K-Ordered Hamiltonian Graphs

J. Rashmi
Department of Mathematics, Bharath University, Chennai, Tamil Nadu, India

ABSTRACT: In this chapter, we review the following results proved in [5, 8, 10]: (i) For k ≥ 3, every (k+1)-hamiltonian connected graph is k-ordered. Determine f(k,n) if n is sufficiently large in terms of k. Let g(k,n) = n/2 + k/2 -1 (ii) f(k,n)=g(k,n) if n ≥ 1, k=3 (iii) f(k,n) ≥ g(k,n) for any n ≥ 2k and f(k,n)≥ g(k,n) if 2k≤n≤3k-6 (iv) if G is a graph of order n with 3≤k≤n/2 and deg(u)+deg(v)≥n+(3k-9)/2 for every pair u, v of non-adjacent vertices of G, then G is k-ordered Hamiltonian.

I. INTRODUCTION

In this section, the concept of K-hamiltonian connected graphs is introduced. The result that, for k≥3, every (k+1)-hamiltonian connected graph is k-ordered is established[2]. Also necessary and sufficient conditions for a graph to be k-ordered are discussed[5]. The notations used are recalled here. We write v1, (x1), v2, (x2), ..., vk, (xk-1), vk to indicate a v1-vk path that contains the k vertices v1, v2,..., vk and possibly up to k additional vertices, namely x1, x2, ..., xk-1. Thus for 1≤i≤k-1, (x1) indicates that some vertex, which we denote by xi, may (or) may not be present on the path. So for each i (1≤i≤k-1), the vertices vi and vi+1 are either adjacent or vi, vi+1 is a path. For example, u, v(x), y indicates the path u, v, y or the path u, v, x, y[6].

Proposition 1.1: Let G be a Hamiltonian graph of order n ≥ 3. If G is k-ordered, 3 ≤ k ≤ n, then G is (k-1)-connected.
Proof: Suppose, to the contract that G is not (k-1) connected. Then there exists a cut set S= v1, v2, ..., v of G, where ≤ k-2. Let v1 and v2 be vertices belonging to distinct components of G-S. Consequently, every v1, v2 path in G contains vertices of S, so G contains no Hamiltonian cycle containing the vertices of the sequence v1, v2, ..., v, v1, v2 in this order. Hence G is not (k-2)-order and so is not k-ordered[1].

Corollary 1.2: If G is a k-ordered Hamiltonian graph, then δ(G)≥k-1.

II. SUFFICIENT CONDITIONS FOR K-ORDERED GRAPHS

Many sufficient conditions have been given for Hamiltonian graphs. Two of the best known are due to Dirac and Ore.

Theorem 1.3 (Dirac): Let G be a graph of order n ≥ 3. If deg v ≥ n/2 for every vertex v of G, then G is Hamiltonian.

Theorem 1.4 (Ore): Let G be a graph of order n ≥ 3. If for every pair u, v of non-adjacent vertices of G, deg u+deg v ≥ n, then G is Hamiltonian.

Lemma 1.5: Let G be a graph of order n ≥ 3 and let S= v1, v2, ..., vk be a sequence of k distinct vertices of G, where 3 ≤ k ≤ n. If

deg u+deg v ≥ n+2k-7

for every pair u, v of non-adjacent vertices of G, then G contains a v1-vk path of the type v1, (x1), v2, (x2), ..., vk, (xk), where u, v, and x1 are path of the type v1, (x1), v2, (x2), ..., vk, (xk).

Proof: Consider the vertices v1 and v2. If v1 and v2 are adjacent, then G contains the path v1, v2: otherwise, deg v1 +deg v2 ≥ n+2k-7≥n-1, which implies the existence of a vertex x1 mutually adjacent to v1 and v2 and so v1, x1, v2 is a path in G. In any case, G contains a path v1, (x1), v2. Now, consider the vertex v3. If v2v3 ∉ E(G), then G contains the path v1, (x1), v2, v3. Suppose then that v2v3 ∈ E(G). Thus deg v2+deg v2 ≥ n+2k-7 by hypothesis. Since G has order n, the vertices v2 and v3 are mutually adjacent to at least 2k-5 vertices. if k≥4, then G contains a vertex v3 distinct from v1 and x1 (if it exists) such that v3 is mutually adjacent to v2 and v3. Hence G contains the path v1, (x1), v2, (x2), v3. Suppose then that k=3 and that G contains to vertices distinct from v1 and x1 that is mutually adjacent to v2 and
If $v_3$ is adjacent to $v_1$, then $v_1, v_1, (x_1), v_2$ is a path in $G$. If $v_3$ is not adjacent to $v_1$, then $v_2$ and $v_3$ are mutually.[6] adjacent to 1 and to no other vertex, while $v_2$ is adjacent to $v_1$. However, then $v_1, v_2, x_1, v_1$ is a path in $G$.

Proceeding inductively, assume that we have constructed a path $v_1, (x_1), v_2, (x_2), \ldots, (x_{j-1}), v_j$ in $G$. Suppose first that $j \geq k-1$. It is to show that $G$ contains a path $v_1, (x_1), v_2, (x_2), \ldots, (x_j), v_j$. If $v_{j+1}$ is adjacent to $v_j$, then $G$ contains such a path. Suppose then that $v_{j+1} \notin E(G)$. If $G$ contains a vertex $x_{j+1}$ distinct from the vertices on the path $v_1, (x_1), v_2, (x_2), \ldots, (x_j)$ such that $x_{j+1}$ is mutually adjacent to $v_1$ and $v_j$ then $G$ contains a path of the desired type, otherwise $v_1, v_j$ and $v_j$ are mutually adjacent to at most $2j$ vertices and $G$ contains at least $n+2k-2j$ vertices. Since $k \geq j+1$, this produces a contradiction and hence the desired claim is verified.

Finally, suppose that $j = k$, that is, assume that there is a path $v_1, (x_1), v_2, (x_2), \ldots, (x_{k-1})v_k$, $v_{k+1}$ is constructed in $G$. It is to show that $G$ contains either a path $v_1, (x_1), v_2, (x_2), \ldots, v_{k-1}, (x_{k-1}), v_k$, or a path $v_k, v_{k-1}, v_{k-2}, (x_{k-2}), v_{k+1}$. If either $v_k v_{k+1} \notin E(G)$ or $v_{k+1} v_k \notin E(G)$, then the desired result is obtained, thus assume that $v_k$ is adjacent to neither $v_{k+1}$ nor $v_{k-1}$. Also, if $v_{k-1}$ and $v_k$ are mutually adjacent to some vertex other than the (at most) $2k-5$ vertices on the path $(x_1), v_2, (x_2), \ldots, v_{k-2}, (x_{k-2})$, the proof is complete; so assume that this is not the case. Hence $v_{k-1}$ and $v_k$ are mutually adjacent to at most $2k-5$ vertices. Since $G$ has order $n$, it follows that $v_{k-1}$ and $v_k$ are mutually adjacent to exactly $2k-5$ vertices (so all of the vertices $x_1, x_2, \ldots, x_{k-2}$ exist) and every other vertex of $G$ is adjacent to exactly one of $v_{k-1}$ and $G$ contains the path $v_1, x_1, v_2, x_2, \ldots, v_{k-2}, x_{k-2}, v_k$, producing the desired path. Hence the result.[8]

REFERENCES

[1] D.J. Oberly and D.P. Summer, Every connected, locally connected non trivial graph with no induced claw is Hamiltonian, J. Graph Theory 3 (1979), 351-356.