

SUBORDINATION RESULTS FOR CERTAIN SUBCLASSES OF UNIVALENT MEROMORPHIC FUNCTIONS

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Abstract: In this paper we consider $\frac{(1-\lambda)zg'(z)}{\lambda g(z)+g(z)}$ ($0 \leq \lambda < 1$) of meromorphic functions $g \in \Sigma$ such that

$\operatorname{Re} \left(\frac{(1-\lambda)zg'(z)}{\lambda g(z)+g(z)} \right) > \alpha$ be the class of λ -starlike functions of order α ($0 \leq \alpha < 1$), and analogous to the class $S_b^*(\phi)$, for $g \in \Sigma$, we define the class $F^*[b, \lambda](\phi)$ of meromorphic functions $g \in \Sigma$ then, obtaining some subordination results on functions in this class.

Keywords : Analytic functions, univalent functions, Starlike functions of complex order, Meromorphic functions.
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I. INTRODUCTION

Let A be the class of functions f normalized by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (1.1)$$

which are analytic in the open unit disc $U = \{z \in \mathbb{C}: |z| < 1\}$ and let S be the subclass of A consisting of functions which are also univalent in U .

Let $\phi(z)$ be an analytic function with positive real part on U that satisfies $\phi(0)=1$, $\phi'(0) > 0$ and which maps the unit disc U onto a region starlike with respect to 1 and symmetric with respect to the real axis.

Ma and Minda [2] introduced and studied the class $S^*(\phi)$ consists of functions $f \in S$ for which

$$\frac{zf'(z)}{f(z)} < \phi(z) \quad (z \in U).$$

Following Ma and Minda [2], Ravichandran et. al [9] defined a more general class $S_b^*(\phi)$ of starlike functions of complex order consists of functions $f \in S$ for which

**International Journal of Innovative Research in Science,
Engineering and Technology**

(ISO 3297: 2007 Certified Organization)

Vol. 2, Issue 9, September 2013

$$1 + \frac{1}{b} \left(\frac{zf'(z)}{f(z)} - 1 \right) < \phi(z) \quad (z \in U),$$

where $b \neq 0$ is a complex number.

Let Σ be the class of meromorphic functions g normalized by

$$g(z) = \frac{1}{z} + \sum_{k=0}^{\infty} b_k z^k, \tag{1.2}$$

which are analytic and univalent in the punctured open unit disc $U^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = U - \{0\}$, where $U = \{z \in \mathbb{C} : |z| < 1\}$.

The function $g \in \Sigma$ is said to be meromorphic starlike of order α ($0 \leq \alpha < 1$), if $g(z) \neq 0$ and

$$-\text{Re} \left(\frac{zg'(z)}{g(z)} \right) > \alpha \quad (z \in U^*),$$

and this class is denoted by $MS^*(\alpha)$ or $\Sigma^*(\alpha)$. Where first introduced by Pommerenke

[7], denoting it by $\Sigma(\alpha)$.

Similarly the function $g \in \Sigma$ is said to be meromorphic convex of order α ($0 \leq \alpha < 1$), if $g'(z) \neq 0$ and

$$-\text{Re} \left\{ 1 + \frac{zg''(z)}{g'(z)} \right\} > \alpha \quad (z \in U^*),$$

and this class is denoted by $MC(\alpha)$. Where also, first introduced by Pommerenke [7], denoting it by $K(\alpha)$.

In this paper for functions $g \in \Sigma$, $g(z) \neq 0$ given as in (1.2) we consider $\frac{(1-\lambda)zg'(z)}{\lambda g'(z) + g(z)}$ ($0 \leq \lambda < 1$) such that $-\text{Re} \left(\frac{(1-\lambda)zg'(z)}{\lambda g'(z) + g(z)} \right) > \alpha$ be the class of meromorphic λ -starlike functions of order α ($0 \leq \alpha < 1$).

Taking $\lambda=0$ it reduce to the class $\frac{zg'(z)}{g(z)}$ with

$$-\text{Re} \left(\frac{zg'(z)}{g(z)} \right) > \alpha \quad (0 \leq \alpha < 1)$$

of meromorphic starlike functions of order α , and then analogous to the class $S_b^*(\phi)$, for $g \in \Sigma$, we define the class $F^*[b, \lambda](\phi)$ of meromorphic functions $g \in \Sigma$ as follows

Definition 1.1

Let $\phi(z)$ be an analytic function with positive real part on U that satisfies $\phi(0) = 1$, $\phi'(0) > 0$ and which maps the unit disc U onto a region starlike with respect to 1 and symmetric with respect to the real axis. Let $F^*[b, \lambda](\phi)$ be the class of functions g , $g \in \Sigma$, $g(z) \neq 0$ satisfying

**International Journal of Innovative Research in Science,
Engineering and Technology**

(ISO 3297: 2007 Certified Organization)

Vol. 2, Issue 9, September 2013

$$1 - \frac{1}{b} \left(\frac{(1 - \lambda) z g'(z)}{\lambda g(z) + g(z)} + 1 \right) < \phi(z),$$

where $0 \leq \lambda < 1$ and $b \neq 0$ is a complex number.

Taking $\lambda = 0$ our class $F^*[b, \lambda](\phi)$ reduce to the class $M_b^*(\phi)$ defined by Mohammed and Darus [6].

Using mainly the method of subordination we obtain some results for the class $F^*[b, \lambda](\phi)$.

Let F and G be analytic functions in the unit disc U . The function F is subordinate to G written $F \prec G$ if G is univalent, $F(0) = G(0)$ and $F(U) \subset G(U)$. In general, given two functions F and G which are analytic in U , the function F is said to be subordinate to G if there exist a function w analytic in U with $w(0)=0$ and $(\forall z \in U), |w(z)| < 1$, such that $F(z) = G(w(z))$.

The general theory of differential subordinations was introduced by Miller and Mocanu [3] (see also [4] and [5]).

Our results and their proofs are motivated by similar manners as results of Srivastava and Lashin [10], Ravichandran et. al [9], Ramachandran [8], Ali and Ravichandran [1] and Mohammed and Darus [6].

To prove our main results, the following lemmas are needed

Lemma 1.2 [5] Let ϕ be a convex function defined on U , $\phi(0)=1$. Define $F(z)$ by

$$F(z) = z \exp \left(\int_0^z \frac{\phi(x)-1}{x} dx \right).$$

Let $q(z) = 1+c_1z+c_2z^2 + \dots$ be analytic in U . Then

$$1 + \frac{zq'(z)}{q(z)} < \phi(z),$$

if and only if for all $|s| \leq 1$ and $|t| \leq 1$, we have

$$\frac{q(tz)}{q(sz)} < \frac{sF(tz)}{tF(sz)}.$$

Lemma 1.3 [4] Let $q(z)$ be univalent in the unit disc U and ϑ and φ be analytic in a domain D containing $q(U)$ with $\varphi(w) \neq 0$ when $w \in q(U)$. Set $Q(z)=zq'(z) \varphi(q(z))$ and $h(z) = \vartheta(q(z))+Q(z)$.

Suppose that either $h(z)$ is convex, or $Q(z)$ is starlike univalent in U . In addition, assume that $\text{Re} \left[\frac{zh'(z)}{Q(z)} \right] > 0$ for $z \in U$. If $p(z)$ is analytic in U with $p(0) = q(0)$, $p(U) \subseteq D$ and

**International Journal of Innovative Research in Science,
Engineering and Technology**

(ISO 3297: 2007 Certified Organization)

Vol. 2, Issue 9, September 2013

$$\vartheta (p(z))+zp'(z) \varphi (p(z)) < \vartheta (q(z)) + zq'(z) \varphi(q(z)),$$

then $p(z) < q(z)$ and $q(z)$ is the best domenant.

II. SUBORDINATION RESULTS

First we prove the following result for functions in the class $F^*[b, \lambda](\phi)$.

Theorem 2.1

A function $g \in F^*[b, \lambda](\phi)$ for $g(z) \neq 0$ if and only if

$$[z(g(z))^{\frac{1-\lambda}{1+\lambda}}]^{\frac{1}{b}} = \exp \left(\int_0^z \frac{1-\phi(w(\zeta))}{\zeta} d\zeta \right),$$

where $w(z)$ is analytic in U satisfying $w(0)=0$ and $|w(z)| \leq 1$.

Proof:

First let $g \in F^*[b, \lambda](\phi)$, applying Definition 1.1. We get

$$1 - \frac{1}{b} \left(\frac{(1-\lambda)zg'(z)}{\lambda g(z)+g(z)} + 1 \right) < \phi(z), \tag{2.2}$$

therefore, there is a function $w(z)$ analytic in U with $w(0) = 0$ and $|w(z)| \leq 1$ such that

$$1 - \frac{1}{b} \left(\frac{(1-\lambda)zg'(z)}{\lambda g(z)+g(z)} + 1 \right) = \phi(w(z)), \tag{2.3}$$

or

$$\frac{1}{b} \left(\frac{1-\lambda}{1+\lambda} \frac{g'(z)}{g(z)} + \frac{1}{z} \right) = \frac{1-\phi(w(z))}{z}. \tag{2.4}$$

Integrating both sides of (2.4) from 0 to z gives

$$\frac{1}{b} \left(\frac{1-\lambda}{1+\lambda} \ln g(z) + \ln z \right) = \int_0^z \frac{1-\phi(w(\zeta))}{\zeta} d\zeta, \tag{2.5}$$

so

$$\frac{1}{b} \ln [z(g(z))^{\frac{1-\lambda}{1+\lambda}}] = \int_0^z \frac{1-\phi(w(\zeta))}{\zeta} d\zeta, \tag{2.6}$$

Hence the result

$$[z(g(z))^{\frac{1-\lambda}{1+\lambda}}]^{\frac{1}{b}} = \exp \left(\int_0^z \frac{1-\phi(w(\zeta))}{\zeta} d\zeta \right) \tag{2.7}$$

Conversely, suppose that

**International Journal of Innovative Research in Science,
Engineering and Technology**

(ISO 3297: 2007 Certified Organization)

Vol. 2, Issue 9, September 2013

$$[z(g(z))^{\frac{1-\lambda}{1+\lambda}}]^{\frac{1}{b}} = \exp \left(\int_0^z \frac{1-\phi(w(\zeta))}{\zeta} d\zeta \right) \tag{2.8}$$

Now differentiating (2.8) with respect to z we obtain

$$\frac{1}{b} [z(g(z))^{\frac{1-\lambda}{1+\lambda}}]^{\frac{1}{b}-1} \left[(g(z))^{\frac{1-\lambda}{1+\lambda}} + \frac{1-\lambda}{1+\lambda} z(g(z))^{\frac{1-\lambda}{1+\lambda}-1} g'(z) \right] = \exp \left(\int_0^z \frac{1-\phi(w(\zeta))}{\zeta} d\zeta \right) \frac{1-\phi(w(z))}{z}$$

(2.9)

begin{split}

from (2.8) and (2.9) we obtain

$$\frac{1}{b} \left(\frac{1-\lambda}{1+\lambda} \frac{g'(z)}{g(z)} + \frac{1}{z} \right) = \frac{1-\phi(w(z))}{z} \tag{2.10}$$

Hence

$$1 - \frac{1}{b} \left(\frac{(1-\lambda)zg'(z)}{\lambda g(z)+g(z)} + 1 \right) < \phi(z) \tag{2.11}$$

Therefore

$$g \in F^*[b, \lambda](\phi) \tag{2.12}$$

This completes the proof of Theorem 2.1.

Taking $\lambda=0$ in Theorem 2.1 gives Theorem 2.1 in [6], then we have the following corollary.

Corollary 2.2

A function $g \in F^*[b, \lambda](\phi)$ for $g(z) \neq 0$ if and only if

$$(zg(z))^{\frac{1}{b}} = \exp \left(\int_0^z \frac{1-\phi(w(\zeta))}{\zeta} d\zeta \right) \tag{2.13}$$

where $w(z)$ is analytic in U satisfying $w(0)=0$ and $|w(z)| \leq 1$.

Using Lemma 1.2 we obtain the following necessary and sufficient conditions for functions to belong to $F^*[b, \lambda](\phi)$.

Theorem 2.3

Let $\phi(z)$ and $F(z)$ be as in Lemma 1.2. The function $g \in F^*[b, \lambda](\phi)$ if and only if for all

$|s| \leq 1$ and $|t| \leq 1$,

$$\left[\frac{s(g(sz))^{\frac{1-\lambda}{1+\lambda}}}{t(g(tz))^{\frac{1-\lambda}{1+\lambda}}} \right]^{\frac{1}{b}} < \frac{sF(tz)}{tF(sz)} \tag{2.14}$$

**International Journal of Innovative Research in Science,
Engineering and Technology**

(ISO 3297: 2007 Certified Organization)

Vol. 2, Issue 9, September 2013

Proof:

Let $g \in F^*[b, \lambda](\phi)$

Define the function $p(z) \neq 0$ by

$$\frac{1}{p(z)} = [z(g(z))^{\frac{1-\lambda}{1+\lambda}}]^{\frac{1}{b}} \tag{2.15}$$

Differentiating (2.15) with respect to z we get

$$-\frac{p'(z)}{(p(z))^2} = \frac{1}{b} [z(g(z))^{\frac{1-\lambda}{1+\lambda}}]^{\frac{1}{b}-1} \left[(g(z))^{\frac{1-\lambda}{1+\lambda}} + \frac{1-\lambda}{1+\lambda} z(g(z))^{\frac{1-\lambda}{1+\lambda}-1} g'(z) \right] \tag{2.16}$$

Dividing (2.16) by (2.15) gives

$$-\frac{zp'(z)}{p(z)} = \frac{1}{b} \frac{1}{(g(z))^{\frac{1-\lambda}{1+\lambda}}} (g(z))^{\frac{1-\lambda}{1+\lambda}} \left[\frac{1-\lambda}{1+\lambda} \frac{zg'(z)}{g(z)} + 1 \right], \tag{2.17}$$

or

$$1 + \frac{zp'(z)}{p(z)} = 1 - \frac{1}{b} \left[\frac{1-\lambda}{1+\lambda} \frac{zg'(z)}{g(z)} + 1 \right], \tag{2.18}$$

then by Definition 1.1

$$1 + \frac{zp'(z)}{p(z)} = 1 - \frac{1}{b} \left[\frac{1-\lambda}{1+\lambda} \frac{zg'(z)}{g(z)} + 1 \right] < \phi(z). \tag{2.19}$$

And applying Lemma 1.2, we get that

$$1 + \frac{zp'(z)}{p(z)} = 1 - \frac{1}{b} \left[\frac{1-\lambda}{1+\lambda} \frac{zg'(z)}{g(z)} + 1 \right] < \phi(z), \tag{2.20}$$

if and only if for all $|s| \leq 1$ and $|t| \leq 1$,

$$\frac{p(tz)}{p(sz)} = \left[\frac{s(g(sz))^{\frac{1-\lambda}{1+\lambda}}}{t(g(tz))^{\frac{1-\lambda}{1+\lambda}}} \right]^{\frac{1}{b}} < \frac{sF(tz)}{tF(sz)} \tag{2.21}$$

This completes the proof of Theorem 2.3

If we take $\lambda = 0$ in Theorem 2.3 then we have the following corollary due to [6].

Corollary 2,4

Let $\phi(z)$ and $F(z)$ be as in Lemma 1.2. The function $g \in F^*[b, \lambda](\phi)$ if and only if for all $|s| \leq 1$ and $|t| \leq 1$,

**International Journal of Innovative Research in Science,
Engineering and Technology**

(ISO 3297: 2007 Certified Organization)

Vol. 2, Issue 9, September 2013

$$\left(\frac{sg(sz)}{tg(tz)}\right)^{\frac{1}{b}} \prec \frac{sF(tz)}{tF(sz)}. \tag{2.22}$$

Theorem 2.5

Let $q(z)$ be univalent and $q(z) \neq 0$ in U and such that $\frac{zq'(z)}{q(z)}$ is starlike univalent in U , and

$$\text{Re}\left[1 + \frac{zq''(z)}{q(z)} - \frac{zq'(z)}{q(z)} - \frac{q(z)}{\gamma}\right] > 0 \quad (z \in U, \gamma \neq 0). \tag{2.23}$$

If $g(z) \in \Sigma$ for $g(z) \neq 0$ and

$$\frac{\{(1-\lambda)zg'(z)\}}{\lambda g(z)+g(z)} - \gamma \left(1 + \frac{zg''(z)}{g(z)} - \frac{zg'(z)}{g(z)}\right) \prec q(z) - \gamma \frac{zq'(z)}{q(z)}. \tag{2.24}$$

Then

$$\frac{\{(1-\lambda)zg'(z)\}}{\lambda g(z)+g(z)} \prec q(z). \tag{2.25}$$

Proof:

Define the function $p(z)$ by

$$p(z) = \frac{(1-\lambda)zg'(z)}{\lambda g(z)+g(z)} \quad (z \in U), \tag{2.26}$$

differentiating (2.26) with respect to z , we obtain

$$p'(z) = \frac{1-\lambda}{1+\lambda} \frac{g(z)[zg''(z)+g'(z)]-z(g'(z))^2}{(g(z))^2}, \tag{2.27}$$

and

$$\begin{aligned} \frac{z p'(z)}{p(z)} &= \frac{zg(z)[zg''(z)+g'(z)]-z^2(g'(z))^2}{(g(z))^2} \frac{g(z)}{zg'(z)} \\ &= \frac{zg''(z)}{g'(z)} + 1 - \frac{zg'(z)}{g(z)} \end{aligned} \tag{2.28}$$

so

$$\begin{aligned} p(z) - \gamma \frac{z p'(z)}{p(z)} &= \frac{1-\lambda}{1+\lambda} \frac{zg'(z)}{g(z)} - \gamma \frac{zg''(z)}{g'(z)} + \gamma \frac{zg'(z)}{g(z)} - \gamma \\ &= \frac{(1-\lambda)zg'(z)}{\lambda g(z)+g(z)} - \gamma \left(\frac{zg''(z)}{g'(z)} + 1\right) + \gamma \frac{zg'(z)}{g(z)}. \end{aligned} \tag{2.29}$$

**International Journal of Innovative Research in Science,
Engineering and Technology**

(ISO 3297: 2007 Certified Organization)

Vol. 2, Issue 9, September 2013

By setting

$$\theta(w)=w$$

and

$$\phi(w) = -\frac{\gamma}{w},$$

it can be easily observed that $\theta(w)$ is analytic in the complex plane \mathbb{C} , $\phi(w)$ is analytic in the complex plane $\mathbb{C}\setminus\{0\}$ and that

$$\phi(w) \neq 0 \quad (w \in \mathbb{C}\setminus\{0\}).$$

Also, by letting

$$Q(z) = zq'(z)\phi(q(z)) = -\gamma \frac{zq'(z)}{q(z)},$$

and

$$h(z) = \theta(q(z)) + Q(z) = q(z) - \gamma \frac{zq'(z)}{q(z)}.$$

Thus

$$\begin{aligned} \frac{zQ'(z)}{Q(z)} &= z \left\{ -\gamma \frac{q(z)[zq''(z) + q'(z)] - z(q'(z))^2}{(q(z))^2} \right\} \left\{ \frac{-q(z)}{\gamma zq'(z)} \right\} \\ &= \frac{zq''(z)}{q'(z)} + 1 - \frac{zq'(z)}{q(z)}, \end{aligned} \tag{2.30}$$

and

$$\begin{aligned} \frac{z \left(\frac{zq'(z)}{q(z)} \right)'}{\frac{zq'(z)}{q(z)}} &= \frac{q(z)[zq''(z) + q'(z)] - z(q'(z))^2}{(q(z))^2} \frac{q(z)}{q'(z)} \\ &= \frac{zq''(z)}{q'(z)} + 1 - \frac{zq'(z)}{q(z)}. \end{aligned} \tag{2.31}$$

Since $\frac{zq'(z)}{q(z)}$ is starlike and univalent in U so

$$\operatorname{Re} \left\{ \frac{z \left(\frac{zq'(z)}{q(z)} \right)'}{\frac{zq'(z)}{q(z)}} \right\} = \operatorname{Re} \left\{ \frac{zq''(z)}{q'(z)} + 1 - \frac{zq'(z)}{q(z)} \right\} > 0. \tag{2.32}$$

Then, $Q(z)$ is starlike and univalent in U .

Now

**International Journal of Innovative Research in Science,
Engineering and Technology**

(ISO 3297: 2007 Certified Organization)

Vol. 2, Issue 9, September 2013

$$\begin{aligned} \frac{zh'(z)}{Q(z)} &= z \left\{ q'(z) - \gamma \frac{q(z)[zq''(z)+q'(z)]-z(q'(z))^2}{(q(z))^2} \right\} \left\{ \frac{-q(z)}{\gamma z q'(z)} \right\} \\ &= \frac{zq''(z)}{q'(z)} + 1 - \frac{zq'(z)}{q(z)} - \frac{q(z)}{\gamma}. \end{aligned} \tag{2.33}$$

And from (2.23) we know that

$$\operatorname{Re}\left[1 + \frac{zq''(z)}{q(z)} - \frac{zq'(z)}{q(z)} - \frac{q(z)}{\gamma}\right] > 0$$

therefore, by an application of lemma 1.3 we get that

$$p(z) = \frac{\{(1-\lambda)zg'(z)\}}{\lambda \underline{g}(z) + \overline{g}(z)} \prec q(z), \tag{2.34}$$

and $q(z)$ is the best dominant.

This completes the proof of Theorem 2.5.

If we take $\lambda=0$, then we have the following corollary.

Corollary 2.6

Let $q(z)$ be univalent and $q(z) \neq 0$ in U and such that $\frac{zq'(z)}{q(z)}$ is starlike univalent in U ,

and

$$\operatorname{Re}\left[1 + \frac{zq''(z)}{q(z)} - \frac{zq'(z)}{q(z)} - \frac{q(z)}{\gamma}\right] > 0 \quad (z \in U, \gamma \neq 0). \tag{2.35}$$

If $g(z) \in \Sigma$ for $g(z) \neq 0$ and

$$\frac{zg'(z)}{g(z)} - \gamma \left(1 + \frac{zg''(z)}{g(z)} - \frac{zg'(z)}{g(z)}\right) \prec q(z) - \gamma \frac{zq'(z)}{q(z)}, \tag{2.36}$$

then

$$\frac{zg'(z)}{g(z)} \prec q(z) \tag{2.37}$$

and $q(z)$ is the best dominant.

International Journal of Innovative Research in Science, Engineering and Technology

(ISO 3297: 2007 Certified Organization)

Vol. 2, Issue 9, September 2013

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