^T$$

Let the mapping  $F : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$  assumed to satisfy the following assumptions (I.1)  $F(X)$  is continuously differentiable in an open neighborhood  $D$  of  $X^*$ . There exists a solution vector  $X^*$  of (I.1) in  $D$  such that  $F(X^*) = 0$  and  $F'(X^*) \neq 0$  then the standard method for finding the solution to equations (I.1) is the classical Newton method [2-5] given by

$$X^{(k+1)} = X^{(k)} - F(X^{(k)})/F'(X^{(k)}); \quad k = 0, 1, \dots \quad (I.2)$$

Though, Newton formula (I.2) is simple and fast with quadratic convergence but it may fail miserably if at any stage of computation, the Jacobian matrix of  $F(X)$  at any iterative point is singular or almost singular i.e.  $|F'(X)| = 0$  Therefore, it has poor convergence and stability problems as it is very sensitive to initial guess. Classical methods by the simple modification of iteration processes.

## II. DEVELOPMENT OF THE METHOD

First we shall outline the derivation of the method in  $n$ -dimensions. If we multiply by  $e^{p(x_1 - x_{10})}$ ,  $e^{p(x_2 - x_{20})}$ ,  $\dots, e^{p(x_n - x_{n0})}$ ; each of above expression the result will not be change. Consider the following system of  $n$ -nonlinear equations in  $n$ -unknowns  $x_1, x_2, \dots, x_n$ :

$$\begin{aligned} e^{p(x_1 - x_{10})} f_1(x_1, x_2, \dots, x_n) &= 0; \\ e^{p(x_2 - x_{20})} f_2(x_1, x_2, \dots, x_n) &= 0; \\ &\vdots \\ &\vdots \\ e^{p(x_n - x_{n0})} f_n(x_1, x_2, \dots, x_n) &= 0; \end{aligned}$$

Taking an estimate  $(x_{10}, x_{20}, \dots, x_{n0})$  of a solution (I.1), we try to compute step size  $(h_1, h_2, \dots, h_n)$  such that  $X = (x_{10} + h_1, x_{20} + h_2, \dots, x_{n0} + h_n)^T$  (II.1)

Let us denote the approximation as  $X = (x_1, x_2, \dots, x_n)^T$  and the step size as  $H = (h_1, h_2, \dots, h_n)^T$

Using Taylor's theorem for  $n$ -variables in equation (I.1), we get

$$F(X) = F(X_0 + H) = F(X_0) + HF'(X_0) + (1/2!)H^2F''(X_0) + \dots \quad (II.2)$$

$$X = X_0 + H \text{ where } X_0 = (x_{10}, x_{20}, \dots, x_{n0})^T$$

$$0 = f_1(X)$$

$$= f_1(X_0 + H)$$

$$\begin{aligned} &\approx [f_1(X_0) + h_1 \partial f_1 / \partial x_1 + h_2 \partial f_1 / \partial x_2 + \dots + h_n \partial f_1 / \partial x_n + O(h^2)] e^{ph_1} \\ &= [f_1(X_0) + h_1 \partial f_1 / \partial x_1 + h_2 \partial f_1 / \partial x_2 + \dots + h_n \partial f_1 / \partial x_n + O(h^2)] (1 + ph_1 + O(h^2)) \\ &= f_1(X_0) + h_1 \partial f_1 / \partial x_1 + h_2 \partial f_1 / \partial x_2 + \dots + h_n \partial f_1 / \partial x_n + ph_1 f_1(X_0) + O(h^2) \end{aligned}$$

Similarly

$$\begin{aligned} f_2(X_0) + h_1 \partial f_2 / \partial x_1 + h_2 \partial f_2 / \partial x_2 + \dots + h_n \partial f_2 / \partial x_n + ph_2 f_2(X_0) + O(h^2) &= 0 \\ f_3(X_0) + h_1 \partial f_3 / \partial x_1 + h_2 \partial f_3 / \partial x_2 + \dots + h_n \partial f_3 / \partial x_n + ph_3 f_3(X_0) + O(h^2) &= 0 \\ \dots & \\ f_n(X_0) + h_1 \partial f_n / \partial x_1 + h_2 \partial f_n / \partial x_2 + \dots + h_n \partial f_n / \partial x_n + ph_n f_n(X_0) + O(h^2) &= 0 \end{aligned}$$

Converting the system in matrix form, we have

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \dots & \dots & \dots \\ \frac{\partial f_n}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ \dots \\ h_n \end{bmatrix} + \begin{bmatrix} pf_1(X_0) & 0 & \dots & 0 \\ 0 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & pf_n(X_0) \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ \dots \\ h_n \end{bmatrix} + \begin{bmatrix} f_1(X_0) \\ f_2(X_0) \\ \dots \\ f_n(X_0) \end{bmatrix} = 0$$

$$\begin{aligned} [J_{F(X_0)} + \text{diag}(pf_i(X_0))] \begin{bmatrix} h_1 \\ h_2 \\ \dots \\ h_n \end{bmatrix} &= - \begin{bmatrix} f_1(X_0) \\ f_2(X_0) \\ \dots \\ f_n(X_0) \end{bmatrix} \\ \begin{bmatrix} h_1 \\ h_2 \\ \dots \\ h_n \end{bmatrix} &= - \frac{F(X_0)}{[J_{F(X_0)} + \text{diag}(pf_i(X_0))]} \end{aligned}$$

As  $X = X_0 + H$ , we have

$$X = X_0 - [J_{F(X_0)} + \text{diag}(pf_i(X_0))]^{-1} F(X_0) \quad (II.3)$$

Where  $p$  is any real non-zero parameter.

If  $p = 0$  then given method is Newton's method. Therefore  $p$  is a non-zero real number chosen so that the denominator has maximum value.

$$P = \begin{cases} \square - m^2 \text{ if } |J_{F(X_0)}| < 0 \text{ and } |\text{diag}(pf_i(X_0))| > 0 \\ -m^2 \text{ if } |J_{F(X_0)}| > 0 \text{ and } |\text{diag}(pf_i(X_0))| < 0 \\ m^2 \text{ if } |J_{F(X_0)}| < 0 \text{ and } |\text{diag}(pf_i(X_0))| < 0 \\ m^2 \text{ if } |J_{F(X_0)}| > 0 \text{ and } |\text{diag}(pf_i(X_0))| > 0 \end{cases}$$

### III. CONVERGENCE ANALYSIS

We shall present the mathematical proof for the order of convergence of formula (II.3) the order of convergence for the remaining algorithms can be proved in same lines.

Theorem III.1. Let  $r \in D$  be a simple zero of a sufficiently differentiable function  $F : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$  for interval  $D$ . Let  $X_0$  be an initial guess sufficiently closed to  $r$  and  $[J_{F(X_0)} + \text{diag}(pf_i(X_0))] \neq 0$  in  $D$ . Then the sequence generated by formula (3.3) is quadratical convergent.

Proof. : Let  $r$  be a simple zero of  $F(X) = 0$  (i.e.  $F(r) = 0$ ) and  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be defined

$$\text{by } \phi(X) = X - [J_{F(X)} + \text{diag}(pf_i(X))]^{-1} F(X)$$

This can be written as

$$[J_{F(X)} + \text{diag}(pf_i(X))] (\phi(X) - X) + F(X) = 0$$

(III.1)

Taking  $\sum$

We have

$$\sum_1^n [J_{F(X)} + \text{diag}(pf_i(X))] (\phi_i(X) - X) + f_i(X) = 0$$

(III.2)

Putting  $X = r$  in (III.2) we have

$$\phi_j(r) = r_j$$

(III.3)

Differentiate (III.2) w.r.t.  $x_q$  we have

$$\sum_1^n \frac{\partial [J_{F(X)} + \text{diag}(pf_i(X))]}{\partial x_q} (\phi_i(X) - x_i) + \sum_1^n [J_{F(X)} + \text{diag}(pf_i(X))] \frac{\partial \phi_i(X)}{\partial x_q} - \delta_{jq} + \partial f_i(X) / \partial x_q = 0$$

(III.4)

Using (III.3) in (III.4) and putting  $X = r$  we have

$$\partial \phi_j(r) / \partial x_q = 0$$

(III.5)

Differentiate (III.4) w.r.t.  $x_r$  we have

$$\begin{aligned} \sum_1^n \frac{\partial^2 [J_{F(X)} + \text{diag}(pf_i(X))]}{\partial x_r \partial x_q} (\phi_i(X) - x_i) + \sum_1^n \partial [J_{F(X)} + \text{diag}(pf_i(X))] / \partial x_q \frac{\partial \phi_i(X)}{\partial x_r} - \delta_{jr} \\ + \sum_1^n \partial [J_{F(X)} + \text{diag}(pf_i(X))] / \partial x_r \frac{\partial \phi_i(X)}{\partial x_q} - \delta_{jq} + \sum_1^n [J_{F(X)} + \text{diag}(pf_i(X))] \frac{\partial^2 \phi_i(X)}{\partial x_r \partial x_q} + \partial^2 f_i(X) / \partial x_r \partial x_q = 0 \end{aligned}$$

(III.6)

Putting  $X = r$  in (III.6) and using (III.5) and (III.3) we have

$$\partial^2 \phi_i(r) / \partial x_r \partial x_q \neq 0$$

(III.7)

therefore the equation (II.3) is quadratically convergent.

#### IV. NUMERICAL RESULTS

In this section, we shall check the performance of the Present families say S1(II.3) and S2(II.3) by taking  $m = 0.5$  and  $m = 0.05$ . The comparisons are carried out with Newton method and with GTM1 and GTM2[8]. A mat lab program has been written to implement these methods. We use the following stopping criteria for computer programs. We use  $\epsilon = e^{-10}$

(i)  $|F(X_n)| < \epsilon$

For every method, we analyze the number of iterations needed to converge to the required solution. The numerical results are reported in the table 1

We consider the following problems for a system of nonlinear equations.

Problem (a)

$$\begin{aligned} x_1^2 - 2x_1 - x_2 + 0.5 &= 0 \\ x_1^2 + 4x_2^2 - 4 &= 0 \end{aligned}$$

Problem (b)

$$\begin{aligned} x_1^2 + x_2^2 - 1 &= 0 \\ x_1^2 - x_2^2 + 0.5 &= 0 \end{aligned}$$

Problem (c)

$$\begin{aligned} x_1^2 - x_2^2 + 3\log(x_1) &= 0 \\ 2x_1^2 - x_1 x_2 - 5x_1 + 1 &= 0 \end{aligned}$$

Problem (d)

$$\begin{aligned} e^{x_1} + x_1 x_2 - x_2 - 0.5 &= 0 \\ \sin(x_1 x_2) + x_1 + x_2 - 1 &= 0 \end{aligned}$$

Problem (e)

$$\begin{aligned} x_1 + 2x_2 - 3 &= 0 \\ 2x_1^2 + x_2^2 - 5 &= 0 \end{aligned}$$

Problem (f)

$$\begin{aligned} x_1^2 - x_2^2 - 4 &= 0 \\ x_1^2 + x_2^2 - 16 &= 0 \end{aligned}$$

Problem (g)

$$\begin{aligned} x_1^2 - x_2^2 - 1 &= 0 \\ x_1^3 x_2^2 - 1 &= 0 \end{aligned}$$

- Solution (a)  $r = (1.9006767263670658, 0.31121856541929427)^T$
- Solution (b)  $r = (0.5000000000000000, 0.8660254378443865)^T$   
 $r = (-0.5000000000000000, -0.8660254378443865)^T$
- Solution (c)  $r = (1.3192058033298924, -1.6035565551874148)^T$
- Solution (d)  $r = (0, 1)^T$
- Solution (e)  $r = (1.4880338717125849, 0.75598306414370757)^T$
- Solution (f)  $r = (3.1622776601683820, 2.44948974278312840)^T$
- Solution (g)  $r = (1.2365057033915010, 0.7272869822289620)^T$

Table I Numerical results of problems (a) to (g) using different methods.

F(X)	X	NM	GMT1	GMT2	S1	S2
(a)	$(3,2)^T$	9	5	6	4	5
(a)	$(1.6,0)^T$	8	6	5	3	4
(b)	$(.7,-2)^T$	7	5	5	3	2
(b)	$(-1,-2)^T$	8	6	6	5	4
(c)	$(.91,-2)^T$	8	6	5	4	4
(c)	$(1.5,-1.5)^T$	7	5	5	3	3
(d)	$(.9, .9)^T$	8	5	5	4	4
(d)	$(-0.1, 0.2)^T$	7	5	5	3	3
(e)	$(.9, .5)^T$	8	6	5	4	4
(e)	$(1.5, 1)^T$	7	5	4	3	3
(f)	$(0, 0)^T$	Fails	Fails	Fails	4	Div
(f)	$(3, 2)^T$	3	Fails	2	3	3
(g)	$(0, 0)^T$	Fails	Fails	Fails	15	26
(g)	$(1, 2)^T$	12	12	12	12	12

### V. CONCLUSIONS

The presented families are simple to understand, easy to program and have the same rate of convergence as Newton method and GTM1, GTM2 have. The behavior of existing iterative scheme and proposed modification can be compared by their corresponding correction factors. The correction factor  $F(X^k)/F'(X^k)$ , which appears in the Newton method and its variants, is now modified by

$$\frac{F(X^k)}{j_{F(X^k)} + \text{diag}(pf_i(X^k))}; \quad p \neq 0$$

This factor is always well defined, even if Jacobian determinate of  $|F(X^k)| = 0$  at some points in the vicinity of required root. Moreover, they have the same efficiency indices as that of existing method. Therefore, these techniques have a definite practical utility. However, if  $|F'(X^k)| = 0$  and any of  $f_i(X^k)$  is zero at initial guess, then the method will be work.

### REFERENCES

- [1] Otolorin, O., (2005). A new Newton like method for roots of analytic function, International Journal of Mathematical education in Science and Education, 36(5), 539-546.
- [2] Ostrowski, A. M., (1966). Solution of equations and system of equations. Academic Press NY, London.
- [3] Ortega, J. M. and Rheinboldt, W. C., (1970). Iterative solution of non-linear equations in several variables, Academic Press, Inc.
- [4] Traub, J. F., (1982). Iterative method for the solution of equations, Chelsea Publishing Company, New York.
- [5] Dennis Jr, J. E. and Schnabel, R. B., (1983). Numerical methods for unconstrained optimization and nonlinear equations, Prentice Hall, Englewood Cliffs, NJ.
- [6] Cronin-Scanlon, J., (1967). Advanced calculus, Boston, M.A., D.c.Heath.
- [7] Macleod, A. J., (1984). A generalization of Newton Raphson. International Journal of Mathematical Education in Science and Education, 15(1), 117-120.
- [8] Sharma, J. R. and Sharma, R. (2011), some third order methods for solving system of nonlinear equations, World Academy of science, Engineering and Technology, 60, 1294-1301.