

# Characterization Theorems for Modular and Distributive Soft Lattices

E.K.R.Nagarajan<sup>1</sup> and P.Geetha<sup>2</sup>

<sup>1</sup> Associate Professor and Head, Centre for Research and Post Graduate Studies in Mathematics,  
Ayya Nadar Janaki Ammal College (Autonomous), Sivakasi, Tamil Nadu, India.

<sup>2</sup> Assistant Professor, Department of Mathematics,  
V.V.Vanniaperumal College for Women, Virudhunagar, Tamil Nadu, India.

**Abstract:** Soft set theory was introduced by Molodtsov in 1999 as a mathematical tool for dealing with problems that contain uncertainty. Faruk Karaaslan et al.[6] defined the concept of soft lattices, modular soft lattices and distributive soft lattice over a collection of soft sets. In this paper, we define the concept of principle of duality in soft lattices and discuss some related properties of modular and distributive soft lattices. We also establish characterization theorems for modular and distributive soft lattices by their soft sublattices.

**Key words:** Soft sets, soft lattices, soft sublattices, dual soft lattices, modular soft lattices, distributive soft lattices.

## I. INTRODUCTION

Soft set theory was introduced by Molodtsov [9] in 1999 as a mathematical tool for dealing with uncertainty. Maji et al.[8] defined some operations on soft sets and proved related properties. Irfan Ali et al.[5] studied some new operations on soft sets. Li [7], Nagarajan et al.[10] defined the soft lattices using soft sets. Faruk Karaaslan et al.[6] defined the concept of soft lattices over a collection of soft sets by using the operations of soft sets defined by Cagman et al.[1]. In this paper, we define the concept of principle of duality in soft lattices and discuss some related properties of modular and distributive soft lattices. We also illustrate them with some examples. In addition, we establish characterization theorems for modular and distributive soft lattices by their soft sublattices.

## II. SOME CONCEPTS IN SOFT SETS AND SOFT LATTICES

In this section, we have presented the basic definitions and results of soft set theory [9, 6] and soft lattices [6] for subsequent discussions. Throughout this work,  $U$  refers to the initial universe,  $P(U)$  is the power set of  $U$ ,  $E$  is a set of parameters and  $A \subseteq E$ .

**Definition 2.1** A function  $f_A : E \rightarrow P(U)$  such that  $f_A(x) = \emptyset$  if  $x \notin A$ , is called a soft set over  $U$ . The set of all soft sets over  $U$  is denoted by  $S(U)$ .

**Definition 2.2** Let  $f_A \in S(U)$ . If  $f_A(x) = \emptyset$  for all  $x \in E$ , then  $f_A$  is called an empty soft set, denoted by  $f_{\emptyset}$ . If  $f_A(x) = U$  for all  $x \in A$ , then  $f_A$  is called  $A$ -universal soft set denoted by  $f_{\bar{A}}$ . If  $A = E$  then the

## International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 2, Issue 11, November 2013

$A$  -universal soft set is called universal soft set denoted by  $f_{\tilde{E}}$ .

**Definition 2.3** Let  $f_A, f_B \in S(U)$ . Then,  $f_A$  is a soft subset of  $f_B$ , denoted by  $f_A \tilde{\subseteq} f_B$ , if  $f_A(x) \subseteq f_B(x)$  for all  $x \in E$ .  $f_A$  and  $f_B$  are equal, denoted by  $f_A = f_B$ , if and only if  $f_A(x) = f_B(x)$  for all  $x \in E$ .

**Definition 2.4** Let  $f_A \in S(U)$ . Then, soft complement of  $f_A$  is defined by  $f_A^{\tilde{c}} = f_{A^{\tilde{c}}}$  such that  $f_{A^{\tilde{c}}}(x) = f_A^{\tilde{c}}(x) = U \setminus f_A(x)$  for all  $x \in E$ .

**Definition 2.5** Let  $f_A, f_B \in S(U)$ . Then, soft union of  $f_A$  and  $f_B$  is defined by  $f_A \tilde{\cup} f_B = f_{A \tilde{\cup} B}$  such that  $f_{A \tilde{\cup} B}(x) = f_A(x) \cup f_B(x)$  for all  $x \in E$ .

**Definition 2.6** Let  $f_A, f_B \in S(U)$ . Then, soft intersection of  $f_A$  and  $f_B$  is defined by  $f_A \tilde{\cap} f_B = f_{A \tilde{\cap} B}$  such that  $f_{A \tilde{\cap} B}(x) = f_A(x) \cap f_B(x)$  for all  $x \in E$ .

**Definition 2.7** Let  $L \subseteq S(U)$ , and  $\vee$  and  $\wedge$  be two binary operations on  $L$ . If the set  $L$  together with  $\vee$  and  $\wedge$  satisfies the following conditions then  $(L, \vee, \wedge)$  is called a soft lattice.

1.  $f_A \vee f_B = f_B \vee f_A$  and  $f_A \wedge f_B = f_B \wedge f_A$  for all  $f_A, f_B \in L$ .
2.  $f_A \vee (f_B \vee f_C) = (f_A \vee f_B) \vee f_C$  and  $f_A \wedge (f_B \wedge f_C) = (f_A \wedge f_B) \wedge f_C$  for all  $f_A, f_B, f_C \in L$ .
3.  $f_A \vee (f_A \wedge f_B) = f_A$  and  $f_A \wedge (f_A \vee f_B) = f_A$  for all  $f_A, f_B \in L$ .

**Theorem 2.8** Let  $(L, \vee, \wedge)$  be a soft lattice and  $f_A, f_B \in L$ . Then,  $f_A \wedge f_B = f_A \Leftrightarrow f_A \vee f_B = f_B$ .

**Theorem 2.9** Let  $(L, \vee, \wedge)$  be a soft lattice and  $f_A, f_B \in L$ . Then a relation  $\leq$  that is defined by  $f_A \leq f_B \Leftrightarrow f_A \wedge f_B = f_A$  or  $f_A \vee f_B = f_B$  is an ordering relation on  $L$ .

**Lemma 2.10** Let  $(L, \vee, \wedge)$  be a soft lattice and  $f_A, f_B \in L$ . Then,  $f_A \vee f_B$  and  $f_A \wedge f_B$  are the least upper bound and the greatest lower bound of  $f_A$  and  $f_B$ , respectively.

**Theorem 2.11** A soft lattice  $(L, \vee, \wedge)$  is a poset.

**Theorem 2.12** Let  $L \subseteq S(U)$ . Then, the algebraic structure  $(L, \vee, \wedge, \leq)$  is a soft lattice.

**Note 2.13** According to the theorem 2.12, a soft lattice  $(L, \vee, \wedge)$  has the same character with  $(L, \vee, \wedge, \leq)$ . Therefore, we shall identify any soft lattice  $(L, \vee, \wedge)$  with  $(L, \vee, \wedge, \leq)$  and we use these two concepts

## International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 2, Issue 11, November 2013

interchangeably.

**Definition 2.14** Let  $(L, \vee, \wedge, \leq)$  be a soft lattice and  $S \subseteq L$ . If  $S$  is a soft lattice with the operations of  $L$ , then  $S$  is called a soft sublattice of  $L$ .

**Theorem 2.15** Let  $(L, \vee, \wedge, \leq)$  be a soft lattice and  $S \subseteq L$ . If  $f_A \vee f_B \in S$  and  $f_A \wedge f_B \in S$  for all  $f_A, f_B \in S$ , then  $S$  is a soft sublattice.

### III. MODULAR AND DISTRIBUTIVE SOFT LATTICES

In this section, we introduce the concept of duality principle and give the definition of modular soft lattice and distributive soft lattice. Also we discuss several related properties and characterization theorems on modular and distributive soft lattices. Throughout this section,  $U$  refers to the initial universe. The set of all soft sets over  $U$  is denoted by  $S(U)$ .  $L \subseteq S(U)$  and  $(L, \vee, \wedge, \leq)$  is a soft lattice.

#### Definition 3.1 Duality Principle

Let  $(L, \vee, \wedge, \leq)$  be a soft lattice. If we define a relation  $\leq'$  in  $L$  for all  $f_A, f_B \in L$ ,  $f_A \leq' f_B \Leftrightarrow f_B \leq f_A$  in  $(L, \vee, \wedge, \leq)$ , then  $\leq'$  is also a partial ordering on  $L$ . The partial ordering  $\leq'$  is called the reversal partial ordering of  $\leq$ .

Clearly, for all  $f_A, f_B \in L$ ,  $\text{l.u.b } \{f_A, f_B\}$  in  $(L, \vee, \wedge, \leq')$  =  $\text{g.l.b } \{f_A, f_B\}$  in  $(L, \vee, \wedge, \leq)$  and  $\text{g.l.b } \{f_A, f_B\}$  in  $(L, \vee, \wedge, \leq')$  =  $\text{l.u.b } \{f_A, f_B\}$  in  $(L, \vee, \wedge, \leq)$ . The operations  $\vee$  and  $\wedge$  with respect to  $\leq$  in  $L$  are the  $\wedge$  and  $\vee$  with respect to  $\leq'$  in  $L$ . As  $f_A \leq' f_B$  in  $(L, \vee, \wedge, \leq')$  means  $f_B \leq f_A$  in  $(L, \vee, \wedge, \leq)$ , any formula involving  $\vee, \wedge, \leq$  in  $(L, \vee, \wedge, \leq)$  remains valid in  $(L, \vee, \wedge, \leq')$  if  $\leq, \vee, \wedge$  are replaced by  $\geq, \wedge, \vee$ .

**Remark 3.2** In any soft lattice  $(L, \vee, \wedge, \leq)$  for all  $f_A, f_B \in L$ ,  $f_A \leq f_B \Rightarrow f_A \vee f_B = f_B$  is valid. Hence its dual formula  $f_A \geq f_B \Rightarrow f_A \wedge f_B = f_B$  is valid.

**Definition 3.3** Let  $(L, \vee, \wedge, \leq)$  be a soft lattice. Then  $L$  is said to be a modular soft lattice if

$$f_A \leq f_C \Rightarrow f_A \vee (f_B \wedge f_C) = (f_A \vee f_B) \wedge f_C \text{ for all } f_A, f_B, f_C \in L .$$

or equivalently,

$$f_A \geq f_C \Rightarrow f_A \wedge (f_B \vee f_C) = (f_A \wedge f_B) \vee f_C \text{ for all } f_A, f_B, f_C \in L .$$

**Remark 3.4** To show that the soft lattice  $L$  is a Modular soft lattice it is enough to prove that

$$f_A \leq f_C \Rightarrow (f_A \vee f_B) \wedge f_C \leq f_A \vee (f_B \wedge f_C) \text{ for all } f_A, f_B, f_C \in L , \text{ as the other inequality}$$

$$f_A \vee (f_B \wedge f_C) \leq (f_A \vee f_B) \wedge f_C \text{ for all } f_A, f_B, f_C \in L \text{ is always true for all soft lattices.}$$

**Example 3.5** Let  $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}, E = \{e_1, e_2, e_3, e_4\}, A = \{e_1\}, B = \{e_2\}, C = \{e_3, e_4\}$ ,

## International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 2, Issue 11, November 2013

$D = \{e_1, e_2, e_3, e_4\}$  where  $A, B, C, D \subseteq E$  and  $L = \{f_\emptyset, f_A, f_B, f_C, f_D\}$ . Then  $L \subseteq S(U)$  is a soft lattice with the operations  $\tilde{\cup}$  and  $\tilde{\cap}$ . Assume that,

$$f_\emptyset = \emptyset$$

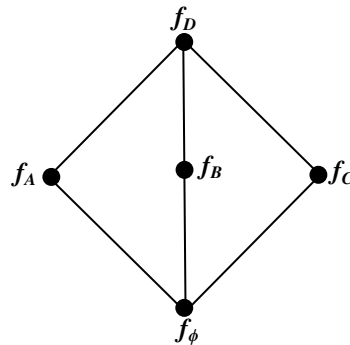
$$f_A = \{(e_1, \{u_1, u_2\})\}$$

$$f_B = \{(e_2, \{u_3, u_4\})\}$$

$$f_C = \{(e_3, \{u_5, u_6\}), (e_4, \{u_1\})\}$$

$$f_D = \{(e_1, \{u_1, u_2\}), (e_2, \{u_3, u_4\}), (e_3, \{u_5, u_6\}), (e_4, \{u_1, u_2\})\}$$

$(L, \tilde{\cup}, \tilde{\cap}, \tilde{\subseteq})$  is a Modular soft lattice. The Hasse diagram of it appears in figure 1. This Modular soft lattice is denoted by  $SM_3$ .



**Figure 1: Modular Soft Lattice ( $SM_3$ )**

**Example 3.6** Let  $U = \{u_1, u_2, u_3, u_4, u_5\}$ ,  $E = \{e_1, e_2, e_3\}$ ,  $A = \{e_1\}$ ,  $B = \{e_2\}$ ,  $C = \{e_1, e_3\}$ ,  $D = \{e_1, e_2, e_3\}$  where  $A, B, C, D \subseteq E$  and  $L = \{f_\emptyset, f_A, f_B, f_C, f_D\}$ . Then  $L \subseteq S(U)$  is a soft lattice with the operations  $\tilde{\cup}$  and  $\tilde{\cap}$ . Assume that,

$$f_\emptyset = \emptyset$$

$$f_A = \{(e_1, \{u_1\})\}$$

$$f_B = \{(e_2, \{u_2, u_3\})\}$$

$$f_C = \{(e_1, \{u_1, u_2\}), (e_3, \{u_3, u_4, u_5\})\}$$

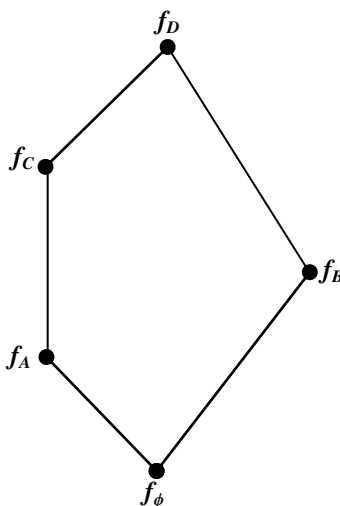
$$f_D = \{(e_1, \{u_1, u_2\}), (e_2, \{u_2, u_3\}), (e_3, \{u_3, u_4, u_5\})\}$$

$(L, \tilde{\cup}, \tilde{\cap}, \tilde{\subseteq})$  is a Non-modular soft lattice. The Hasse diagram of it appears in figure 2. This Non-modular soft lattice is denoted by  $SN_5$ .

**International Journal of Innovative Research in Science,  
Engineering and Technology**

(An ISO 3297: 2007 Certified Organization)

Vol. 2, Issue 11, November 2013



**Figure 2: Non-Modular Soft Lattice (SN<sub>5</sub>)**

**Definition 3.7** Let  $(L, \vee, \wedge, \leq)$  be a soft lattice. Then  $L$  is said to be a distributive soft lattice if for all  $f_A, f_B, f_C \in L$

$$1. f_A \vee (f_B \wedge f_C) = (f_A \vee f_B) \wedge (f_A \vee f_C)$$

$$2. f_A \wedge (f_B \vee f_C) = (f_A \wedge f_B) \vee (f_A \wedge f_C)$$

[These equations are called distributive equations.]

**Remark 3.8** Because of duality principle, a soft lattice  $L$  is said to be a distributive soft lattice if any one of the distributive equations is true for all  $f_A, f_B, f_C \in L$ .

**Remark 3.9** To show that the soft lattice  $L$  is a distributive soft lattice it is enough to prove that,  $f_A \wedge (f_B \vee f_C) \leq (f_A \wedge f_B) \vee (f_A \wedge f_C)$  for all  $f_A, f_B, f_C \in L$  as the other inequality  $(f_A \wedge f_B) \vee (f_A \wedge f_C) \leq f_A \wedge (f_B \vee f_C)$  is always true for all  $f_A, f_B, f_C \in L$ .

**Example 3.10** Let  $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ ,  $E = \{e_1, e_2, e_3\}$ ,  $A = \{e_1\}$ ,  $B = \{e_1, e_2\}$ ,  $C = \{e_1, e_3\}$ ,  $D = \{e_1, e_2, e_3\}$  where  $A, B, C, D \subseteq E$  and  $L = \{f_\emptyset, f_A, f_B, f_C, f_D\}$ . Then  $L \subseteq S(U)$  is a soft lattice with the operations  $\tilde{\cup}$  and  $\tilde{\cap}$ . Assume that

$$f_\emptyset = \emptyset$$

$$f_A = \{(e_1, \{u_1, u_2\})\}$$

$$f_B = \{(e_1, \{u_1, u_2, u_3\}), (e_2, \{u_4, u_6\})\}$$

**International Journal of Innovative Research in Science,  
Engineering and Technology**

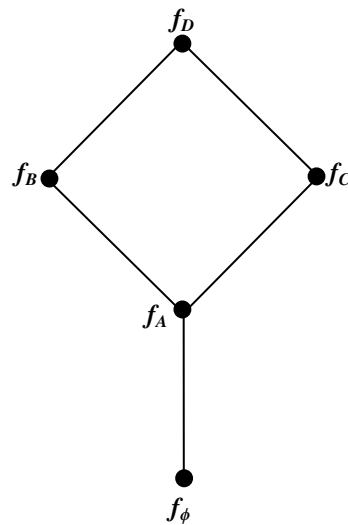
(An ISO 3297: 2007 Certified Organization)

**Vol. 2, Issue 11, November 2013**

$$f_C = \{(e_1, \{u_1, u_2\}), (e_3, \{u_5, u_6\})\}$$

$$f_D = \{(e_1, \{u_1, u_2, u_3\}), (e_2, \{u_4, u_6\}), (e_3, \{u_5, u_6\})\}$$

$(L, \cup, \cap, \tilde{\cup}, \tilde{\cap})$  is a distributive soft lattice. The Hasse diagram of it appears in figure 3.



**Figure 3: Distributive Soft Lattice**

**Lemma 3.11** The soft sublattice (dual) of a modular soft lattice is modular.

*Proof.* Let  $L$  be a modular soft lattice. Then  $f_A \leq f_C \Rightarrow f_A \vee (f_B \wedge f_C) = (f_A \vee f_B) \wedge f_C$  for all  $f_A, f_B, f_C \in L$ . Let  $S$  be a soft sublattice of  $L$ . Take  $f_A, f_B, f_C \in S$ . Since  $S \subseteq L$ ,  $f_A, f_B, f_C \in L$ . Therefore,  $f_A \leq f_C \Rightarrow f_A \vee (f_B \wedge f_C) = (f_A \vee f_B) \wedge f_C$  is true for all  $f_A, f_B, f_C \in S$ . Hence the soft sublattice  $S$  is modular soft lattice. Therefore, the soft sublattice of a modular soft lattice is modular.

Let  $L$  be a modular soft lattice. Then,  $f_A \leq f_C \Rightarrow f_A \vee (f_B \wedge f_C) = (f_A \vee f_B) \wedge f_C$  for all  $f_A, f_B, f_C \in L$ . The dual of  $L$  is  $f_A \geq f_C \Rightarrow f_A \wedge (f_B \vee f_C) = (f_A \wedge f_B) \vee f_C$  for all  $f_A, f_B, f_C \in L$ . Therefore, the dual of the modular soft lattice is modular.

**Lemma 3.12** A soft lattice  $L$  is modular if and only if every triplet of elements  $f_A, f_B, f_C (f_A \leq f_C)$  in  $L$  has a median.

(or) A Soft lattice  $L$  is modular iff for all  $f_A, f_B, f_C \in L$  with  $f_A \leq f_C$ ,  $(f_A \wedge f_B) \vee (f_B \wedge f_C) \vee (f_C \wedge f_A) = (f_A \vee f_B) \wedge (f_B \vee f_C) \wedge (f_C \vee f_A)$ .

*Proof.* Suppose that  $L$  is a soft lattice in which every triplet of elements  $f_A, f_B, f_C (f_A \leq f_C)$  has a median.

**International Journal of Innovative Research in Science,  
Engineering and Technology**

*(An ISO 3297: 2007 Certified Organization)*

**Vol. 2, Issue 11, November 2013**

To prove that  $L$  is a modular soft lattice.

$$\begin{aligned} \text{If } f_A \leq f_C, \text{ then } (f_A \wedge f_B) \vee (f_B \wedge f_C) \vee (f_C \wedge f_A) &= (f_A \wedge f_B) \vee (f_B \wedge f_C) \vee f_A \\ &= f_A \vee (f_B \wedge f_C) \dots\dots\dots(1) \end{aligned}$$

$$\begin{aligned} (f_A \vee f_B) \wedge (f_B \vee f_C) \wedge (f_C \vee f_A) &= (f_A \vee f_B) \wedge (f_B \vee f_C) \wedge f_C \\ &= (f_A \vee f_B) \wedge f_C \dots\dots\dots(2) \end{aligned}$$

From (1) and (2), we get,  $f_A \vee (f_B \wedge f_C) = (f_A \vee f_B) \wedge f_C$

Hence  $L$  is a modular soft lattice.

Conversely, suppose that  $L$  is a modular soft lattice.

$$\text{Then, } f_A \leq f_C \Rightarrow f_A \vee (f_B \wedge f_C) = (f_A \vee f_B) \wedge f_C$$

To prove that every triplet of elements  $f_A, f_B, f_C$  ( $f_A \leq f_C$ ) in  $L$  has a median.

$$\begin{aligned} \text{Now, } f_A \vee (f_B \wedge f_C) &= f_A \vee (f_A \wedge f_B) \vee (f_B \wedge f_C) \\ &= (f_A \wedge f_C) \vee (f_A \wedge f_B) \vee (f_B \wedge f_C) \\ &= (f_A \wedge f_B) \vee (f_B \wedge f_C) \vee (f_C \wedge f_A) \dots\dots\dots(3) \end{aligned}$$

$$\begin{aligned} (f_A \vee f_B) \wedge f_C &= (f_A \vee f_B) \wedge (f_B \vee f_C) \wedge f_C \\ &= (f_A \vee f_B) \wedge (f_B \vee f_C) \wedge (f_A \vee f_C) \\ &= (f_A \vee f_B) \wedge (f_B \vee f_C) \wedge (f_C \vee f_A) \dots\dots\dots(4) \end{aligned}$$

From (3) and (4), we get,  $(f_A \wedge f_B) \vee (f_B \wedge f_C) \vee (f_C \wedge f_A) = (f_A \vee f_B) \wedge (f_B \vee f_C) \wedge (f_C \vee f_A)$

**Theorem 3.13** A soft lattice  $L$  is modular if and only if it has no soft sublattice isomorphic to  $SN_5$ .

*Proof.* Let  $L$  be a modular soft lattice.

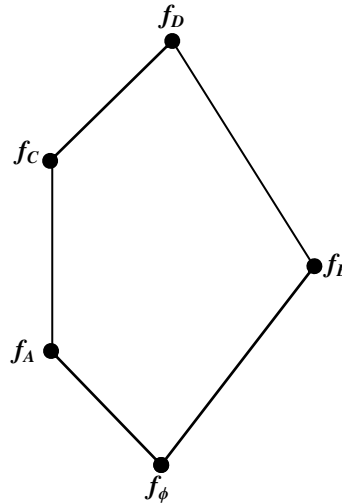
Then every soft sublattice of a modular soft lattice is modular. ....(1)

Consider the soft lattice  $SN_5$  in figure 4.

**International Journal of Innovative Research in Science,  
Engineering and Technology**

(An ISO 3297: 2007 Certified Organization)

Vol. 2, Issue 11, November 2013



**Figure 4.**

Here  $f_A \leq f_C$ ,  $f_A \vee (f_B \wedge f_C) = f_A \vee f_\emptyset = f_A$  while  $(f_A \vee f_B) \wedge f_C = f_D \wedge f_C = f_C$ .

So  $f_A \vee (f_B \wedge f_C) \neq (f_A \vee f_B) \wedge f_C$ . Hence,  $SN_5$  is not a modular soft lattice.

Since every sublattice of a modular soft lattice is modular,  $L$  cannot have a soft sublattice isomorphic to  $SN_5$ .

Conversely, assume that  $L$  has no soft sublattice isomorphic to  $SN_5$ .

To prove,  $L$  is a modular soft lattice. That is,  $f_A \leq f_C \Rightarrow f_A \vee (f_B \wedge f_C) = (f_A \vee f_B) \wedge f_C$  for all  $f_A, f_B, f_C \in L$ . Suppose  $L$  is not a modular soft lattice, then there exist elements  $f_A, f_B, f_C \in L$  such that  $f_A \leq f_C$  and  $f_A \vee (f_B \wedge f_C) < (f_A \vee f_B) \wedge f_C$ .

Let  $f_P = f_B \wedge f_C$ ,  $f_X = f_A \vee (f_B \wedge f_C)$ ,  $f_Y = f_B$ ,  $f_Z = (f_A \vee f_B) \wedge f_C$ ,  $f_Q = (f_A \vee f_B)$

Let  $S = \{f_P, f_X, f_Y, f_Z, f_Q\}$

We claim that  $S$  is a soft sublattice of  $L$  isomorphic to  $SN_5$ .

By assumption, we have  $f_P \leq f_X \leq f_Z \leq f_Q$  and  $f_P \leq f_Y \leq f_Q$ .

- $f_P \leq f_X \wedge f_Y \leq f_Z \wedge f_Y = (f_A \vee f_B) \wedge f_C \wedge f_B = ((f_A \vee f_B) \wedge f_B) \wedge f_C = f_B \wedge f_C = f_P$ .

So  $f_X \wedge f_Y = f_Z \wedge f_Y = f_P$ .

- $f_Q \geq f_Z \vee f_Y \geq f_X \vee f_Y = f_A \vee (f_B \wedge f_C) \vee f_B = f_A \vee ((f_B \wedge f_C) \vee f_B) = f_A \vee f_B = f_Q$ .

So  $f_Z \vee f_Y = f_X \vee f_Y = f_Q$ .

Therefore,  $f_P$  and  $f_Q$  are the bound elements of the soft sublattice  $S$ .

It remains to prove that these five elements  $f_P, f_X, f_Y, f_Z, f_Q$  are all distinct. (ie), to prove



## International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 2, Issue 11, November 2013

$$f_P \neq f_Y, f_Q \neq f_Y, f_P \neq f_X, f_Q \neq f_Z, f_X \neq f_Y, f_Z \neq f_Y$$

$$\text{Suppose } f_P = f_Y \Rightarrow f_X \wedge f_Y = f_Y \Rightarrow f_X \vee f_Y = f_X \Rightarrow f_Q = f_X$$

which is impossible for  $f_P \leq f_X \leq f_Z \leq f_Q$ . Therefore,  $f_P \neq f_Y$

$$\text{Suppose } f_Q = f_Z \Rightarrow f_Z \vee f_Y = f_Z \Rightarrow f_Z \wedge f_Y = f_Y \Rightarrow f_P = f_Y$$

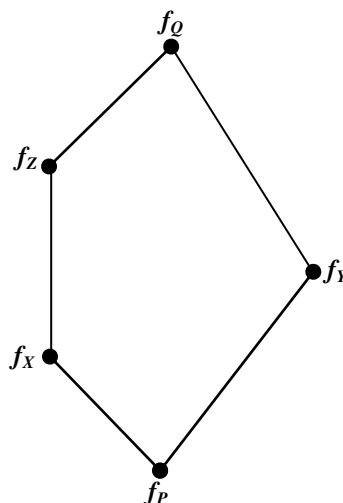
which is a contradiction. Therefore,  $f_Q \neq f_Z$

$$\text{Suppose } f_X = f_Y. \text{ Here } f_P = f_X \wedge f_Y \Rightarrow f_P = f_Y \wedge f_Y \Rightarrow f_P = f_Y$$

which is a contradiction. Therefore,  $f_X \neq f_Y$

By dualizing the above statement, we get  $f_Q \neq f_Y, f_P \neq f_X, f_Z \neq f_Y$ .

These observations lead to the representation of  $S = \{f_P, f_X, f_Y, f_Z, f_Q\}$  by a Hasse diagram given in figure 5.



**Figure 5.**

Therefore,  $S$  is a soft sublattice of  $L$  isomorphic to  $SN_5$ , which is a contradiction.

Hence  $L$  is a modular soft lattice.

**Lemma 3.14** *The soft sublattice (dual) of a distributive soft lattice is distributive.*

*Proof.* Let  $L$  be a distributive soft lattice. Then  $f_A \wedge (f_B \vee f_C) = (f_A \wedge f_B) \vee (f_A \wedge f_C)$  for all  $f_A, f_B, f_C \in L$ . Let  $S$  be a soft sublattice of  $L$ . Take  $f_A, f_B, f_C \in S$ . Since  $S \subseteq L$ ,  $f_A, f_B, f_C \in L$ . Therefore,  $f_A \wedge (f_B \vee f_C) = (f_A \wedge f_B) \vee (f_A \wedge f_C)$  is true for all  $f_A, f_B, f_C \in S$ . Hence the soft sublattice  $S$  is a distributive soft lattice. Therefore, the soft sublattice of a distributive soft lattice is distributive.

Let  $L$  be a distributive soft lattice. Then  $f_A \wedge (f_B \vee f_C) = (f_A \wedge f_B) \vee (f_A \wedge f_C)$  for all  $f_A, f_B, f_C \in L$ . The dual of  $L$  is  $f_A \vee (f_B \wedge f_C) = (f_A \vee f_B) \wedge (f_A \vee f_C)$  for all  $f_A, f_B, f_C \in L$ . Therefore, the dual of

## International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

**Vol. 2, Issue 11, November 2013**

the distributive soft lattice is distributive.

**Lemma 3.15** A soft lattice  $L$  is distributive if and only if every one of its triplets has a median. (or) A soft lattice  $L$  is distributive  $\Leftrightarrow (f_A \vee f_B) \wedge (f_B \vee f_C) \wedge (f_C \vee f_A) = (f_A \wedge f_B) \vee (f_B \wedge f_C) \vee (f_C \wedge f_A)$  for all  $f_A, f_B, f_C \in L$ .

*Proof.* Let  $L$  be a distributive soft lattice.

Let  $f_A, f_B, f_C \in L$

$$\begin{aligned} &(f_A \vee f_B) \wedge (f_B \vee f_C) \wedge (f_C \vee f_A) \\ &= \{(f_A \vee f_B) \wedge (f_B \vee f_C)\} \wedge (f_C \vee f_A) \\ &= \{((f_A \vee f_B) \wedge (f_B \vee f_C)) \wedge (f_C)\} \vee \{((f_A \vee f_B) \wedge (f_B \vee f_C)) \wedge (f_A)\} \\ &= \{(f_A \vee f_B) \wedge ((f_B \vee f_C) \wedge (f_C))\} \vee \{(f_A \wedge (f_A \vee f_B)) \wedge (f_B \vee f_C)\} \\ &= \{(f_A \vee f_B) \wedge f_C\} \vee \{f_A \wedge (f_B \vee f_C)\} \\ &= \{(f_A \wedge f_C) \vee (f_B \wedge f_C)\} \vee \{(f_A \wedge f_B) \vee (f_A \wedge f_C)\} \\ &= (f_A \wedge f_B) \vee (f_B \wedge f_C) \vee (f_C \wedge f_A) \vee (f_C \wedge f_A) \\ &= (f_A \wedge f_B) \vee (f_B \wedge f_C) \vee (f_C \wedge f_A). \end{aligned}$$

Therefore, if  $L$  is a distributive soft lattice then every triplet of elements in  $L$  has a median.

Conversely, Suppose that  $L$  is a soft lattice in which every triplet of elements has a median.

To prove that  $L$  is a distributive soft lattice.

Let  $f_A, f_B, f_C \in L$ .

$$\begin{aligned} f_A \wedge (f_B \vee f_C) &= (f_A \wedge (f_A \vee f_C)) \wedge (f_B \vee f_C) \\ &= f_A \wedge (f_A \vee f_C) \wedge (f_B \vee f_C) \\ &= (f_A \wedge (f_A \vee f_B)) \wedge (f_A \vee f_C) \wedge (f_B \vee f_C) \\ &= f_A \wedge ((f_A \wedge f_B) \vee (f_A \wedge f_C)) \vee (f_B \wedge f_C) \\ &= f_A \wedge ((f_B \wedge f_C) \vee (f_A \wedge f_B)) \vee (f_C \wedge f_A) \end{aligned}$$

Since,  $L$  is modular,  $f_A \geq (f_A \wedge f_B) \vee (f_A \wedge f_C)$ , we get

$$\begin{aligned} f_A \wedge (f_B \vee f_C) &= \{f_A \wedge (f_B \wedge f_C)\} \vee \{(f_A \wedge f_B) \vee (f_C \wedge f_A)\} \\ &= \{(f_A \wedge f_B \wedge f_C) \vee (f_A \wedge f_B)\} \vee (f_A \wedge f_C) \\ &= (f_A \wedge f_B) \vee (f_A \wedge f_C) \end{aligned}$$

Hence  $L$  is a distributive soft lattice.

**Theorem 3.16** A soft lattice  $L$  is distributive if and only if it has no soft sublattice isomorphic to  $SN_5$  or  $SM_3$ .

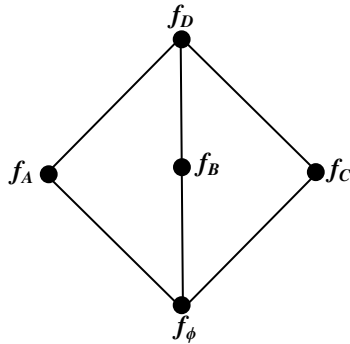
*Proof.* Let  $L$  be a distributive soft lattice. The soft lattice  $SN_5$  is not a modular soft lattice and so it cannot be a

**International Journal of Innovative Research in Science,  
Engineering and Technology**

(An ISO 3297: 2007 Certified Organization)

Vol. 2, Issue 11, November 2013

distributive soft lattice. Consider  $SM_3$  in figure 6.



**Figure 6.**

$$f_A \vee (f_B \wedge f_C) = f_A \vee f_\emptyset = f_A ; (f_A \vee f_B) \wedge (f_A \vee f_C) = f_D \vee f_D = f_D$$

Therefore,  $f_A \vee (f_B \wedge f_C) \neq (f_A \vee f_B) \wedge (f_A \vee f_C)$ . Hence  $SM_3$  is not a distributive soft lattice.

If  $L$  is distributive soft lattice then every soft sublattice of  $L$  is also distributive. So  $L$  cannot have a soft sublattice isomorphic to  $SN_5$  or  $SM_3$ .

Conversely, Suppose that  $L$  is a soft lattice which has no soft sublattice isomorphic to  $SN_5$  or  $SM_3$ .

By theorem 3.13, it is enough if we prove that, a modular soft lattice is non-distributive then it has a soft sublattice isomorphic to  $SM_3$ . Let  $L$  be a modular non-distributive soft lattice. Then  $L$  has a triplet  $f_P, f_Q, f_R$  of elements such that  $(f_P \wedge f_Q) \vee (f_Q \wedge f_R) \vee (f_R \wedge f_P) < (f_P \vee f_Q) \wedge (f_Q \vee f_R) \wedge (f_R \vee f_P)$

$$\text{Let } f_X = (f_P \wedge f_Q) \vee (f_Q \wedge f_R) \vee (f_R \wedge f_P) \quad f_Y = (f_P \vee f_Q) \wedge (f_Q \vee f_R) \wedge (f_R \vee f_P).$$

Then  $f_X \leq f_Y$  and  $L$  is modular soft lattice.

$$\text{Let } f_A = f_X \vee (f_P \wedge f_Y) = (f_X \vee f_P) \wedge f_Y, \quad f_B = f_X \vee (f_Q \wedge f_Y) = (f_X \vee f_Q) \wedge f_Y$$

$$f_C = f_X \vee (f_R \wedge f_Y) = (f_X \vee f_R) \wedge f_Y$$

Let  $S = \{f_X, f_Y, f_A, f_B, f_C\}$ . We claim that  $S$  is a soft sublattice of  $L$  isomorphic to  $SM_3$ .

$$\text{To prove } f_A \wedge f_B = f_B \wedge f_C = f_C \wedge f_A = f_X \quad \text{and} \quad f_A \vee f_B = f_B \vee f_C = f_C \vee f_A = f_Y$$

$$\text{Here, } f_P \wedge f_Y = f_P \wedge \{(f_P \vee f_Q) \wedge (f_Q \vee f_R) \wedge (f_R \vee f_P)\} = f_P \wedge (f_Q \vee f_R)$$

$$f_Q \wedge f_Y = f_Q \wedge \{(f_P \vee f_Q) \wedge (f_Q \vee f_R) \wedge (f_R \vee f_P)\} = f_Q \wedge (f_R \vee f_P)$$

$$\text{Now, } f_A \vee f_B = f_X \vee \{(f_P \wedge (f_Q \vee f_R)) \vee (f_Q \wedge (f_R \vee f_P))\} \dots\dots\dots(1)$$

$$\text{since } f_P \wedge (f_Q \vee f_R) \leq f_P \leq f_R \vee f_P$$

$$\begin{aligned} (f_P \wedge (f_Q \vee f_R)) \vee (f_Q \wedge (f_R \vee f_P)) &= ((f_P \wedge (f_Q \vee f_R)) \vee f_Q) \wedge (f_R \vee f_P) \\ &= (f_Q \vee ((f_P \wedge (f_Q \vee f_R)) \wedge (f_R \vee f_P))) \end{aligned}$$

**International Journal of Innovative Research in Science,  
Engineering and Technology**

(An ISO 3297: 2007 Certified Organization)

**Vol. 2, Issue 11, November 2013**

$$= (f_Q \vee f_P) \wedge (f_Q \vee f_R) \wedge (f_R \vee f_P)$$

Therefore, (1) becomes  $f_A \vee f_B = f_X \vee ((f_Q \vee f_P) \wedge (f_Q \vee f_R) \wedge (f_R \vee f_P))$   
 $= f_X \vee f_Y = f_Y$

Hence  $f_A \vee f_B = f_Y$ . Similarly we prove,  $f_B \vee f_C = f_C \vee f_A = f_Y$

By the principle of duality, we have  $f_A \wedge f_B = f_X$  and  $f_B \wedge f_C = f_C \wedge f_A = f_X$

Therefore,  $f_X$  and  $f_Y$  are the bound elements of the soft sublattice  $S$ .

It remains to prove that these five elements  $f_X, f_A, f_B, f_C, f_Y$  are all distinct. That is to prove

$$f_X \neq f_A, f_X \neq f_B, f_X \neq f_C, f_Y \neq f_A, f_Y \neq f_B, f_Y \neq f_C \text{ and } f_A \neq f_B, f_B \neq f_C, f_C \neq f_A$$

Suppose  $f_X = f_A$

$$\Rightarrow f_A \wedge f_B = f_A$$

$$\Rightarrow f_A \vee f_B = f_B$$

$$\Rightarrow f_Y = f_B$$

Now,  $f_A \wedge f_C = f_A$

$$\Rightarrow f_A \vee f_C = f_C$$

$$\Rightarrow f_Y = f_C$$

Therefore,  $f_Y = f_Y \wedge f_Y = f_B \wedge f_C = f_X$  which is a contradiction. Hence  $f_X \neq f_A$ .

Similarly, we have  $f_X \neq f_B, f_X \neq f_C$

By the principle of duality, we have  $f_Y \neq f_A, f_Y \neq f_B, f_Y \neq f_C$ .

If  $f_A = f_B$  then  $f_X = f_A \wedge f_A = f_A$  which is a contradiction. Hence  $f_A \neq f_B$ .

Similarly we have  $f_B \neq f_C, f_C \neq f_A$ .

These observations lead to the representation of  $S = \{f_X, f_A, f_B, f_C, f_Y\}$  by a Hasse diagram given in figure 7.

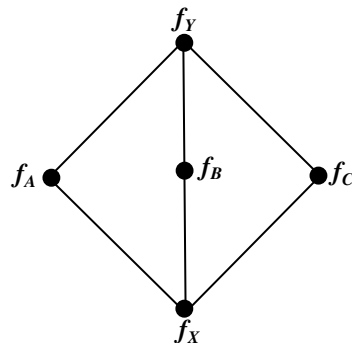


Figure 7.

# International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 2, Issue 11, November 2013

Therefore,  $S$  is a soft sublattice of  $L$  isomorphic to  $SM_3$ . Thus a modular non-distributive soft lattice has an isomorphic copy of  $SM_3$ . Therefore, if  $L$  has no soft sublattice isomorphic to  $SN_5$  or  $SM_3$  then  $L$  is a distributive soft lattice.

## IV.CONCLUSION

In this paper, we have given characterization theorems for modular soft lattices and distributive soft lattices. We are studying about these soft lattices and are expected to give some more results.

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