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Transient Couette Flow of a Dusty Fluid between Parallel Porous Plates with the Ion slip Effect and Heat Transfer under Exponential Decaying Pressure Gradient

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ABSTRACT: In the present paper, the transient Magnetohydrodynamic (MHD) Couette flow of a dusty viscous incompressible electrically conducting fluid under the influence of an exponential decaying pressure gradient is investigated with heat transfer and the ion slip effect is considering. The parallel plates are porous and subjected to a uniform suction from above and injection from below while an axial uniform magnetic field is applied perpendicular to the plates. The governing momentum and energy equations are solved numerically using the method of finite difference. The effect of the different physical parameters of the model on the velocity and temperature fields of both fluid and dust particles are reported.

KEYWORDS: Unsteady Flow, MHD, Couette flow, Ion slip, Dusty Fluid.

I. INTRODUCTION

The unsteady flow of dusty electrically conducting fluids through channels in the presence of an external transverse magnetic field has important applications of many engineering problems concerned with dust collection nuclear reactor cooling, sedimentation, atmospheric fallout, powder technology, environmental pollution, performance of solid fuel rock nozzles, filtration, and geothermal systems, ...etc [1,2]. The flow of dusty fluids was examined by many researchers. Saffman [3] formulated a model of dusty fluid under the simplifying assumptions with considering the effect of dust particles on the stability of laminar flow. The unsteady flow with heat transfer of a viscous, incompressible dusty fluid between two infinite horizontal parallel porous plates was studied by Attia et al. [4]. Dubey et al. [5], discussed the effect of dusty viscous fluid on unsteady laminar free convection flow through porous medium along a moving porous hot vertical plate in the presence of heat source and thermal diffusion with mass transfer. Gireesha et al. [6], discussed the laminar flow of an incompressible viscous dusty fluid with uniform distribution of dust particles in porous medium between horizontal plate and a long wavy wall.

Many research papers have investigated the magnetohydrodynamic (MHD) flow of dusty/non-dusty fluids under the influence of an external magnetic field. Where the Hall and ion slip terms were considered in applying Ohm's law. Chamkha [7] obtained exact solutions for the transient hydrodynamic dusty gas flow in a circular pipe with inclusion of a uniform magnetic field and considering the viscous effects of the particle-phase. Attia et al. [8], studied the effects of Hall current, and ion slip on unsteady magnetohydrodynamic Couette flow with heat transfer of a dusty viscous incompressible electrically conducting fluid with Joule heating considered. The influence of the Hall current on unsteady magneto-hydrodynamic viscous flow with heat transfer in a Darcy porous medium between two infinite

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parallel porous plates considering the temperature dependent physical properties was studied [9-12]. Hatami et al. [13], used the Differential Quadrature numerical method (DQM) to solve the unsteady magneto-hydrodynamic (MHD) Couette flows between two parallel infinite plates. Attia [14] studied the influence of the Hall current and ion slip on the MHD flow of a dusty fluid with heat transfer under exponential decaying pressure gradient. Other some authors investigated the flow of non-Newtonian of dusty fluids under some physical effects [15-17].

In the current paper, transient Couette flow with heat transfer of an electrically conducting, viscous, incompressible, dusty fluid is studied, both the Hall current and ion slip is considering under an exponential decaying pressure gradient. The particle phase is assumed to incompressible, non-conducting and pressureless. The governing momentum and energy equations including the viscous and Joule dissipation terms are solved numerically using the method of finite differences. The influence of the magnetic field, the Hall current, and the ion slip on both the velocity and temperature fields of the fluid and particle phases are discussed.

II. GOVERNING EQUATIONS

Consider the dusty fluid flows between two infinite horizontal porous plates located at the $y = \pm h$ planes where the upper one is moving with a constant velocity U_0 and the lower is kept stationary. A uniform suction from above and a uniform injection from below are applied perpendicular to the plates where the y-component of the velocity of the fluid is assumed constant and denoted by v_0 . An external uniform magnetic field B_0 is applied in the positive y-direction. The dust particles are assumed to be non-conducting, spherical in shape and distributed uniformly throughout the fluid and to be big enough, so that they are not pumped out through the holes of the porous plates with zero velocity in y-direction. The two plates are non-conducting and kept at two constant temperatures T_1 for the lower plate and T_2 for the upper plate with $T_2 > T_1$. An exponential decaying uniform pressure gradient $dp/dx = -Ce^{-at}$ is applied in the x-direction which together with the movement of the upper plate representing the driving force, where G , and a are constants. The fluid motion starts impulsively from rest at $t=0$, and the no-slip condition at the plates implies that the fluid and dust particles velocities have neither a z nor an x-component at $y = \pm h$. The initial temperatures of the fluid and dusty particles are assumed to be equal to the temperature of the lower plate T_1 . Due to the infinite dimensions in the x and z-directions, the physical quantities do not vary in these directions and the problem is one-dimensional.

The Velocity and Temperature Distributions

The momentum equation:

$$\rho \frac{Dv}{Dt} = -\nabla P + \mu \nabla^2 v + JxB_0 - KN(v - v_p) \quad (1)$$

where, ρ is the density of clean fluid, μ is the viscosity of clean fluid, v is the velocity vector of the fluid, $v = u(y, t)\mathbf{i} + v\mathbf{j} + w(y, t)\mathbf{k}$, v_p is the velocity of dust particles, $v_p = up(y, t)\mathbf{i} + wp(y, t)\mathbf{k}$, J is the current density, N is the number of dust particles per unit volume, K is the Stokes constant $=6\pi\mu a$, and a is the average radius of dust particles.

The first three terms in the right side of Eq. (1) are, respectively, the pressure gradient, viscosity, and Lorentz force terms. The last term represents the force due to the relative motion between fluid and dust particles. Considering the Hall and ion slip terms, the current density J from the generalized Ohm's law is given by [18]:

$$J = \sigma(E + VxB_0 - \beta(JxB_0) + \frac{\beta Bi}{B_0}(JxB_0)xB_0) \quad (2)$$

where, σ is the electric conductivity of the fluid, β is the Hall factor and Bi is the ion slip parameter. Also, in the present study magnetic field is applied in one direction, ($B = B_0\mathbf{j}$). Under the assumptions made in the present problem, Eqs. (1) and (2) become:

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$$\rho \frac{\partial u}{\partial t} + \rho v_o \frac{\partial u}{\partial y} = -\frac{dP}{dx} + \mu \frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_o^2}{(1+BiBe)^2 + Be^2} ((1+BiBe)u + Bew) - KN(u - u_p) \quad (3)$$

$$\rho \frac{\partial w}{\partial t} + \rho v_o \frac{\partial w}{\partial y} = \mu \frac{\partial^2 w}{\partial y^2} - \frac{\sigma B_o^2}{(1+BiBe)^2 + Be^2} ((1+BiBe)w - Bew) - KN(w - w_p) \quad (4)$$

where, $Be = \sigma B_o$, is the Hall parameter, The motion of dust particles is governed by Newton's second law applied in the x and z -directions:

$$m_p \frac{\partial u_p}{\partial t} = KN(u - u_p) \quad (5)$$

$$m_p \frac{\partial w_p}{\partial t} = KN(w - w_p) \quad (6)$$

Where, m_p is the average mass of dust particles. The initial conditions are given as:

$$t \leq 0 : u = u_p = w = w_p = 0 \quad (7a)$$

$$t > 0, y = -h : u = u_p = w = w_p = 0 \quad (7b)$$

$$t > 0, y = h : u = U_o, u_p = w = w_p = 0 \quad (7c)$$

The energy equations describing the temperature distributions for both the fluid and dust particles give [19]:

$$\rho c \frac{\partial T}{\partial t} + \rho c v_o \frac{\partial T}{\partial y} = k \frac{\partial^2 T}{\partial y^2} + \mu \left(\left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right) + \frac{\sigma(1+BiBe)B_o^2}{(1+BiBe)^2 + Be^2} (u^2 + w^2) + \frac{\rho_p c_s}{\gamma_T} (T_p - T) \quad (8)$$

$$\frac{\partial T_p}{\partial t} = -\frac{1}{\gamma_T} (T_p - T), \quad (9)$$

where, T and T_p are, respectively, the temperature of the fluid and the dust particles, c is the specific heat capacity of the fluid at constant volume, k is the thermal conductivity of the fluid, ρ_p is the mass of dust particles per unit volume of the fluid, γ_T is the temperature relaxation time, and c_s is the specific heat capacity of the particles. The last three terms on the right side of Eq. (8) represent the viscous dissipation, the Joule dissipation, and the heat conduction between the fluid and dust particles respectively. The temperature relaxation time depends, in general, on the geometry, and since the dust particles are assumed to be spherical in shape, the last term in Eq. (8) is equal to $4\pi a N k (T_p - T)$ [19]. Then:

$$\gamma_T = \frac{3Pr \gamma_p c_s}{2c} \quad (10)$$

Where, γ_p is the velocity relaxation time $= 2\rho_s a^2 / 9\mu$, Pr is the Prandtl number $= \mu c / k$, and ρ_s is the material density of dust particles $= 3\rho_p / 4\pi a^3 N$. T_p must satisfy the initial and boundary conditions:

$$t \leq 0 : T = T_p = T_1, \quad (11a)$$

$$t > 0, y = -h : T = T_p = T_1, \quad (11b)$$

$$t > 0, y = h : T = T_p = T_2. \quad (11c)$$

Introducing the following dimensionless variables and parameters:

$$(\hat{x}, \hat{y}) = \frac{(x, y)}{h}, \hat{t} = \frac{t U_o}{h}, (\hat{u}, \hat{w}) = \frac{(u, w)}{U_o}, (\hat{u}_p, \hat{w}_p) = \frac{(u_p, w_p)}{U_o}, \hat{P} = \frac{P}{\rho U_o^2}, \hat{T} = \frac{T - T_1}{T_2 - T_1}, \hat{T}_p = \frac{T_p - T_1}{T_2 - T_1}$$

$S = v_o / U_o$, is the suction parameter, $Ha = B_o h \sqrt{\sigma / \mu}$, is the Hartmann number,

$Re = U_o \rho h / \mu$, is the Reynolds number, $E_c = U_o^2 / c(T_2 - T_1)$, is the Eckert number,

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$G = m_p \mu / \rho h^2 K$, is the particle mass parameter, $R = KNh^2 / \mu$, is the particle concentration parameter,

$L_o = \rho h^2 / \mu \gamma_T$, is the temperature relaxation time parameter.

Equations (4)-(11) lead to (the hats are dropped for simplicity):

$$\frac{\partial u}{\partial t} + S \frac{\partial u}{\partial y} = C e^{-\alpha} + \frac{1}{Re} \frac{\partial^2 u}{\partial y^2} - \frac{H_a^2}{Re((1+BiBe)^2 + Be^2)} ((1+BiBe)u + Bew) - \frac{R}{Re} (u - u_p) \quad (12)$$

$$\frac{\partial w}{\partial t} + S \frac{\partial w}{\partial y} = \frac{1}{Re} \frac{\partial^2 w}{\partial y^2} - \frac{H_a^2}{Re((1+BiBe)^2 + Be^2)} ((1+BiBe)w - Beu) - \frac{R}{Re} (w - w_p) \quad (13)$$

$$G \frac{\partial u_p}{\partial t} = u - u_p \quad (14)$$

$$G \frac{\partial w_p}{\partial t} = w - w_p \quad (15)$$

$$t \leq 0: u = u_p = w = w_p = 0 \quad (16a)$$

$$t > 0, y = -h: u = u_p = w = w_p = 0 \quad (16b)$$

$$t > 0, y = h: u = 1, u_p = w = w_p = 0 \quad (16c)$$

$$\frac{\partial T}{\partial t} + S \frac{\partial T}{\partial y} = \frac{1}{Re Pr} \frac{\partial^2 T}{\partial y^2} + \frac{E_c}{Re} \left(\left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right) + \frac{(1+BiBe)H_a^2 E_c}{Re((1+BiBe)^2 + Be^2)} (u^2 + w^2) + \frac{2R}{3Pr} (T_p - T) \quad (17)$$

$$\frac{\partial T_p}{\partial t} = -L_o (T_p - T), \quad (18)$$

$$t \leq 0: T = T_p = 0, \quad (19a)$$

$$t > 0, y = -1: T = T_p = 0, \quad (19b)$$

$$t > 0, y = 1: T = T_p = 1. \quad (19c)$$

III. RESULTS AND DISCUSSION

Equations (12)-(19) represent a system of partial differential equations which can be solved numerically using the finite difference method. The Crank-Nicolson implicit method[20] is used at two successive time levels and the finite difference equations relating the variables are obtained by writing the equations at the mid point of the computational cell and then replacing the different terms by their second order central difference approximation in the y-direction. The diffusion terms are replaced by the average of the central differences at two successive time-levels. Finally, the resulting block tri-diagonal system is solved using the generalized Thomas-algorithm [20]. Computations

are made for $C=-5$, $\alpha=1$, $Re=1$, $R=0.5$, $G=0.8$, $L_o=0.7$, $Pr=1$, and $E_c=0.2$.

Figures 1-3 present the profiles of the velocity components u , u_p , w , and w_p , and temperatures T and T_p , respectively for various values of time t and for $Ha=1$, $Be=3$, $Bi=3$, and $S=1$. It is clear from Figs. 1a, 2a, and 3a that the velocity component u reaches the steady state faster than w which, in turn, reaches the steady state faster than T , which is expected, as u is the source of w , and both u and w are sources for the temperature. The same finding is apparent in Figs. 1b, 2b, and 3b for w , w_p and T_p . Comparing Figs. 1a, 2a and 3a with 1b, 2b and 3b, respectively, shows that the velocity components and temperature of the fluid phase reach the steady state faster than that of the particle phase which can be attributed to the fact that the velocity of fluid is the source for the velocity of the dust particles.

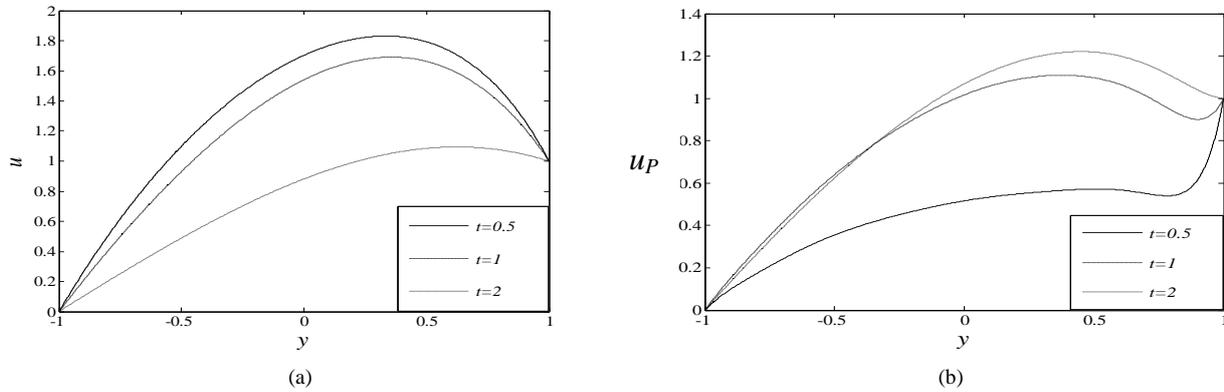


Fig. 1 Time variation of fluid and dust velocity components u, u_p : (a) u ; (b) u_p . ($Ha=1, Be=3, Bi=3$ and $S=1$)

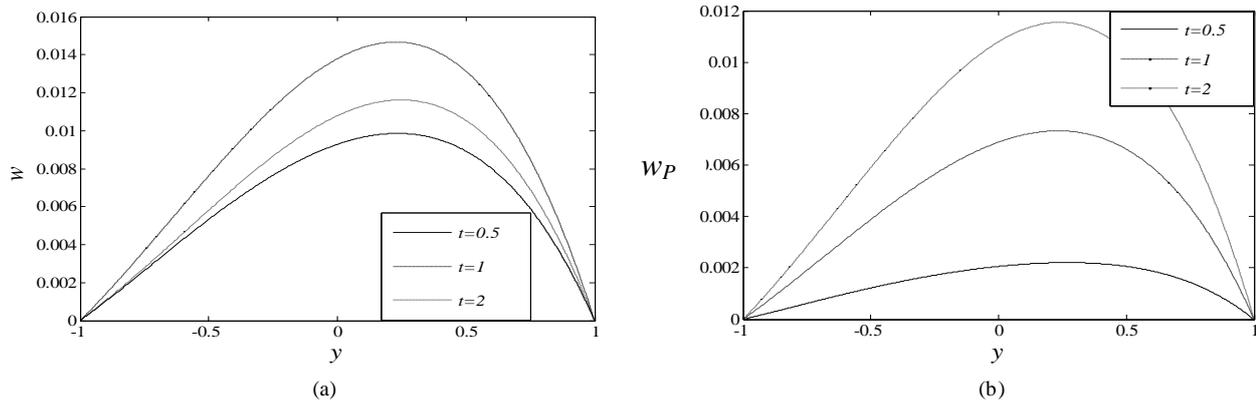


Fig. 2 Time variation of fluid and dust velocity components w, w_p : (a) w ; (b) w_p . ($Ha=1, Be=3, Bi=3$ and $S=1$)

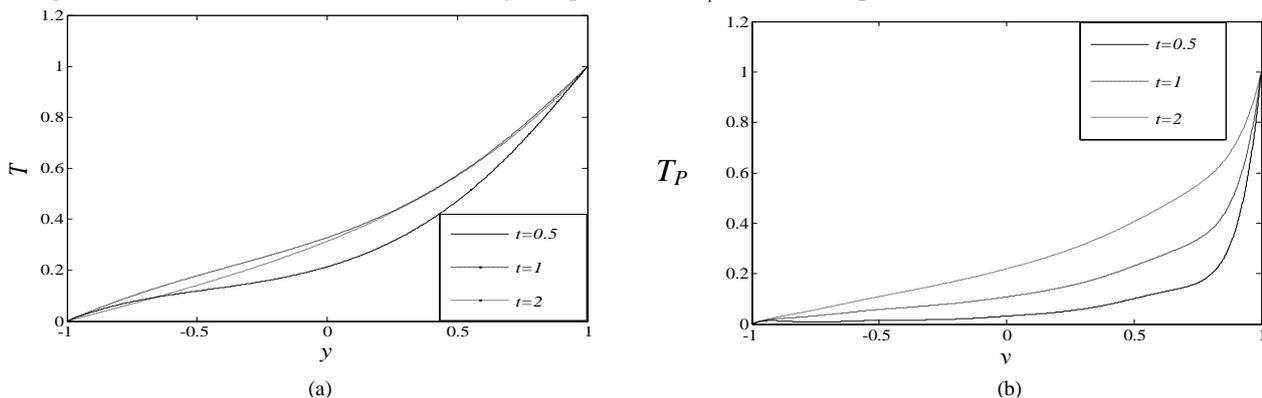


Fig. 3 Time variation of fluid and dust temperatures T and T_p : (a) T ; (b) T_p . ($Ha=1, Be=3, Bi=3$ and $S=1$)

Figures 4-6 present the time progress of the velocity components and temperature at the centre of the channel ($y=0$), respectively, for the fluid and particle phases for various values of the Hall parameter Be and the ion slip parameter Bi and for $Ha=1$ and $S=0$. It is indicated from Figs. 4a and 4b that increasing the parameter Be or Bi increases the velocities u and u_p . This is due to the effective conductivity ($\sigma/((1+BeBi)^2+Be^2)$) which decreases with increasing Be or Bi and, in turn reduces the magnetic resistive force on u . It is shown in Figs. 5a and 5b, that the velocity components w and w_p decrease with increasing Be or Bi as a result of decreasing the source term of w while increasing its damping term. Figures 6a and 6b depict that, unless Bi is large, increasing Be or Bi decreases the

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temperatures T and T_p since an increment in Be or Bi decreases the Joule dissipation which is proportional to $(1/((1+BeBi)^2+Be^2))$. On the other hand, for large values of Bi , increasing Be decreases T and T_p at small time and increases when time develops. This is because, for small times, u and w are small and an increment in Be or Bi decreases the Joule dissipation which is also proportional to $(1/((1+BeBi)^2+Be^2))$. For large time, increasing Be increases both u and w and, in turn, increases the Joule and viscous dissipations. For large time, increasing Bi , although it decreases w , it increases the velocity u of the main flow and subsequently increases the viscous and Joule dissipations. This results in crossing the curves of T with time for higher values of Be and Bi . The same observations are indicated in Fig. 6b. Comparing Figs. 6a and 6b shows that the temperature of the fluid reaches the steady state faster than the temperature of the particles.

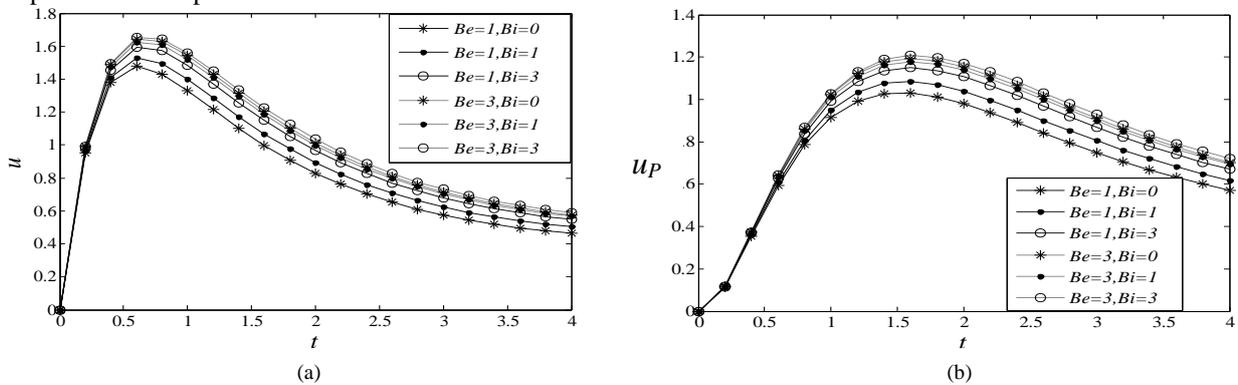


Fig. 4 Effect of the parameters Be and Bi on fluid and dust velocity components u, u_p at $y=0$: (a) u ; (b) u_p .

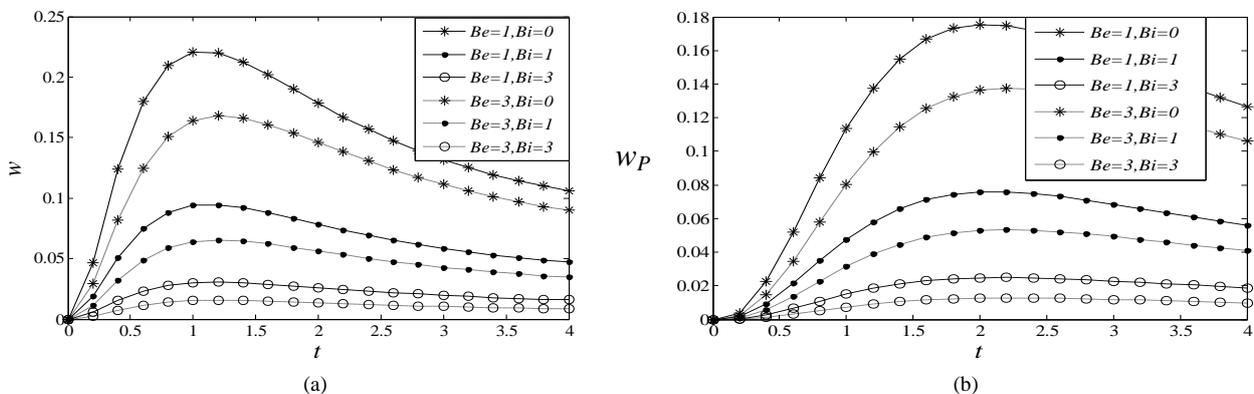


Fig. 5 Effect of the parameters Be and Bi on fluid and dust velocity components w, w_p at $y=0$: (a) w ; (b) w_p .

Figures 7-9 show the evolution of the velocity components and temperature at the centre of the channel ($y=0$) with time, respectively, for the fluid and particle phases for various values of the Hartmann number Ha and the ion slip parameter Bi and for $Be=3$ and $S=0$. Figures 7a and 7b indicate that the effect of Bi on u and u_p depends on Ha . For small values of Ha increasing Bi decreases u and u_p as a result of increasing the damping force on u which is proportional to Bi . Increasing Bi more increases the effective conductivity and, in turn, decreases the damping force on u which increases u and u_p . On the other hand, for larger values of Ha , u becomes small, and increasing Bi reduces the effective conductivity and therefore increases u and u_p . The effect of Bi on u and u_p becomes more pronounced for higher values of Ha . Figures 8a and 8b indicate that increasing the ion slip parameter Bi decreases w and w_p for all values of Ha and that its effect is more clear for higher values of Ha . An overshooting in the velocity components u and w for higher values of Ha and Bi is observed which is interesting. Figures 9a and 9b indicate that the influence of Bi on T and T_p becomes more apparent for higher values of Ha . Increasing Bi decreases T and T_p because of its influence in reducing the Joule dissipation.

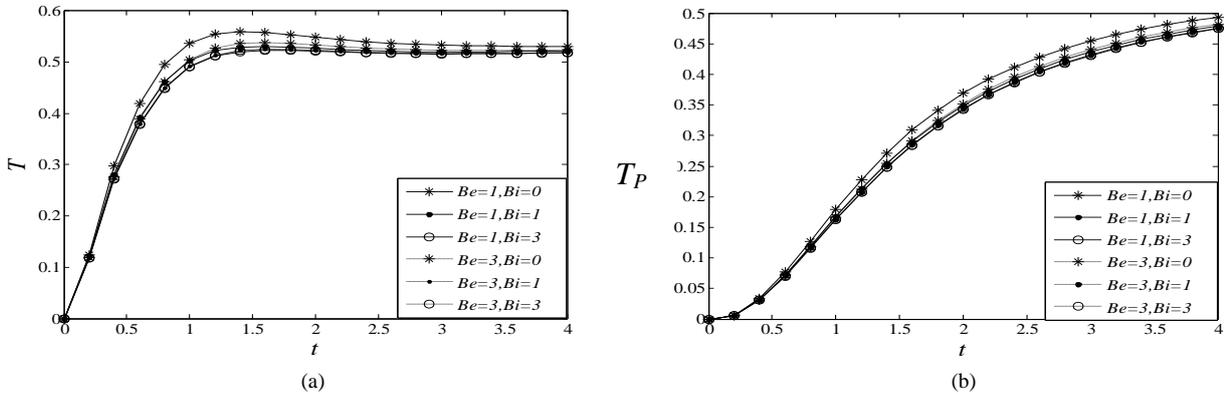


Fig. 6 Effect of the parameters Be and Bi on fluid and dust temperatures T and T_p : (a) T ; (b) T_p .

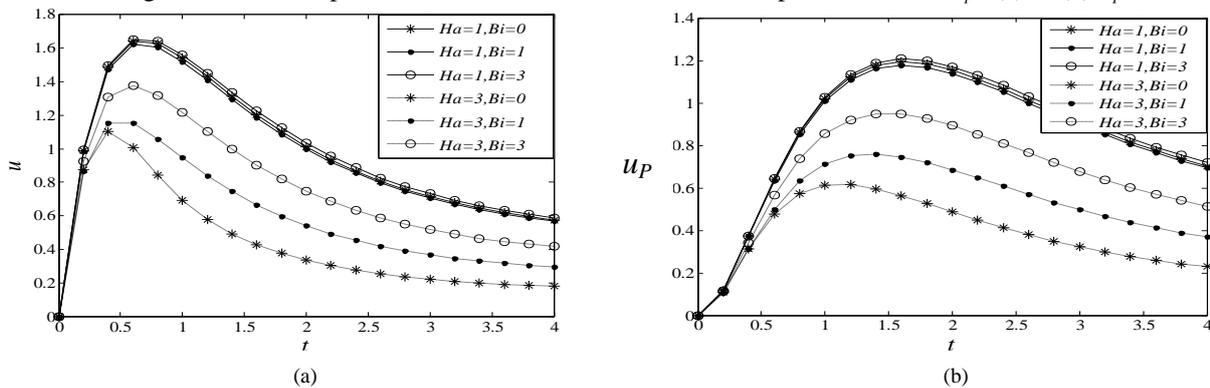


Fig. 7 Effect of the parameters Ha and Bi on fluid and dust velocity components u , u_p at $y=0$: (a) u ; (b) u_p .

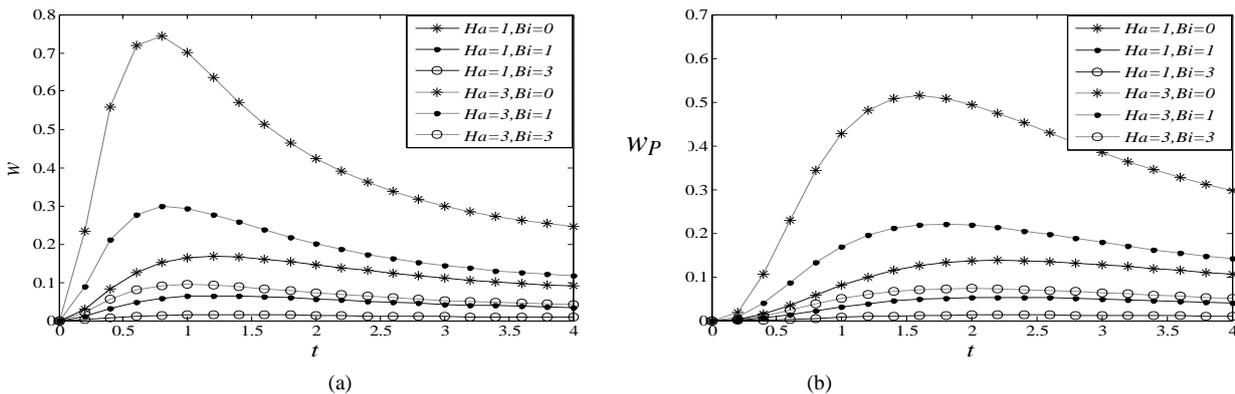


Fig. 8 Effect of the parameters Ha and Bi on fluid and dust velocity components w , w_p at $y=0$: (a) w ; (b) w_p .

IV. CONCLUSION

The transient Couette flow with heat transfer of a dusty conducting fluid under the influence of an applied uniform magnetic field was examined taking the Hall and ion slip effects into consideration and in the presence of uniform suction and injection. The effect of the magnetic field, the Hall parameter, and the ion slip parameter on the velocity and temperature distributions for both the fluid and particle phases was discussed. It was detected that increasing the ion slip parameter increases the velocity components u and u_p , but decreases the velocity components w and w_p and temperatures T and T_p . It is interesting to observe that, for large value of the Hall parameter, the variation of T and T_p

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with the ion slip parameter depends on time. The influence of the Hall current on all the velocity and temperature distributions depends on the ion slip. The effect of the ion slip on the velocity components and temperatures of the fluid and dust particles becomes more apparent for higher values of Ha as expected.

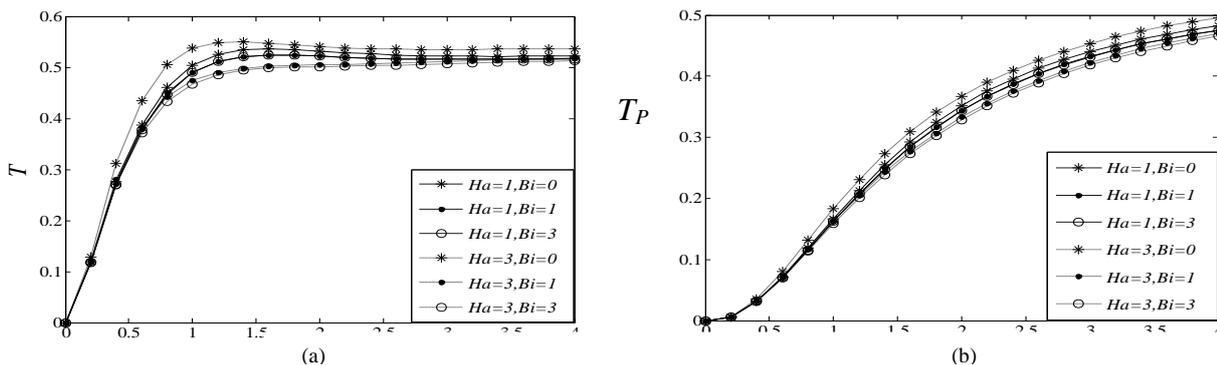


Fig. 9 Effect of the parameters Ha and Bi on fluid and dust temperatures T and T_p : (a) T ; (b) T_p .

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