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On a Two Dimensional Finsler Space whose Geodesics are Semi-Cubical Parabolas

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ABSTRACT: In this present paper we have obtain the fundamental metric function of two dimensional Finsler space whose geodesics are semi-cubical parabolas. We showed that such space is locally Minkowskian.

KEYWORDS: Keywords: Finsler space, Two Dimensional, Geodesics, Locally Minkowskian.

I. INTRODUCTION

The fundamental idea of a Finsler space may be traced back to the famous lecture of Riemann' at Göttingen in 1854. He discussed various possibilities by means of which an n-dimensional manifold may be equipped with a metric and paid special attention to a metric defined by the positive square root of a positive definite quadratic differential form. In this way he laid down the foundation of Riemannian geometry. He also suggested that the positive fourth root of a fourth order differential form might serve as a metric function. This function has three properties in common. It is positive definite, homogeneous of the first degree in the differential and is also convex in the later. Riemann was not hopeful about the geometrical interpretation of the results in such space and therefore he made the following comments:

"*Investigation of this more general would actually require no essential different principles, but it would be rather time consuming and throw relatively little new light on the study of space especially since the results cannot be expressed geometrically*."

It is remarkable that the first systematic study of manifolds equipped with such a metric was delayed by more than 60 years. Twenty four years old young German Paul Finsler started the study of Finsler Geometry and submitted his epoc-making thesis to Gottingen University in 1918. He studied this geometry from the stand point of a generalization of the calculus of variation. Finsler derived the idea almost directly from the calculus of variations, with particular reference to the new geometrical background which was introduced by his teacher 'Cartheodory' in connection with problems in parametric form.

The history of development of Finsler Geometry can be divided into four periods.

The first period of the history of Finsler geometry began in 1904, when three geometers J.H. Taylor [21], J.L. Syange [20] and L. Berwald [16] simultaneously started the work in this field. Berwald is the first man who has introduced the concept of connection in the theory of Finsler spaces. He developed a theory with particular reference to the theory of curvature in which the Ricci's lemma does not hold good. J.H. Taylor and J.L. Synge introduced a special parallelism. In 1928, Taylor gave the name 'Finsler Space' to the manifold equipped with this generalized metric.

The second period started in 1934, when E. Cartan [12] published his thesis on Finsler geometry. He showed that it was indeed possible to define connection coefficients and hence covariant derivatives, such that the Ricci's lemma is satisfied. On this basis Cartan developed the theory of curvature, torsion and discussed geometry of Finsler spaces by his approach. Many mathematician such as E.T. Devies [13], S. Golab [15], H. Hombu [16], O. Varga [22], V.V. Wagner [23] studied Finsler geometry along Cartan's approach. The above mentioned theories explored certain devices, which basically involves the consideration of a space whose elements were not the points of the underlying manifold, but the line elements, which form a (2n – 1) dimensional variety.

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The third period of the history of Finsler geometry began in 1951 when H.Rund [18] introduced a new process of parallelism from the stand point of Minkowskian geometry. It was emphasized that the local metric of a Finsler space is a Minkowskian one and that the arbitrary imposition of a Euclidean metric would to some extent, obscure some of the most interesting characteristics of the Finsler space. Latter on E.T. Davies [13] and A. Deicke [14] have indicated that Rund's and Cartan's parallelism were the same. Several Mathematicians such as W. Barthel [10] A Deicke [14], D. Laugwitz [17], R.Sulanke [19] have studied Finsler spaces on Rund's approach.

The fourth period started in 1963, when the Finsler geometry came under the influence of Topology. H. Akabar Zadeh [9] developed the modern theory of Finsler spaces based on the theory of connections in fibre bundles. The reason of modernization is to establish a global definition of connections in Finsler spaces and to re-examine the Cartan's sytem of axioms. The study on this line started when Matsumoto organized symposium on the models of Finsler spaces in 1970. Several mathematicians are working in this field. Some of them are S.S. Chern, D.Bao, Z.Shen, R.L. Bryant, Burogo, etc. The era of applications of Finsler geometry began with the work of R.S. Ingarden [1]. Several Mathematicians have applied the theory of Finsler spaces in various fields of physics and Biology up till now. We also apply this theory to thermodynamics and ecology.

It is an interesting problem to find the fundamental function of a two dimensional Finsler space whose geodesics constitute a given family of curves. M. Matsumoto [**5**],[**6**],[**8**][1)] obtained the fundamental function of a two dimensional Finsler space whose geodesics are special conic sections. The aim of the present paper is to obtain the fundamental metric function of two dimensional Finsler space whose geodesics are semi-cubical parabolas. We show that such space is locally Minkowskian.

PRELIMINARIES

Let $F^n = (M^n, L(x, y))$ be an n-dimensional Finsler space on an underlying smooth manifold M^n with the fundamental function $L(x,y)$. The fundamental tensor $g_{ij}(x,y)$, the angular metric tensor $h_{ij}(x,y)$ and the normalized supporting element $l_i(x,y)$ are defined respectively by

$$g_{ij} = h_{ij} + l_i l_j, \qquad h_{ij} = LL_{(i)(j)}, \qquad l_i = L_{(i)}$$

where $L_{(i)} = \dfrac{\partial L}{\partial y^i}$ and $L_{(i)(j)} = \dfrac{\partial^2 L}{\partial y^i \partial y^j}$.

The geodesic, the extremal of the length integral $s = \int_{t_0}^{t} L(x,y)dt$ where $t \geq t_0$, $y^i = \dot{x}^i = \dfrac{dx^i}{dt}$ along an oriented curve $C{:}x^i = x^i(t)$ from point P = $x^i(t_o)$ to point Q= $x^i(t)$, $t > 0$, is given by the Euler equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^i}\right) - L_i = 0, \tag{1.1}$$

where $L_i = \dfrac{\partial L}{\partial x^i}$.

In terms of $F(x, y) = L^2(x, y)/2$, (1.1) is written in the form

1) Numbers in square brackets refer to the references at the end of the paper.

$$\frac{d^2x^i}{dt^2} + 2G^i(x,\dot{x}) = h(t)\frac{dx^i}{dt} \tag{1.2}$$

where we put

$$2g_{ij}\, G^i\,(x,y) = y^r\left(\frac{\partial^2 F}{\partial x^r \partial yJ}\right) - \frac{\partial F}{\partial x^J}\,, \tag{1.3}$$

and $h\,(t) = (d^2s/dt^2)/\,(ds/dt)$.

Now we consider two dimensional Finsler space and use the notation (x,y) and (p,q) respectively, instead of (x^1,x^2) and (y^1,y^2). The fundamental function $L(x,y;p,q)$ are positively homogeneous of degree one in p and q. Therefore, we have

$$L_x = pL_{xp} + qL_{xq}, \qquad L_y = pL_{yp} + qL_{yq}\,. \tag{1.3 a}$$

Also from homogeneity of L_p and L_q we have

$$\frac{L_{pp}}{q^2} = \frac{-L_{pq}}{pq} = \frac{L_{qq}}{p^2} = W(say)\,, \tag{1.3 b}$$

where W is called Weierstrass invariant. Consequently the two equations represented by (1.1) reduce to the single equation

$$L_{xq} – L_{yp} + (p\dot{q} - q\dot{p})W = 0\,, \tag{1.4}$$

which is called Weierstrass form of geodesic equation. Now consider the associated fundamental function $A(x,y,z)$, $z=y' = \dfrac{dy}{dx}$ defined as follows:

$$A(x,y,z) = L(x,y;\,1,z), \quad L(x,y;p,q) = A\,(x,y,\frac{q}{p})p. \tag{1.5}$$

Therefore $L_p=A–zA_z$, $L_q=A_z$, $L_{yp} = A_y – zA_{yz}$, $L_{xq} = A_{xz}$, $L_{pp} = (z^2/p)\,Azz$, $L_{pq} = (-z/p)\,A_{z\,z}$, $L_{qq}= (1/p)\,A_{zz}$.

On using these values in (1.4), we have

$$A_{zz}\,y'' + A_{yz}\,y' + A_{xz} – A_y=0,\; z=y', \tag{1.6}$$

which is called the Rashevsky form of geodesic equation.

We observe that $y'=q/p$ gives $y''= (p\dot{q} - q\dot{p})\,/\,p^3$. Hence from (1.2) we have another form of geodesic equation

$$y'' = \frac{2}{p^3}\left(qG^1 - pG^2\right). \tag{1.7}$$

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Now in *n*-dimensional Finsler space F^n we have the Berwald connection $B\Gamma = \left(G^i_{jk}, G^i_j\right)$; defined by $G^i_j = \partial G^i / \partial y^j$, $G^i_{jk} = \partial G^i_j / \partial y^k$ and two kinds of covariant differentiation of Finslerian vector field $V^i(x,y)$ given by

$$V^i_{(j)} = \frac{\partial v^i}{\partial x^j} - \frac{\partial v^i}{\partial y^r} G^r_J + V^r G^i_{rJ}, \; V^i_{,j} = \frac{\partial v^i}{\partial y^j},$$

called *h* and *v*–covariant derivative of V^i respectively. Since $B\Gamma$ is *L*-metrical means $L_{(i)}=0$, we have

$$L_i = l_r G^r_i . \tag{1.8}$$

Further from (1.8) we have

$$L_{i,j} = (1/L) h_{rj} G^r_i + l_r G^r_{ij} . \tag{1.9}$$

Next from (1.3) we have $2g_{ij}G^i y^J = \left(\frac{\partial F}{\partial x^r}\right) y^r$, that is

$$2G^i l_i = L^r y_r . \tag{1.10}$$

Now we shall return to the two-dimensional case since the matrix (h_{ij}) is of rank one, we get $\varepsilon = \pm 1$ and the vector field $m_i(x,y)$ satisfying

$$h_{ij} = \varepsilon\, m_i m_j, \tag{1.11}$$

so we have

$$g_{ij} = l_i l_j + \varepsilon\, m_i m_j , \tag{1.12}$$

Therefore, we get easily

$$l_i l^i = 1,\ l_i m^i = m_i l^i = 0,\ m_i m^i = \varepsilon . \tag{1.13}$$

Thus we obtain the orthonormal frame field (l^i, m^i), called the Berwald frame. Therefore, we have scalar fields $h(x, y; p, q)$ and $k(x, y; p, q)$ such that

$$(m_1, m_2) = h\,(-l^2, l^1), \quad (m^1, m^2) = k(-l_2, l_1), \quad hk = \varepsilon. \tag{1.14}$$

The equations (1.12) and (1.14) give

$$g(=det\ g_{ij}) = g_{11}g_{22} - g^2_{12} = \varepsilon\,(l_1 m_2 - l_2 m_1)^2 = \varepsilon h^2, \tag{1.15}$$

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and

$$\frac{1}{h}(m^1, m^2) = \frac{1}{g}(-l_2, l_1). \tag{1.16}$$

We already have $h_{11}=L.L_{pp}=LWq^2$ and $h_{11}=\varepsilon(m_1)^2=\varepsilon h^2(l^2)^2=\varepsilon h^2\left(\frac{q}{l}\right)^2$. Consequently we have

$$L^3W = \varepsilon h^2 = g. \tag{1.17}$$

Now we try to find the expression for G^i in the Berwald frame [**4**] the equations (1.9) and (1.14) give

$$L_{xq} - L_{yp} = \frac{1}{L}\left(h_{rj}G_i^r - h_{ri}G_j^r\right)$$

$$= \frac{\varepsilon m_r}{L}\left(m_j G_i^r - m_i G_j^r\right)$$

$$= \frac{\varepsilon m_r}{L}\left(m_2 G_1^r - m_1 G_2^r\right)$$

$$= \frac{\varepsilon m_r}{L} h\left(l^1 G_1^r + l^2 G_2^r\right)$$

$$= \frac{\varepsilon m_r}{L^2} h\left(G_i^r y^i\right).$$

Due to homogeneity of G^r in y^i, we have $G_i^r y^i = 2G^r$, therefore

$$2G^i m^i = (L_{xq}-L_{yp})L^2/\varepsilon h.$$

Using $2G^i(l_i l^j + \varepsilon\, m_i\, m^j) = (L_r y^r)l^j + m^j(L_{xq} - L_{yp})L^2/h$ and (1.10) leads to

$$2G^i = (L_r y^r)l^i + (L^2/h)\,(L_{xq}-L_{yp})m^i. \tag{1.18}$$

If we put

$$L_r y^r = L_o \text{ and } L_{xq}-L_{yp} = M \tag{1.19}$$

in (1.18), and using (1.16) and (1.17) we get

$$2G^1 = \frac{1}{L}\left(L_0 p - \frac{M}{W}L_q\right),\ 2G^2 = \frac{1}{L}\left(L_0 q - \frac{M}{W}L_p\right). \tag{1.20}$$

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II. FROM GEODESICS TO THE FINSLER METRIC

Let us consider a family of curves $\{C(a, b)\}$ on the (x, y)–plane R^2, given by the equation

$$y = f(x, a, b), \tag{2.1}$$

with two parameters (a, b). Differentiating (2.1) with respect to x, we get

$$z\ (=y') = f_x(x, a, b). \tag{2.2}$$

Solving (2.1) and (2.2) for a, b, we find

$$a = \alpha(x, y, z),\ b = \beta(x, y, z). \tag{2.3}$$

In view of (2.3) the differentiation of (2.2) leads to

$$z' = f_{xx}(x, a, b) = u(x, y, z). \tag{2.4}$$

which is precisely the second order differential equation of y characterizing $\{C(a, b)\}$.

Now we are concerned with the Rashevsky form (1.6) of geodesic equation $L_{qq} = A_{zz}/p$ and $A_{zz} = Wp^3$, hence from (1.5) and (1.17) we have

$$L^3W = A^3A_{zz} = g. \tag{2.5}$$

Thus it is suitable to call A_{zz} the associated Weierstrass invariant. If we put $B=A_{zz}$, then the differentiation of (1.6) with respect to z gives

$$B_x + B_y z + B_z u + Bu_z = 0, \tag{2.6}$$

which is first order quasi-linear partial differential equation. Its auxiliary equations are given by

$$\frac{dx}{1} = \frac{dy}{z} = \frac{dz}{u} = \frac{dB}{-Bu_z} \tag{2.7}$$

Now defining $U(x,a,b)$ and $V(x,y,z)$ by

$$U(x; a, b) = exp\int u_z(x, f, f_x)dx,\quad V(x, y, z) = U(x; \alpha, \beta), \tag{2.8}$$

we obtain

$$B(x,y,z) = \frac{H(\alpha,\beta)}{V(x,y,z)}, \tag{2.9}$$

where H is an arbitrary non-zero function of two arguments.

From $A_{zz} = B$ we get A in the form

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$$\begin{cases} A(x,y,z) = A^*(x,y,z) + C(x,y) + D(x,y)z, \\ A^* = \int\left\{\int B(x,y,z)dz\right\}dz = \int_{z_0}^{z} (z-t).B(x,y,t)dt. \end{cases} \tag{2.10}$$

where C and D are arbitrary but must be chosen so that A may satisfy (1.6), that is

$$C_y – D_x = A^*_{zz}u + A^*_{yz}z + A^*_{xz} - A^*_y. \tag{2.11}$$

If a pair (C_0, D_0) has been chosen so as to satisfy (2.11), then $(C–C_0)_y=(D–D_0)_x$, so that we have locally a function $E(x, y)$ satisfying $E_x=C – C_0$ and $E_y=D – D_0$. Thus (2.10) is written as

$$A=A^*+C_0+D_0z+E_x+E_y z .$$

Therefore (1.5) leads to fundamental function

$$\begin{cases} L(x,y;p,q) = L_0(x,y;p,q) + e(x,y,p,q), \\ L_0 = A^*(x,y,\frac{q}{p})p + C_0(x,y)p + D_0(x,y)q \end{cases} \tag{2.12}$$

where e is the derived form given by

$$e(x, y; p, q) =E_xdx+E_ydy. \tag{2.13}$$

Thus, we see that the Finsler metric is uniquely determined when the functions H and E of two arguments are chosen. Further, for different choice of the function H we obtain Finsler spaces which are projective to each other because each one has the same geodesics $\{C(a, b)\}$.

II. FAMILY OF SEMI-CUBICAL PARABOLAS

Let us consider the family of semi-cubical parabolas $\{C(a, b)\}$ given by the equation

$$by^2 = (x – a)^3, \quad y, b>0, \tag{3.1}$$

on the semiplane R^2_+ having the vertex $(a,0)$ on the x-axis.

From (3.1) we have

$$2byz = 3(x–a)^2, z = y'. \tag{3.2}$$

Consequently the functions $\alpha(x, y, z)$, $\beta(x, y, z)$ and $u(x, y, z)$ of the preceding sections are given by

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$$a = x - \frac{3y}{2z} = \alpha(x, y, z),\ b = \frac{27y}{8z^3} = \beta(x, y, z), \tag{3.3}$$

$$z' = \frac{z^2}{3y} = u(x, y, z). \tag{3.4}$$

From (3.4) we get

$$3yy'' = (y')^2, \tag{3.5}$$

which characterizes the family $\{C(a, b)\}$.

Now we find the function $U(x; a, b)$ and $V(x; y, z)$ defined by (2.8). Differentiating (3.4) we get

$$u_z = \frac{2z}{3y} = \frac{1}{(x-a)} \text{ and } U(x; a, b) = exp\int \frac{1}{(x-a)}dx = (x-a),\ V(x, y, z) = \frac{3y}{2z}.$$

Thus, (2.9) implies that

$$B(x,y,z) = H(\alpha,\beta)\frac{2z}{3y} = \frac{1}{y^{2/3}\beta^{1/3}} H(\alpha,\beta).$$

On account of the arbitrariness of H, we may write B as $B=H(\alpha,\beta)/y^{2/3}$ and A^* of (2.10) is written in the form

$$A^*(x, y, z) = \frac{1}{y^{2/3}}\iint H(\alpha,\beta)(dz)^2, \tag{3.6}$$

or

$$A^*(x, y, z) = \frac{1}{y^{2/3}}\int_1^z (z-t)F(t)dt, \qquad F(t) = H\left(x - \frac{3y}{2t}, \frac{27y}{8t^3}\right). \tag{3.6$'$}$$

We have taken the limit of integration from 1 to z instead of 0 to z because t is in the denominator of F (t).

Now if we put $F_1(t) = H_\alpha\left(x - \frac{3y}{2t}, \frac{27y}{8t^3}\right)$, $F_2(t) = H_\beta\left(x - \frac{3y}{2t}, \frac{27y}{8t^3}\right)$, then from (2.10) and (2.11) we have

$$C_y - D_x = \frac{z^2}{3y^{5/3}}H(\alpha,\beta) - \frac{2}{3y^{5/3}}\int_1^z tF(t)dt - \frac{1}{2y^{2/3}}\int_1^z F_1(t)dt + \frac{27}{8y^{2/3}}\int_1^z \frac{F_2(t)}{t^2}dt. \tag{3.7}$$

Therefore we have:

Theorem 1. Every associated fundamental function $A(x, y, z)$ of a Finsler space $\left(R_+^2, L(x, y; p, q)\right)$ having the semi-cubical parabolas (3.1) as the geodesics is given by

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$$A(x, y, z)= A^*(x, y, z)+C(x, y)+D(x, y)z,$$

where A^* is defined by (3.6)′, H is an arbitrary function of (α,β) given by (3.3) and the function (C, D) must be chosen so as to satisfy (3.7).

Example. In particular, we first put $H(\alpha,\beta) = (\beta)^n$ for a real number n, then $F(t) = \left(\frac{27y}{8t^3}\right)^n$, hence from (3.6)′ and (3.7) we have

$$A^* = \frac{1}{y^{2/3}}\int_1^z (z-t)\left(\frac{27y}{8t^3}\right)^n dt, \tag{3.8}$$

or

$$A^* = \frac{\left(\frac{27}{8}\right)^n y^{n-2/3}}{(1-3n)(2-3n)}\left[z^{2-3n} - (2-3n)z + (1-3n)\right] \tag{3.8′}$$

and

$$C_y - D_x = \left(\frac{27}{8}\right)^n \frac{y^{n-5/3}}{3}. \tag{3.9}$$

If we choose $C = -\left(\frac{27}{8}\right)^n \frac{y^{n-2/3}}{(2-3n)}$ and $D = \left(\frac{27}{8}\right)^n \frac{y^{n-2/3}}{(1-3n)}$, we have

$$A(x, y, z) = \left(\frac{27}{8}\right)^n \frac{y^{n-2/3} z^{2-3n}}{(1-3n)(2-3n)}. \tag{3.10}$$

Therefore it follows from (1.5) that, the fundamental function

$$L(x, y; p, q) = y^{n-2/3} q^{2-3n} p^{3n-1},\ n\neq 1/3,\ 2/3 \tag{3.11}$$

where $\left(\frac{27}{8}\right)^n \frac{1}{(1-3n)(2-3n)}$ was omitted.

Case I. If $n = \frac{1}{3}$, then (3.8) and (3.9) gives

$$A^* = \frac{3}{2} y^{-1/3}(z \log|z| - z + 1), \quad C_y - D_x = \frac{1}{2} y^{-4/3}.$$

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Choosing $D = -C = \frac{3}{2}y^{-1/3}$, we have $A(x, y, z) = \frac{3}{2}y^{-1/3}z\log|z|$,

consequently we omit 3/2 and obtain fundamental function

$$L(x, y. p, q) = y^{-1/3}\, q \log\left|\frac{q}{p}\right|. \tag{3.12}$$

Case II. If n $= \frac{2}{3}$, we have similarly $A^* = \frac{9}{4}(z - \log|z| - 1)$, C_y–D_x= $\frac{3}{4y}$.

Choosing $C = \frac{9}{4}$ and $D = -\frac{3x}{4y} - \frac{9}{4}$, we have $A(x, y, z) = -\frac{3}{4}(3\log|z| + \frac{xz}{y})$. Therefore, omitting -3/4 we obtain the metric

$$L(x, y; p, q) = 3p \log\left|\frac{q}{p}\right| + \frac{xq}{y}. \tag{3.13}$$

Now we shall return to the general case with the Finsler metric (3.11). If we refer to the new co-ordinate system $(\bar{x}, \bar{y}) = (x, y^{3/2})$, then we have (p, q) $= \left(\bar{p}, \frac{3}{2}\sqrt{\bar{y}}\,\bar{q}\right)$, and the metric (3.11) can be written in the form

$$\bar{L}(\bar{x}, \bar{y}; \bar{p}, \bar{q} = \left(\frac{3}{2}\right)^{2-3n}(\bar{q})^{2-3n}(\bar{p})^{3n-1}. \tag{3.11$'$}$$

Since $\bar{L}$ does not depend on $\bar{x}$ and $\bar{y}$, this is a simple metric, called a locally Minkowskian metric and ($\bar{x}, \bar{y}$) is an adapted co-ordinate system to the structure. Further its main scalar I is constant. Since (3.11)′ is of the form (*i*) or (*iv*) of Theorem *3.5.3.2* of [**1**], we have directly as follows:

(i) $\varepsilon = 1$, $I^2 > 4$, $(2\text{-}3n)(3n-1) < 0$, $\frac{I}{\sqrt{I^2-4}} + 1 = 2(2-3n)$,

(ii) $\varepsilon = -1$, $(2\text{-}3n)(3n-1) > 0$, $\frac{I}{\sqrt{I^2+4}} + 1 = 2(2-3n)$.

Thus we have:

Proposition 1. The Finsler space $(R_+^2, L(x, y, p, q))$ with a metric (3.11) is locally Minkowskian and have the signature ε and the constant main scalar I as follows:

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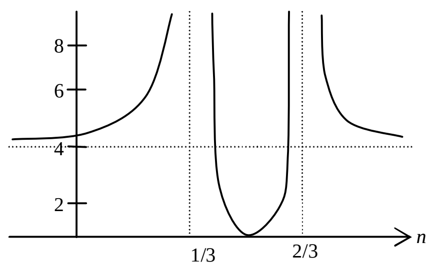
(1) $n < \frac{1}{3}, > \frac{2}{3}: \varepsilon = 1,\ I^2 = \frac{(6n-3)^2}{(3n-2)(3n-1)}$,

(2) $\frac{1}{3} < n < \frac{2}{3}: \varepsilon = -1,\ I^2 = -\frac{(6n-3)^2}{(3n-2)(3n-1)}$.

Remark. $\frac{(6n-3)^2}{(3n-2)(3n-1)} = 4 + \frac{1}{3n-2} - \frac{1}{3n-1}$. The graph of I^2 is shown in figure.



Since a Finsler space of dimension two is Riemannian if and only if $I = 0$. Therefore, *The Finsler space* $\left(R_+^2, L(x,y,p,q)\right)$ *under consideration is Riemannian if and only if n=1/2.*

IV. CONCLUSION

It showed that the fundamental function of a two dimensional Finsler space whose geodesics constitute a given family of curves. We have obtained the fundamental metric function of two dimensional Finsler space whose geodesics are semi-cubical parabolas and we also proved that such space is locally Minkowskian.

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