

A Common Fixed Point Theorem in Complex Valued Metric Space for Four Mappings

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ABSTRACT: In this paper we prove a common fixed point theorem for four mappings in a complex valued metric space.

KEYWORDS: Complex valued metric space, Property (E.A), weakly compatible mappings, Property (CLR g). AMS Subject Classification (2010): 47H10, 54H25.

I. INTRODUCTION

In 2011, A. Azam, B. Fisher and M. Khan [2] introduced the notion of complex valued metric space and established some sufficient conditions for existence of common fixed points of a pair of mappings satisfying a contractive condition. S. Bhatt, S. Chaukiyal and R.C. Dimri [3] have proved a theorem on common fixed point for weakly compatible mappings in a complex valued metric space. R.K. Verma and H.K. Pathak [8] introduced the concept of the property (E.A) in complex valued metric space to prove some common fixed point results for two pairs of weakly compatible mappings. The aim of this paper is to obtain a common fixed point theorem under a contractive condition of two pairs of weakly compatible self-maps with CLR g property in complex valued metric spaces.

II. PRILIMINARIES

Let \mathbb{C} be the set of all complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order relation \preceq on \mathbb{C} as follows:

$$z_1 \preceq z_2 \text{ if and only if } \operatorname{Re}(z_1) \leq \operatorname{Re}(z_2) \text{ and } \operatorname{Im}(z_1) \leq \operatorname{Im}(z_2).$$

Thus $z_1 \preceq z_2$ if one of the followings holds:

- (1) $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) = \operatorname{Im}(z_2)$,
- (2) $\operatorname{Re}(z_1) < \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) = \operatorname{Im}(z_2)$,
- (3) $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) < \operatorname{Im}(z_2)$ and
- (4) $\operatorname{Re}(z_1) < \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) < \operatorname{Im}(z_2)$.

We write $z_1 \prec z_2$ if $z_1 \preceq z_2$ and $z_1 \neq z_2$ i.e., one of (2), (3) and (4) is satisfied and we will write $z_1 < z_2$ if only (4) is satisfied.

Remark 1: We can easily check the followings:

- (i) $a, b \in \mathbb{R}, a \leq b \Rightarrow az \preceq bz \forall z \in \mathbb{C}$.
- (ii) $0 \preceq z_1 \preceq z_2 \Rightarrow |z_1| < |z_2|$.
- (iii) $z_1 \preceq z_2$ and $z_2 < z_3 \Rightarrow z_1 < z_3$.

Azam et al. [2] defined the complex valued metric space in the following way:

Definition 1 ([2]): Let X be a nonempty set. Suppose that the mapping $d : X \times X \rightarrow \mathbb{C}$ satisfies the following conditions:

- (C1) $0 \preceq d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
- (C2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (C3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

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Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.

Example 1: Let $X = \mathbb{C}$. Define the mapping $d : X \times X \rightarrow \mathbb{C}$ by

$$d(z_1, z_2) = i|z_1 - z_2| \quad \forall z_1, z_2 \in \mathbb{C}.$$

One can easily verify that (X, d) is a complex valued metric space.

Definition 2 ([2]): Let (X, d) be a complex valued metric space. Then

- (i) A point $x \in X$ is called an interior point of a set $A \subseteq X$ if there exists $0 < r \in \mathbb{C}$ such that
- $$B(x, r) = \{y \in X : d(x, y) < r\} \subseteq A.$$

A subset $A \subseteq X$ is called open if each element of A is an interior point of A .

- (ii) A point $x \in X$ is called a limit point of $A \subseteq X$ if for every $0 < r \in \mathbb{C}$,
- $$B(x, r) \cap (A - \{x\}) \neq \phi.$$

A subset $A \subseteq X$ is called closed if each element of $X - A$ is not a limit point of A .

- (iii) The family

$$F = \{B(x, r) : x \in X, 0 < r\}$$

is a sub-basis for a Hausdorff topology τ on X .

Definition 3 ([2]): Let (X, d) be a complex valued metric space. Then

- (i) A sequence $\{x_n\}$ in X is said to converge to $x \in X$ if for every $0 < r \in \mathbb{C}$ there exists $N \in \mathbb{N}$ such that $d(x_n, x) < r, \forall n > N$. We denote this by $\lim_{n \rightarrow \infty} x_n = x$ or $x_n \rightarrow x$ as $n \rightarrow \infty$.
- (ii) If for every $0 < r \in \mathbb{C}$ there exists $N \in \mathbb{N}$ such that $d(x_n, x_{n+m}) < r$ for all $n > N, m \in \mathbb{N}$, then $\{x_n\}$ is called a Cauchy sequence in (X, d) .
- (iii) If every Cauchy sequence in X is convergent in X then (X, d) is called a complete complex valued metric space.

Jungck introduced the concepts of commuting mappings in [4], compatible mappings in [5] and weakly compatible mappings in [6]. In a like manner we may define these concepts in complex valued metric spaces as follows:

Definition 4 ([4]): Let (X, d) be a complex valued metric space. The self-maps S and T are said to be commuting if $STx = TSx$ for all $x \in X$.

Definition 5 ([5]): Let (X, d) be a complex valued metric space. The self-maps S and T are said to be compatible if $\lim_{n \rightarrow \infty} d(STx_n, TSx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = t$ for some $t \in X$.

Definition 6 ([6]): Let (X, d) be a complex valued metric space. The self-maps S and T are said to be weakly compatible if $STx = TSx$ whenever $Sx = Tx$, i.e., they commute at their coincidence points.

Definition 7 ([8]): Let (X, d) be a complex valued metric space. The self-maps S and T are said to satisfy the property (E.A) if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = t \quad \text{for some } t \in X.$$

Definition 8 ([10]): Let (X, d) be a metric space. The self-maps f and g are said to satisfy the CLR g property if there exists a sequence $\{x_n\}$ in X such that $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = gu$, for some $u \in X$.

In a similar manner in a complex valued metric space (X, d) , two self-maps f and g are said to satisfy the CLR g property if there exists a sequence $\{x_n\}$ in X such that $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = gu$, for some $u \in X$.

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III. LEMMAS

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 ([2]): Let (X, d) be a complex valued metric space and $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ converges to $x \in X$ if and only if $|d(x_n, x)| \rightarrow 0$ as $n \rightarrow \infty$.

Lemma 2 ([2]): Let (X, d) be a complex valued metric space and $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \rightarrow 0$ as $n \rightarrow \infty$ where $m \in \mathbb{N}$.

Lemma 3([9]): Let (X, d) be a complex valued metric space and $\{x_n\}$ be a sequence in X such that $\lim_{n \rightarrow \infty} x_n = x$. Then for any $a \in X$, $\lim_{n \rightarrow \infty} d(x_n, a) = d(x, a)$.

IV. MAIN RESULT

In this section we present the main result of this paper.

Theorem: Let (X, d) be a complex valued metric space. Suppose that S, T, f and g be self-maps of X satisfying the following conditions:

- (i) The pairs $\{S, f\}$ and $\{T, g\}$ are weakly compatible,
- (ii) $TX \subseteq fX$ and $SX \subseteq gX$,
- (iii) Either the pair $\{S, f\}$ satisfy CLR_f property or the pair $\{T, g\}$ satisfy CLR_g property and
- (iv) $d(Sx, Ty) \lesssim \lambda d(fx, gy) + \frac{\mu d(fx, Sx)d(gy, Ty)}{1+d(fx, gy)}$, $\forall x, y \in X$, where λ, μ are non-negative reals satisfy $\lambda + \mu < 1$.

Then S, T, f and g have a unique common fixed point.

Proof: Suppose that the pair $\{S, f\}$ satisfy CLR_f property.

So there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} fx_n = fu, \text{ for some } u \in X.$$

Since $SX \subseteq gX$, so there exists a sequence $\{y_n\}$ in X , such that $Sx_n = gy_n$.

Thus $\lim_{n \rightarrow \infty} gy_n = fu$.

Now by condition (iv),

$$d(Sx_n, Ty_n) \lesssim \lambda d(fx_n, gy_n) + \frac{\mu d(fx_n, Sx_n)d(gy_n, Ty_n)}{1+d(fx_n, gy_n)}$$

Now letting $n \rightarrow \infty$, we get $d(fu, Ty_n) \lesssim 0$.

Thus $d(fu, Ty_n) = 0$ and hence $\lim_{n \rightarrow \infty} Ty_n = fu$.

Thus $\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} fx_n = fu = \lim_{n \rightarrow \infty} gy_n = \lim_{n \rightarrow \infty} Ty_n$.

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We now show that $fu = Su$.

By condition (iv),

$$d(Su, Ty_n) \lesssim \lambda d(fu, gy_n) + \frac{\mu d(fu, Su) d(gy_n, Ty_n)}{1 + d(fu, gy_n)}$$

Letting $n \rightarrow \infty$, we get $d(Su, fu) \lesssim 0$

Therefore $d(Su, fu) = 0$ and hence $fu = Su$.

Since S, T are weakly compatible,

$$ffu = fSu = Sfu = SSu.$$

Again $SX \subseteq gX$.

Thus there exists $v \in X$, such that $Su = gv$.

We now show that $gv = Tv$.

By the condition (iv),

$$d(Su, Tv) \lesssim \lambda d(fu, gv) + \frac{\mu d(fu, Su) d(gv, Tv)}{1 + d(fu, gv)}$$

$$\lesssim 0 \quad [\because fu = Su = gv]$$

Therefore $d(Su, Tv) = 0$.

Thus $Su = Tv = gv$.

Since g, T are weakly compatible,

$$ggv = gTv = Tgv = TTv.$$

Now we have,

$$Su = fu = Tv = gv = t \text{ (say), for some } t \in X.$$

We now show that t is a common fixed point of S, T, f and g .

By condition (iv),

$$d(t, T) = d(Su, Tt)$$

$$\lesssim \lambda d(fu, gt) + \frac{\mu d(fu, Su) d(gt, Tt)}{1 + d(fu, gt)}$$

$$= \lambda d(fu, gt)$$

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$$= \lambda d(t, Tt) \quad [\because gt = ggv = TTv = Tt]$$

$$\therefore (1 - \lambda)|d(t, Tt)| \leq 0$$

$$\therefore 0 \leq \lambda + \mu < 1, \quad d(t, Tt) = 0 \quad \text{and hence } t = Tt.$$

Now $gt = Tt = t$.

Again $d(St, t) = d(St, Tv)$

$$\begin{aligned} &\lesssim \lambda d(ft, gv) + \frac{\mu d(ft, St)d(gv, Tv)}{1+d(ft, gv)} \\ &= \lambda d(St, t) \quad [\because ft = ffu = SSu = St] \end{aligned}$$

$$\therefore (1 - \lambda)|d(St, t)| \leq 0$$

Therefore $d(St, t) = 0$. i.e. $St = t$.

Thus $St = ft = t$.

Therefore $St = ft = gt = Tt = t$.

Thus t is a common fixed point of S, T, f and g .

For uniqueness, let $w \in X$, such that

$$Sw = Tw = fw = gw = w.$$

Now $d(w, t) = d(Sw, Tt)$

$$\begin{aligned} &\lesssim \lambda d(fw, gt) + \frac{\mu d(fw, Sw)d(gt, Tt)}{1+d(fw, gt)} \\ &= \lambda d(w, t) \end{aligned}$$

$$\therefore (1 - \lambda)|d(w, t)| \leq 0.$$

$$\therefore d(w, t) = 0 \quad \text{and hence } t = w.$$

Thus t is the unique common fixed point of S, T, f and g .

If the pair $\{T, g\}$ satisfy CLR_g property, similarly we can prove the theorem.

Corollary 1: Let (X, d) be a complex valued metric space. Suppose that T, f and g be self-maps of X satisfying the following conditions:

- (i) The pairs $\{T, f\}$ and $\{T, g\}$ are weakly compatible,
- (ii) $TX \subseteq fX$ and $TX \subseteq gX$,
- (iii) Either the pair $\{T, f\}$ satisfy CLR_f property or the pair $\{T, g\}$ satisfy CLR_g property and

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$$(iv) \quad d(Tx, Ty) \lesssim \lambda d(fx, gy) + \frac{\mu d(fx, Tx)d(gy, Ty)}{1+d(fx, gy)}, \forall x, y \in X, \text{ where } \lambda, \mu \text{ are non-negative reals satisfy } \lambda + \mu < 1.$$

Then T, f and g have a unique common fixed point.

Proof: Setting $S = T$ in the above theorem, we get the result.

Corollary 2: Let (X, d) be a complex valued metric space. Suppose that U, V be two weakly compatible self-maps of X , such that

- (i) U, V satisfy CLR_v property and
- (ii) $d(Ux, Vy) \lesssim \lambda d(Vx, Vy) + \frac{\mu d(Ux, Vx)d(Uy, Vy)}{1+d(Vx, Vy)}, \forall x, y \in X$, where λ, μ are non-negative reals with $\lambda + \mu < 1$.

Then U and V have unique common fixed point in X .

Proof: If we take $S = T = U$ and $f = g = V$ in the above theorem we get the result.

Now we show an example in support of the theorem.

Example: Let $X = (0, 5]$.

Let $d: X \times X \rightarrow \mathbb{C}$, defined as $d(x, y) = i|x - y|, \forall x, y \in X$.

We define
$$S(x) = \begin{cases} \frac{1}{5}, & \text{if } x \in (0, 1) \\ 1, & \text{if } x \in [1, 5] \end{cases}$$

$$T(x) = \begin{cases} \frac{1}{4}, & \text{if } x \in (0, 1) \\ 1, & \text{if } x \in [1, 5] \end{cases}$$

$$g(x) = \begin{cases} \frac{1}{5}, & \text{if } x \in (0, 1) \\ \frac{x+1}{2}, & \text{if } x \in [1, 5] \end{cases}$$

$$f(x) = \begin{cases} \frac{1}{4}, & \text{if } x \in (0, 1) \\ 1, & \text{if } x = 1 \\ \frac{x-1}{2}, & \text{if } x \in (1, 5] - \{3\} \\ 3, & \text{if } x = 3 \end{cases}$$

(i) Here the pairs $\{S, f\}$ and $\{T, g\}$ are weakly compatible

(ii)
$$TX = \left\{ \frac{1}{4}, 1 \right\} \subseteq (0, 2] \cup \{3\} = fX$$

and
$$SX = \left\{ \frac{1}{5}, 1 \right\} \subseteq \left\{ \frac{1}{5} \right\} \cup [1, 3] = gX.$$

(iii) Now for sequence $x_n = 3 - \frac{1}{n}, Sx_n = 1, \forall n \in \mathbb{N}$ and

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$$fx_n = \frac{3 - \frac{1}{n} - 1}{2} = 1 - \frac{1}{2n}, \forall n \in \mathbb{N}.$$

Thus $\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} fx_n = 1 = f(1)$

Therefore the pair $\{S, f\}$ satisfy CLR_f property.

Also for sequence $y_n = 1 + \frac{1}{n}$, $Ty_n = 1, \forall n \in \mathbb{N}$ and

$$gy_n = \frac{1 + \frac{1}{n} + 1}{2} = 1 + \frac{1}{2n}, \forall n \in \mathbb{N}.$$

Thus $\lim_{n \rightarrow \infty} Ty_n = \lim_{n \rightarrow \infty} gy_n = 1 = g(1)$.

Therefore the pair $\{T, g\}$ satisfy CLR_g property.

(iv) Clearly for every $x, y \in X$,

$$d(Sx, Ty) \lesssim \lambda d(fx, gy) + \frac{\mu d(fx, Sx)d(gy, Ty)}{1 + d(fx, gy)}, \text{ where } \lambda, \mu \text{ are non-negative reals with } \lambda + \mu < 1.$$

Thus the mappings S, T, f and g satisfy all the conditions of the theorem.

Here 1 is the unique common fixed point of S, T, f and g .

V. CONCLUSION

In the above theorem if we replace the condition (iii) with the condition, “ Either the pair $\{S, f\}$ satisfy E.A. property and fX is closed in X or the pair $\{T, g\}$ satisfy the E.A. property and gX is closed in X ”, then we can also show that S, T, f and g have unique common fixed point.

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