ISSN(Online) : 2319 - 8753
ISSN (Print) : 2347 - 6710

International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 4, Issue 4, April 2015

An $M^{[X]}/G/1$ Retrial G-Queue with Server Breakdown

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ABSTRACT: Batch arrival retrial queue with positive and negative customers is considered. If the server is idle upon the arrival of a batch, one of the customers in the batch receives service immediately and others join the orbit. If the server is busy, all the customers join the orbit. The arrival of negative customer brings the server down and removes the customer in service from the system. The server is subject to random breakdown while it is working. Using supplementary variable technique, expected number of customers in the orbit and expected number of customers in the system are derived. Stochastic decomposition property is established. Some special cases are discussed and numerical results are presented.

KEYWORDS: Batch arrival retrial queue, G queue, Breakdown, Performance Measures, Stochastic Decomposition.

I. INTRODUCTION

Retrial queueing systems are characterized by the feature that arriving customers who cannot receive service immediately may join a virtual queue called orbit to try their request after some random time. Queueing systems with repeated attempts are found suitable for modelling the processes in telephone switching systems, digital cellular mobile networks, packet switching networks, local area networks, stock and flow etc. Review of retrial queueing literature can be found in the bibliographies of Artalejo (1999a, 1999b) and the books by Falin and Templeton (1997) and Artalejo and Gomez Corral (2008). The applications of retrial queues in science and engineering are given in Kulkarni and Liang (1997). Queue with negative arrivals called G queue was first introduced by Gelenbe (1989) with a view to modelling neural networks. Wu and Lian (2013) analysed an M/G/1 retrial G queue with priority resume, Bernoulli vacation and server breakdown. Using Lyapunov functions. Peng et al. (2013) suggested an M/G/1 retrial G- queue with preemptive resume priority and collisions under linear retrial policy. In this article batch arrival retrial G-queue with server breakdown is analysed.

II. MODEL DESCRIPTION

Consider a single server retrial queueing system with positive and negative customers. Positive customers arrive in batches according to Poisson process with rate λ^{+}. Negative customers arrive singly with Poisson arrival rate λ^{-}. At every arrival epoch, a batch of k customers arrive with probability C_k. The generating function of the sequence $\{C_k\}$ is C(z) with first two moments m_1 and m_2. There is no waiting space in front of the server and therefore if an arriving batch of positive customers finds the server idle, then one of the customers receives the service and others join the retrial queue. If the server is busy, then the arriving batch enters the orbit. The retrial time of the customers is generally distributed with distribution function A(x), density function a(x), Laplace-Stieltjes transform $A^{*}(s)$ and hazard rate function $\eta(x)=\frac{a(x)}{1-A(x)}$.The service time of positive customers is generally distributed with distribution function B(x),

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Vol. 4, Issue 4, April 2015

density function b(x), Laplace-Stieltjes transform $B^*(s)$ and hazard rate function $\mu(x)=\frac{b(x)}{1-B(x)}$. The arrival of a negative customer removes the positive customer in service from the system and causes the server breakdown. Also the server is subject to unpredictable breakdown while it is working. The life time of the server is exponentially distributed with rate α.The repair time of the failed server is generally distributed with distribution function R(x), density function r(x) and hazard rate function $\beta(x)=\frac{r(x)}{1-R(x)}$. All stochastic processes involved in the system are assumed to be independent of each other. Throughout the rest of the paper, we denote $\bar{F}(x)=1-F(x)$ the tail of distribution function F(x), $F^*(s)=\int_0^\infty e^{-sx}dF(x)$ and $\tilde{\bar{F}}(s)=\int_0^\infty e^{-sx}\bar{F}(x)dx=\frac{1-F^*(s)}{s}$.

III. STABILITY CONDITION

Let $N(t_n^+)$ be the number of customers in the orbit just after the time t_n. Then the sequence of random variables $Y_n = N(t_n^+)$ form a Markov chain, which is the embedded Markov chain for this queueing system.

Theorem 3.1

The embedded Markov chain $\{Y_n,\ n\in N\}$ is ergodic if and only if

$$(1-B^*(\lambda^-+\alpha))(\lambda^+m_1(1+\beta_1(\lambda^-+\alpha))+\alpha+m_1(\lambda^-+\alpha)(1-A^*(\lambda^+)))<(\lambda^-+\alpha)(1-m_1(1-A^*(\lambda^+))B^*(\lambda^-+\alpha))$$

The theorem can be proved as in Gomez-Corral (1999).

IV. STEADY STATE DISTRIBUTION

In this section, by treating elapsed service time and elapsed repair time of the server as supplementary variables, the steady state probability generating functions of the orbit size distribution are derived.

Define the states of the server as

$$C(t)=\begin{cases}0, & \text{if the server is idle at time } t,\\ 1, & \text{if the server is busy at time } t,\\ 2, & \text{if the server is under repair at time } t.\end{cases}$$

For $t \ge 0$, we define the random variable $\xi(t)$ as follows:

(i) if C(t) = 0, then $\xi(t)$ represents the elapsed retrial time at time t;
(ii) if C(t) = 1, then $\xi(t)$ represents the elapsed service time at time t;
(iii) if C(t) = 2, then $\xi(t)$ represents the elapsed repair time at time t.

Then, the process $\{X(t);\ t\ge 0\} = \{C(t), N(t), \xi(t), t\ge 0\}$ is a Markov process.

For the process $\{X(t);\ t\ge 0\}$, we define the following probability densities

$I_0(t) = P\{C(t) = 0, N(t) = n\}$

$I_n(x, t)\,dx = P\{C(t) = 0, N(t) = n, x \le \xi(t) < x + dx\},\ x \ge 0,\ n \ge 1$

$P_n(x, t)\,dx = P\{C(t) = 1, N(t) = n, x \le \xi(t) < x + dx\},\ x \ge 0,\ n \ge 0$

$R_n(x, t)\,dx = P\{C(t) = 2, N(t) = n, x \le \xi(t) < x + dx\},\ x \ge 0,\ n \ge 0$

Let I_0, $I_n(x)$, $P_n(x)$ and $R_n(x)$ be the steady state probabilities of $I_n(t)$, $I_n(x, t)$, $P_n(x, t)$ and $R_n(x, t)$ respectively.

4.1 The steady state equations

The system of equation that governs the model under supplementary variable technique are given below

IJIRSET

ISSN(Online) : 2319 - 8753
ISSN (Print) : 2347 - 6710

International Journal of Innovative Research in Science, Engineering and Technology

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Vol. 4, Issue 4, April 2015

$$\lambda^{+}I_0 = \int_0^{\infty} P_0(x)\mu(x)dx + \int_0^{\infty} R_0(x)\beta(x)dx, \tag{1}$$

$$\frac{d}{dx} I_n(x) = -(\lambda^{+} + \eta(x))I_n(x), \quad n \ge 1 \tag{2}$$

$$\frac{d}{dx} P_n(x) = -(\lambda^{+} + \lambda^{-} + \alpha + \mu(x))P_n(x) + \lambda^{+} \sum_{k=1}^{n} C_k P_{n-k}(x), \; n \ge 0 \tag{3}$$

$$\frac{d}{dx} R_n(x) = -(\lambda^{+} + \beta(x))R_n(x) + \lambda^{+} \sum_{k=1}^{n} C_k R_{n-k}(x), \quad n \ge 0 \tag{4}$$

with boundary conditions

$$I_n(0) = \int_0^{\infty} P_n(x)\mu(x)dx + \int_0^{\infty} R_n(x)\beta(x)dx, \quad n \ge 1 \tag{5}$$

$$P_0(0) = \lambda^{+}C_1 I_0 + \int_0^{\infty} I_1(x)\eta(x)dx, \tag{6}$$

$$P_n(0) = \lambda^{+}C_{n+1}I_0 + \int_0^{\infty} I_{n+1}(x)\eta(x)dx + \lambda^{+} \sum_{k=1}^{n} C_k \int_0^{\infty} I_{n-k+1}(x)dx, \qquad n \ge 1 \tag{7}$$

$$R_0(0) = \lambda^{-} \int_0^{\infty} P_0(x)dx \tag{8}$$

$$R_n(0) = \lambda^{-} \int_0^{\infty} P_n(x)dx + \alpha \int_0^{\infty} P_{n-1}(x)dx, \qquad n \ge 1 \tag{9}$$

The normalising condition is

$$I_0 + \sum_{n=1}^{\infty} \int_0^{\infty} I_n(x)dx + \sum_{n=0}^{\infty} \int_0^{\infty} P_n(x)dx + \sum_{n=0}^{\infty} \int_0^{\infty} R_n(x)dx = 1 \tag{10}$$

Define the probability generating functions

$$I(x,z) = \sum_{n=1}^{\infty} I_n(x) z^n; \; P(x,z) = \sum_{n=0}^{\infty} P_n(x) z^n \text{ and } R(x,z) = \sum_{n=0}^{\infty} R_n(x) z^n, \; |z| \le 1$$

The following theorem discusses the steady state distribution of the system.

Theorem 4.1

The stationary distribution of the process $\{X(t);\, t \ge 0\}$ has the following generating functions

$$I(x,z) = \lambda^{+}I_0[C(z)B^{*}(\lambda^{+} + \lambda^{-} + \alpha - \lambda^{+}C(z)) - z + C(z)(\lambda^{-} + \alpha z)K(z)R^{*}(\lambda^{+} - \lambda^{+}C(z))]e^{-\lambda^{+}x}(1 - A(x))/T(z)$$

$$P(x,z) = \lambda^{+}I_0 A^{*}(\lambda^{+})(C(z) - 1)e^{-(\lambda^{+} + \lambda^{-} + \alpha - \lambda^{+}C(z))x}(1 - B(x))/T(z)$$

$$R(x,z) = \lambda^{+}(\lambda^{-} + \alpha z)I_0 A^{*}(\lambda^{+})(C(z) - 1)K(z)e^{-(\lambda^{+} - \lambda^{+}C(z))x}(1 - R(x))/T(z)$$

where

$$T(z) = z - (C(z) + (1 - C(z))A^{*}(\lambda^{+}))B^{*}(\lambda^{+} + \lambda^{-} + \alpha - \lambda^{+}C(z)) - (\lambda^{-} + \alpha z)K(z)R^{*}(\lambda^{+} - \lambda^{+}C(z))(C(z) + (1 - C(z))A^{*}(\lambda^{+}))$$

Proof

Multiplying equations (2) to (9) by z^n and summing over all possible values of n, we obtain the following equations:

$$\frac{d}{dx}I(x,z) = -(\lambda^{+} + \eta(x))I(x,z) \tag{11}$$

$$\frac{d}{dx}P(x,z) = -(\lambda^{+} + \lambda^{-} + \alpha + \mu(x) - \lambda^{+}C(z))P(x,z) \tag{12}$$

$$\frac{d}{dx}R(x,z) = -(\lambda^{+} + \beta(x) - \lambda^{+}C(z))R(x,z) \tag{13}$$

$$I(0,z) = \int_0^\infty P(x,z)\mu(x)dx + \int_0^\infty R(x,z)\beta(x)dx - \lambda^{+}I_0 \tag{14}$$

$$P(0,z) = \frac{\lambda^{+}C(z)}{z}I_0 + \frac{1}{z}\int_0^\infty I(x,z)\eta(x)dx + \frac{\lambda^{+}C(z)}{z}\int_0^\infty I(x,z)dx \tag{15}$$

$$R(0,z) = (\lambda^{-} + \alpha z)\int_0^\infty P(x,z)dx \tag{16}$$

Solving the partial differential equations (11) - (13), we get

$$I(x,z) = I(0,z)e^{-\lambda^{+}x}(1 - A(x)) \tag{17}$$

$$P(x,z) = P(0,z)e^{-(\lambda^{+} + \lambda^{-} + \alpha - \lambda^{+}C(z))x}(1 - B(x)) \tag{18}$$

$$R(x,z) = R(0,z)e^{-(\lambda^{+} - \lambda^{+}C(z))x}(1 - R(x)) \tag{19}$$

Using equations (18) and (19) in equation (14), we get

$$I(0,z) = P(0,z)[B^{*}(\lambda^{+} + \lambda^{-} + \alpha - \lambda^{+}C(z)) + (\lambda^{-} + \alpha z)K(z)R^{*}(\lambda^{+} - \lambda^{+}C(z))] - \lambda^{+}I_0 \tag{20}$$

where $K(z) = \dfrac{1 - B^{*}(\lambda^{+} + \lambda^{-} + \alpha - \lambda^{+}C(z))}{(\lambda^{+} + \lambda^{-} + \alpha - \lambda^{+}C(z))}$

Substituting the expression of I(x, z), in equation (15) and simplifying we obtain

$$P(0,z) = \frac{\lambda^{+}C(z)}{z}I_0 + \frac{I(0,z)}{z}(C(z) + (1 - C(z))A^{*}(\lambda^{+})) \tag{21}$$

Solving equations (20) and (21), we get

$$I(0,z) = \lambda^{+}I_0[C(z)B^{*}(\lambda^{+} + \lambda^{-} + \alpha - \lambda^{+}C(z)) - z + C(z)(\lambda^{-} + \alpha z)K(z)R^{*}(\lambda^{+} - \lambda^{+}C(z))]/T(z) \tag{22}$$

$$P(0,z) = \lambda^{+}I_0A^{*}(\lambda^{+})(C(z) - 1)/T(z) \tag{23}$$

Using equations (18) and (23), equation (16) yields

$$R(0,z) = \lambda^{+}(\lambda^{-} + \alpha z)I_0A^{*}(\lambda^{+})(C(z) - 1)K(z)/T(z) \tag{24}$$

Substituting the expressions of I(0, z), P(0, z) and R(0, z), in equations (17), (18), and (19), we get respectively

$$I(x,z) = \lambda^{+}I_0[C(z)B^{*}(\lambda^{+} + \lambda^{-} + \alpha - \lambda^{+}C(z)) - z + C(z)(\lambda^{-} + \alpha z)K(z)R^{*}(\lambda^{+} - \lambda^{+}C(z))]e^{-\lambda^{+}x}(1 - A(x))/T(z) \tag{25}$$

$$P(x,z) = \lambda^{+}I_0A^{*}(\lambda^{+})(C(z) - 1)e^{-(\lambda^{+} + \lambda^{-} + \alpha - \lambda^{+}C(z))x}(1 - B(x))/T(z) \tag{26}$$

$$R(x,z) = \lambda^{+}(\lambda^{-} + \alpha z)I_0A^{*}(\lambda^{+})(C(z) - 1)K(z)e^{-(\lambda^{+} - \lambda^{+}C(z))x}(1 - R(x))/T(z) \tag{27}$$

Theorem 4.2

The partial probability generating function of the orbit size

(i) when the server is idle (ii). when the server is busy (iii). when the server is under repair are given by

$$I(z) = I_0(1 - A^{*}(\lambda^{+}))[(C(z)B^{*}(\lambda^{+} + \lambda^{-} + \alpha - \lambda^{+}C(z)) - z)(\lambda^{+} + \lambda^{-} + \alpha - \lambda^{+}C(z)) + (\lambda^{-} + \alpha z)C(z)(1 - B^{*}(\lambda^{+} + \lambda^{-} + \alpha - \lambda^{+}C(z)))$$

ISSN(Online) : 2319 - 8753
ISSN (Print) : 2347 - 6710

International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 4, Issue 4, April 2015

$$R^*(\lambda^+ - \lambda^+ C(z))]/T_1(z) \quad (28)$$

$$P(z) = \lambda^+ I_0 A^*(\lambda^+)(C(z)-1)(1-B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z)))/T_1(z) \quad (29)$$

$$R(z) = (\lambda^- + \alpha z) I_0 A^*(\lambda^+)(1-B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z)))(R^*(\lambda^+ - \lambda^+ C(z)) - 1)/T_1(z) \quad (30)$$

where

$$T_1(z) = (z-(C(z)+(1-C(z))A^*(\lambda^+))B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z)))(\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z)) - (C(z)+(1-C(z))A^*(\lambda^+))$$
$$(\lambda^- + \alpha z)(1-B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z)))R^*(\lambda^+ - \lambda^+ C(z))$$

Proof

Integrates I(x, z), P(x, z) and R(x, z) with respect to x under the limit 0 to ∞ we get the result in equations (28), (29) and (30).

Corollary 4.1

The partial probability generating function of the orbit size is

$$P_q(z) = I_0 + I(z) + P(z) + R(z)$$
$$= I_0 A^*(\lambda^+)[\lambda^+(z-1)(1-C(z)) + (z-1)(\lambda^- + \alpha B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z)))]/T_1(z) \quad (31)$$

Corollary 4.2

The partial probability generating function of the system size is

$$P_s(z) = I_0 + I(z) + zP(z) + R(z)$$
$$= I_0 A^*(\lambda^+)[\lambda^+(z-1)(1-C(z))B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z)) + (z-1)(\lambda^- + \alpha B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z)))]/T_1(z) \quad (32)$$

V. PERFORMANCE MEASURES

- The probability that the server is idle during retrial time is given by

$$I = \lim_{z\to 1} I(z)$$
$$= I_0(1-A^*(\lambda^+))[(1-B^*(\lambda^- + \alpha))(\lambda^+ m_1(1+\beta_1(\lambda^- + \alpha)) + \alpha + m_1(\lambda^- + \alpha)) + (\lambda^- + \alpha)(m_1 B^*(\lambda^- + \alpha) - 1)]/T_1'(1) \quad (33)$$

- The probability that the server is busy is given by

$$P = \lim_{z\to 1} P(z) = \lambda^+ m_1 I_0 A^*(\lambda^+)(1-B^*(\lambda^- + \alpha))/T_1'(1) \quad (34)$$

- The probability that the server is down is given by

$$R = \lim_{z\to 1} R(z) = \lambda^+(\lambda^- + \alpha) m_1 \beta_1 I_0 A^*(\lambda^+)(1-B^*(\lambda^- + \alpha))/T_1'(1) \quad (35)$$

where

$$T_1'(1) = (B^*(\lambda^- + \alpha) - 1)[\lambda^+ m_1(1+\beta_1(\lambda^- + \alpha)) + \alpha + m_1(\lambda^- + \alpha)(1-A^*(\lambda^+))] + (\lambda^- + \alpha)(1 - m_1(1-A^*(\lambda^+))B^*(\lambda^- + \alpha))$$

Thenormalising equation (10) is equivalent to $I_0 + I + P + R = 1$

Substituting the expressions of I, P and R, we get

$$I_0 = \frac{(B^*(\lambda^- + \alpha) - 1)(\lambda^+ m_1(1+\beta_1(\lambda^- + \alpha)) + \alpha + m_1(\lambda^- + \alpha)(1-A^*(\lambda^+))) + (\lambda^- + \alpha)(1 - m_1(1-A^*(\lambda^+))B^*(\lambda^- + \alpha))}{A^*(\lambda^+)((\lambda^- + \alpha) + \alpha(B^*(\lambda^- + \alpha) - 1))} \quad (36)$$

- The mean number of customers in the orbit is given by

$$L_q = \lim_{z\to 1} \frac{d}{dz} P_q(z) \quad (37)$$

ISSN(Online) : 2319 - 8753
ISSN (Print) : 2347 - 6710

International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 4, Issue 4, April 2015

$$= \frac{Dr^{'}(1)Nr^{''}(1) - Nr^{'}(1)Dr^{''}(1)}{2Dr^{'}(1)^{2}}$$

where Nr(z) and Dr(z) are the Numerator and Denominator of $P_q(z)$

$$Nr^{'}(1) = I_0 A^{*}(\lambda^{+})((\lambda^{-} + \alpha) - \alpha(1 - B^{*}(\lambda^{-} + \alpha)))$$

$$Nr^{''}(1) = I_0 A^{*}(\lambda^{+})[2\lambda^{+}(\alpha \int_0^{\infty} e^{-(\lambda^{-}+\alpha)x} x b_1(x)dx(m_1) - m_1)]$$

$$Dr^{'}(1) = T_1^{'}(1)$$

$$\begin{aligned} Dr^{''}(1) = {} & (B^{*}(\lambda^{-} + \alpha) - 1)[\lambda^{+} m_2 + (\lambda^{-} + \alpha)\lambda^{+} m_2 \beta_2 + 2\lambda^{+} m_1 \beta_1 \alpha + 2m_1(1 - A^{*}(\lambda^{+})) \\ & ((\lambda^{-} + \alpha)\lambda^{+} m_1 \beta_1 + \alpha) + (\lambda^{-} + \alpha)m_2(1 - A^{*}(\lambda^{+}))] - 2\lambda^{+} m_1(1 - m_1(1 - A^{*}(\lambda^{+}))B^{*}(\lambda^{-} + \alpha)) \\ & - (\lambda^{-} + \alpha)m_2(1 - A^{*}(\lambda^{+}))B^{*}(\lambda^{-} + \alpha) + 2h_1(\lambda^{+} m_1 \beta_1(\lambda^{-} + \alpha) + \alpha + \lambda^{+} m_1) \end{aligned}$$

where

$$h_1 = \lambda^{+} m_1 \int_0^{\infty} e^{-(\lambda^{-}+\alpha)x} x\, b_1(x)dx$$

- The mean number of customers in the system is given by

$$L_S = \lim_{z \to 1} \frac{d}{dz} P_S(z)$$

$$= L_q + P \tag{38}$$

VI. STOCHASTIC DECOMPOSITION

The stochastic decomposition property of the system size distribution is verified. The classical stochastic decomposition property shows that the steady state system size at an arbitrary point can be represented as the sum of two independent random variables, one of which is the system size of the corresponding queueing system without server vacations and the other is the orbit size given that the server is on vacations. Stochastic decomposition has also been held for retrial queues.

Theorem 6.2

The number of customers in the system (L_s) can be expressed as the sum of two independent random variables, one of which is the mean number of customers in batch arrival G-queue with server breakdown (L) and the other is the mean number of customers in the orbit given that the server is idle (L_I).

Proof

The probability generating function Φ(z) of the number of customers in batch arrival G-queue with server breakdown is given by

$$\Phi(z) = I_0[\lambda^{+}(z-1)(1 - C(z))B^{*}(\lambda^{+} + \lambda^{-} + \alpha - \lambda^{+}C(z)) + (z-1)(\lambda^{-} + \alpha B^{*}(\lambda^{+} + \lambda^{-} + \alpha - \lambda^{+}C(z)))]/T_2(z) \tag{39}$$

where

$$T_2(z) = (\lambda^{+} + \lambda^{-} + \alpha - \lambda^{+}C(z))(z - B^{*}(\lambda^{+} + \lambda^{-} + \alpha - \lambda^{+}C(z))) - (1 - B^{*}(\lambda^{+} + \lambda^{-} + \alpha - \lambda^{+}C(z)))(\lambda^{-} + \alpha z)R^{*}(\lambda^{+} - \lambda^{+}C(z))$$

The probability generating function ψ(z) of the number of customers in the orbit given that the server is idle is given by

$$\Psi(z) = \frac{I_0 + I(z)}{I_0 + I(1)}$$

ISSN(Online) : 2319 - 8753
ISSN (Print) : 2347 - 6710

International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 4, Issue 4, April 2015

$$=\frac{[I_0 A^*(\lambda^+)((\lambda^+ +\lambda^- +\alpha-\lambda^+ C(z))(z-B^*(\lambda^+ +\lambda^- +\alpha-\lambda^+ C(z)))-(1-B^*(\lambda^+ +\lambda^- +\alpha-\lambda^+ C(z)))(\lambda^- +\alpha z)R^*(\lambda^+ -\lambda^+ C(z)))][(\lambda^- +\alpha)+\alpha(B^*(\lambda^- +\alpha)-1)]}{T_1(z)[(\lambda^- +\alpha)+(B^*(\lambda^- +\alpha)-1)(\lambda^+ m_1+\alpha+\lambda^+ m_1\beta_1(\lambda^- +\alpha))]} \quad (40)$$

From equations (32), (39) and (40) we get

$$P_s(z)=\Phi(z)\Psi(z)$$

Differentiating $P_s(z)$ with respect to z and taking limit as $z\to 1$, we get $L_s = L + L_I$

VII. RELIABILITY ANALYSIS

Let A(t) be the system availability at time t, that is, the probability that the server is idle or working for a customer. Then under steady state condition, the availability of the server is given by

$$A = I_0+\lim_{z\to 1}\left[\int_0^\infty I(x,z)dx+\int_0^\infty P(x,z)dx\right]$$

$$= I_0+I+P$$

$$=\frac{(\lambda^- +\alpha)+(B^*(\lambda^- +\alpha)-1)(\alpha+\lambda^+ m_1\beta_1(\lambda^- +\alpha))}{(\lambda^- +\alpha)+\alpha(B^*(\lambda^- +\alpha)-1)} \quad (41)$$

The steady state failure frequency of the server is given by

$$W_f=\lambda^- P$$

$$=\frac{\lambda^+\lambda^- m_1(1-B^*(\lambda^- +\alpha))}{(\lambda^- +\alpha)+\alpha(B^*(\lambda^- +\alpha)-1)} \quad (42)$$

Theorem7.1

Let τ be the time to the first failure of the server. Then the Laplace transform of reliability function $\zeta(t) = P(\tau > t)$ of the server is given by

$$\tilde{\zeta}(s)=(1-A^*(s+\lambda^+))+(\lambda^+ +sA^*(s+\lambda^+))\tilde{\tilde{B}}(s+\lambda^- +\alpha)+\tilde{I}_0(s)[(\lambda^+ B^*(s+\lambda^- +\alpha)-(s+\lambda^+))(1-A^*(s+\lambda^+))$$
$$-s(s+\lambda^+)A^*(s+\lambda^+)\tilde{\tilde{B}}(s+\lambda^- +\alpha)]/F(1,s) \quad (43)$$

where

$$F(1,s)=(s+\lambda^+)-(\lambda^+ +sA^*(s+\lambda^+))B^*(s+\lambda^- +\alpha)$$

Proof

Considering failure states of the server as absorbing states, we obtain a new system with the following governing equations.

$$\frac{d}{dt}I_0(t)=-\lambda^+ I_0(t)+\int_0^\infty P_0(x,t)\mu(x)dx, \quad (44)$$

$$\left(\frac{d}{dx}+\frac{d}{dt}\right)I_n(x,t)=-(\lambda^+ +\eta(x))I_n(x,t), \quad n\ge 1 \quad (45)$$

$$\left(\frac{d}{dx}+\frac{d}{dt}\right)P_n(x,t)=-(\lambda^+ +\lambda^- +\alpha+\mu(x))P_n(x,t)+\lambda^+\sum_{k=1}^{n}C_k P_{n-k}(x,t), \; n\ge 0 \quad (46)$$

 DOI: 10.15680/IJIRSET.2015.0404014

ISSN(Online) : 2319 - 8753
ISSN (Print) : 2347 - 6710

International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 4, Issue 4, April 2015

$$I_n(0,t) = \int_0^\infty P_n(x,t)\mu(x)dx, \quad n \geq 1 \tag{47}$$

$$P_0(0,t) = \lambda^+ C_1 I_0(t) + \int_0^\infty I_1(x,t)\eta(x)dx, \tag{48}$$

$$P_n(0,t) = \lambda^+ C_{n+1} I_0(t) + \int_0^\infty I_{n+1}(x,t)\eta(x)dx + \lambda^+ \sum_{k=1}^{n} C_k \int_0^\infty I_{n-k+1}(x,t)dx, \ n \geq 1 \tag{49}$$

Let the initial condition be

$I_n(0) = \delta_{n,0}$, $I_n(x,0) = 0$, $P_n(x,0) = 0$.

Taking Laplace transforms of equations (44) to (49), we obtain

$$(s+\lambda^+)\tilde{I}_0(s) - 1 = \int_0^\infty \tilde{P}_0(x,s)\mu(x)dx \tag{50}$$

$$\frac{d}{dx}\tilde{I}_n(x,s) = -(s+\lambda^+ + \eta(x))\tilde{I}_n(x,s), \ n \geq 1 \tag{51}$$

$$\frac{d}{dx}\tilde{P}_n(x,s) = -(s+\lambda^+ + \lambda^- + \alpha + \mu(x))\tilde{P}_n(x,s) + \lambda^+ \sum_{k=1}^{n} C_k \tilde{P}_{n-k}(x,s), \ n \geq 0 \tag{52}$$

$$\tilde{I}_n(0,s) = \int_0^\infty \tilde{P}_n(x,s)\,\mu(x)dx, \ n \geq 1 \tag{53}$$

$$\tilde{P}_0(0,s) = \lambda^+ C_1 \tilde{I}_0(s) + \int_0^\infty \tilde{I}_1(x,s)\,\eta(x)dx \tag{54}$$

$$\tilde{P}_n(0,s) = \lambda^+ C_{n+1}\tilde{I}_0(s) + \int_0^\infty \tilde{I}_{n+1}(x,s)\,\eta(x)dx + \lambda^+ \sum_{k=1}^{n} C_k \int_0^\infty \tilde{I}_{n-k+1}(x,s)dx, \ n \geq 1 \tag{55}$$

Define the probability generating functions

$$\tilde{I}(z,x,s) = \sum_{n=1}^{\infty} \tilde{I}_n(x,z) z^n \text{ and } \tilde{P}(z,x,s) = \sum_{n=0}^{\infty} \tilde{P}_n(x,s) z^n.$$

Then equations (51) to (55) yield

$$\frac{d}{dx}\tilde{I}(z,x,s) = -(\lambda^+ + s + \eta(x))\tilde{I}(z,x,s) \tag{56}$$

$$\frac{d}{dx}\tilde{P}(z,x,s) = -(s+\lambda^+ + \lambda^- + \alpha + \mu(x) - \lambda^+ C(z))\ \tilde{P}(z,x,s) \tag{57}$$

$$\tilde{I}(z,0,s) = \int_0^\infty \tilde{P}(z,x,s)\,\mu(x)dx - (s+\lambda^+)\tilde{I}_0(s) + 1 \tag{58}$$

$$\tilde{P}(z,0,s) = \frac{\lambda^+ C(z)}{z}\tilde{I}_0(s) + \frac{1}{z}\int_0^\infty \tilde{I}(z,x,s)\eta(x)dx + \frac{\lambda^+ C(z)}{z}\int_0^\infty \tilde{I}(z,x,s)dx \tag{59}$$

The solutions of the partial differential equations (56) and (57) are given by

$$\tilde{I}(z,x,s) = \tilde{I}(z,0,s)e^{-(s+\lambda^+)x}(1-A(x)) \tag{60}$$

$$\tilde{P}(z,x,s) = \tilde{P}(z,0,s)e^{-(s+\lambda^+(1-C(z))+\lambda^- + \alpha)x}(1-B(x)) \tag{61}$$

Using equation (61) in equation (58), we get

$$\tilde{I}(z,0,s) = \tilde{P}(z,0,s)B^*(s+\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z)) - (s+\lambda^+)\tilde{I}_0(s) + 1 \tag{62}$$

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Using equation (60) in equation (59) and simplifying we obtain

$$\tilde{P}(z,0,s) = \frac{\lambda^+ C(z)}{z}\tilde{I}_0(s) + \frac{\tilde{I}(z,0,s)}{z(s+\lambda^+)}(C(z)\lambda^+ + (s+\lambda^+(1-C(z)))A^*(s+\lambda^+)) \tag{63}$$

Solving equation (62) and (63), we get

$$\tilde{I}(z,0,s) = (s+\lambda^+)[z+\tilde{I}_0(s)(\lambda^+ C(z)B^*(s+\lambda^+ +\lambda^- +\alpha-\lambda^+ C(z)) - z(s+\lambda^+))]/F(z,s) \tag{64}$$

$$\tilde{P}(z,0,s) = [C(z)\lambda^+ + (s+\lambda^+(1-C(z)))A^*(s+\lambda^+) - \tilde{I}_0(s)(s+\lambda^+)(s+\lambda^+(1-C(z)))A^*(s+\lambda^+)]/F(z,s) \tag{65}$$

where $F(z,s) = z(s+\lambda^+) - (\lambda^+ C(z) + (s+\lambda^+(1-C(z)))A^*(s+\lambda^+))B^*(s+\lambda^+ +\lambda^- +\alpha-\lambda^+ C(z))$

Substituting the expressions of $\tilde{I}(z,0,s)$ and $\tilde{P}(z,0,s)$ in equations (60) and (61) we get

$$\tilde{I}(z,x,s) = (s+\lambda^+)[z+\tilde{I}_0(s)(\lambda^+ C(z)B^*(s+\lambda^+ +\lambda^- +\alpha-\lambda^+ C(z)) - z(s+\lambda^+))]e^{-(s+\lambda^+)x}(1-A(x))/F(z,s) \tag{66}$$

$$\tilde{P}(z,x,s) = [\lambda^+ C(z) + (s+\lambda^+(1-C(z)))A^*(s+\lambda^+) - \tilde{I}_0(s)(s+\lambda^+)(s+\lambda^+(1-C(z)))A^*(s+\lambda^+)]$$
$$e^{-(s+\lambda^+ +\lambda^- +\alpha-\lambda^+ C(z))x}(1-B(x))/F(z,s) \tag{67}$$

From equations (66) and (67) we can obtain

$$\tilde{I}(z,s) = \int_0^\infty \tilde{I}(z,x,s)dx$$
$$= [z+\tilde{I}_0(s)(\lambda^+ C(z)B^*(s+\lambda^+ +\lambda^- +\alpha-\lambda^+ C(z)) - z(s+\lambda^+))](1-A^*(s+\lambda^+))/F(z,s) \tag{68}$$

$$\tilde{P}(z,s) = \int_0^\infty \tilde{P}(z,x,s)dx$$
$$= [\lambda^+ C(z) + (s+\lambda^+(1-C(z)))A^*(s+\lambda^+) - \tilde{I}_0(s)(s+\lambda^+)(s+\lambda^+(1-C(z)))A^*(s+\lambda^+)]\tilde{\tilde{B}}(s+\lambda^+ +\lambda^- +\alpha-\lambda^+ C(z))/F(z,s) \tag{69}$$

Now $\tilde{\zeta}(s)$ is given by

$$\tilde{\zeta}(s) = \tilde{I}(1,s) + \tilde{P}(1,s)$$

$$\tilde{I}(1,s) = [1+\tilde{I}_0(s)(\lambda^+ B^*(s+\lambda^- +\alpha) - (s+\lambda^+))](1-A^*(s+\lambda^+))/F(1,s) \tag{70}$$

$$\tilde{P}(1,s) = [\lambda^+ + sA^*(s+\lambda^+) - \tilde{I}_0(s)s(s+\lambda^+)A^*(s+\lambda^+)]\tilde{\tilde{B}}(s+\lambda^- +\alpha)/F(1,s) \tag{71}$$

Substituting the expressions of $\tilde{I}(1,s)$ and $\tilde{P}(1,s)$ we get the result in equation (43).

Corollary 7.1

The mean time to the first failure (MTTFF) of the server is given by

$$\text{MTTFF} = (1-A^*(\lambda^+)) + \lambda^+\tilde{\tilde{B}}(\lambda^- +\alpha) + \tilde{I}_0(0)\lambda^+(B^*(\lambda^- +\alpha)-1)(1-A^*(\lambda^+))/\lambda^+(1-B^*(\lambda^- +\alpha)) \tag{72}$$

Proof

$$\text{MTTFF} = \int_0^\infty \zeta(t)dt = \lim_{s\to 0}\tilde{\zeta}(s)$$

Taking limit as $s\to 0$ on both sides of equation (43) we get the result in equation (72).

VIII. SPECIAL CASES

Case (i): No negative customers ($\lambda^- \to 0$)

In this case our model reduces to $M^X/G/1$ retrial with server breakdown for this model,

$P_s(z) = I_0 A^*(\lambda^+)[\lambda^+(z-1)(1-C(z))B^*(\lambda^+ +\alpha-\lambda^+ C(z)) + (z-1)\alpha B^*(\lambda^+ +\alpha-\lambda^+ C(z))]/T_3(z)$ and

ISSN(Online) : 2319 - 8753
ISSN (Print) : 2347 - 6710

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$$I_0 = \lambda^+ m_1(1+\beta_1\alpha)(B^*(\alpha)-1)+\alpha B^*(\alpha)-\alpha m_1(1-A^*(\lambda^+))/A^*(\lambda^+)(\alpha+\alpha(B^*(\alpha)-1))$$

where

$$T_3(z) = (\lambda^+ +\alpha-\lambda^+C(z))(z-(C(z)+(1-C(z))A^*(\lambda^+))B^*(\lambda^+ +\alpha-\lambda^+C(z)))-\alpha z(C(z)+(1-C(z))A^*(\lambda^+))$$
$$(1-B^*(\lambda^+ +\alpha-\lambda^+C(z)))R^*(\lambda^+ -\lambda^+C(z))$$

Case (ii): No retrial ($A^*(\lambda^+)\to 1$)

In this case our model reduces to $M^x/G/1$ G-queue with server breakdown and two types of customers with

$$P_S(z) = I_0[\lambda^+(z-1)(1-C(z))B^*(\lambda^+ +\lambda^- +\alpha-\lambda^+C(z))+(\lambda^- +\alpha)(z-B^*(\lambda^+ +\lambda^- +\alpha-\lambda^+C(z)))-(\lambda^- +\alpha z)$$
$$(1-B^*(\lambda^+ +\lambda^- +\alpha-\lambda^+C(z)))]/T_2(z) \text{ and}$$

$$I_0 = [(B^*(\lambda^- +\alpha)-1)(\lambda^+ m_1(1+\beta_1(\lambda^- +\alpha))+\alpha)+(\lambda^- +\alpha)]/((\lambda^- +\alpha)+\alpha(B^*(\lambda^- +\alpha)-1))$$

IX. NUMERICAL RESULTS

Numerical results are calculated by assuming the distributions of retrial time, service time and repair time as exponential with rates η,μ and β.

For the parameters $\lambda^+ = 1$, $\lambda^- = 0.5$, $\alpha = 0.7$, $\mu = 6$, $\beta = 3$, $\eta = 40$, $c_1 = 0.5$, $c_2 = 0.5$, the performance measures I_0-the probability that the system is empty, I-the probability that the server is idle in non-empty system, P-the probability that the server is busy, R-the probability that the server is under repair, A-the availability of the server, F-the failure frequency of the server and Ls-the mean number of customers in the system are calculated and presented in tables 3.1 to 3.6 respectively for various rates of λ^+, λ^-, α, η, μ and β.

Table 3.1 reveals that I_0 and A monotonically decrease and I, P, R, F and Ls increase as λ^+ increases. Table 3.2 indicates that increase in λ^- decreases I_0, P, A and Ls and increases other performance measures. From Table 3.3 and 3.6 we observe that the parameters α and β have no effect on the performance measures P and F. For increasing values of α, the values of I, R and Ls increase and I_0 and A decrease. As β increases I, R and Ls decrease I_0 and A increase. From Table 3.4 we observe that the parameter η has no effect on the performance measures P, A and F. For increasing values of η, I_0 increases and I and Ls decrease. Table 3.5 depicts the effect of μ on the performance measures. It is noted that I_0 and A increase and I, P, R, F and Ls decrease for increasing values of μ.

Table 9.1 Performance measures by varying λ^+

λ^+	I_0	I	P	R	A	F	Ls
0.5000	0.8282	0.0103	0.1154	0.0462	0.9538	0.0577	0.2173
1.0000	0.6523	0.0246	0.2308	0.0923	0.9077	0.1154	0.6331
1.5000	0.4724	0.0430	0.3462	0.1385	0.8615	0.1731	1.4160
2.0000	0.2885	0.0654	0.4615	0.1846	0.8154	0.2308	3.1777
2.5000	0.1005	0.0918	0.5769	0.2308	0.7692	0.2885	11.1806

Table 9.2 Performance measures by varying λ^-

λ^-	I_0	I	P	R	A	F	Ls
1.0000	0.6396	0.0246	0.2143	0.1214	0.8786	0.2143	0.6157
2.0000	0.6191	0.0247	0.1875	0.1687	0.8313	0.3750	0.5859
3.0000	0.6031	0.0247	0.1667	0.2056	0.7944	0.5000	0.5614
4.0000	0.5903	0.0248	0.1500	0.2350	0.7650	0.6000	0.5408
5.0000	0.5798	0.0248	0.1364	0.2591	0.7409	0.6818	0.5233

ISSN(Online) : 2319 - 8753
ISSN (Print) : 2347 - 6710

International Journal of Innovative Research in Science, Engineering and Technology

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Table 9.3 Performance measures by varying α

α	I_0	I	P	R	A	F	Ls
1.0000	0.6269	0.0269	0.2308	0.1154	0.8846	0.1154	0.7099
2.0000	0.5423	0.0346	0.2308	0.1923	0.8077	0.1154	1.0178
3.0000	0.4577	0.0423	0.2308	0.2692	0.7308	0.1154	1.4397
4.0000	0.3731	0.0500	0.2308	0.3462	0.6538	0.1154	2.0528
5.0000	0.2885	0.0577	0.2308	0.4231	0.5769	0.1154	3.0257

Table 9.4 Performance measures by varying η

η	I_0	I	P	R	A	F	Ls
2.0000	0.1846	0.4923	0.2308	0.0923	0.9077	0.1154	5.4167
4.0000	0.4308	0.2462	0.2308	0.0923	0.9077	0.1154	1.6042
6.0000	0.5128	0.1641	0.2308	0.0923	0.9077	0.1154	1.1467
8.0000	0.5538	0.1231	0.2308	0.0923	0.9077	0.1154	0.9688
10.0000	0.5785	0.0985	0.2308	0.0923	0.9077	0.1154	0.8741

Table 9.5 Performance measures by varying μ

μ	I_0	I	P	R	A	F	Ls
2.0000	0.1160	0.0440	0.6000	0.2400	0.7600	0.3000	9.0833
4.0000	0.5033	0.0300	0.3333	0.1333	0.8667	0.1667	1.1740
6.0000	0.6523	0.0246	0.2308	0.0923	0.9077	0.1154	0.6331
8.0000	0.7312	0.0218	0.1765	0.0706	0.9294	0.0882	0.4359
10.0000	0.7800	0.0200	0.1429	0.0571	0.9429	0.0714	0.3338

Table 9.6 Performance measures by varying β

β	I_0	I	P	R	A	F	Ls
1.0000	0.4631	0.0292	0.2308	0.2769	0.7231	0.1154	1.5083
2.0000	0.6050	0.0258	0.2308	0.1385	0.8615	0.1154	0.7745
3.0000	0.6523	0.0246	0.2308	0.0923	0.9077	0.1154	0.6331
4.0000	0.6760	0.0240	0.2308	0.0692	0.9308	0.1154	0.5756
5.0000	0.6902	0.0237	0.2308	0.0554	0.9446	0.1154	0.5449

X. CONCLUSION

Batch arrival retrial G queue with server breakdown is analysed. The models considered are investigated using supplementary variable technique to obtain performance measures and reliability indices. Stochastic decomposition law is verified and special cases are discussed. Numerical analysis are carried out to analyse the effect of parameters on the system performance.

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