

Some Generalization on Fixed Point Theorems Related To Product of 3-Fuzzy Metric Spaces

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ABSTRACT: In the present paper some generalization on fixed point and common fixed point theorems in product of two complete Fuzzy 3-metric spaces are established which are motivated by Gahler [13-15], Sharma, Sharma and Isekey [30], Sharma, S.[31], Shrivastava R., Singhal J.[36], Rahmat, Salmi and Noorani [30].

KEYWORDS: Fuzzy metric spaces, fuzzy 2- metric spaces, fuzzy 3- metric spaces, fixed point, Common fixed point, Cartesian product, product space.

I. INTRODUCTION

In 1965, the concept of fuzzy sets was introduced by Zadeh [38]. After that many authors have expansively developed the theory of fuzzy sets and applications. Especially, Deng [8], Erceg [10], Kaleva and Seikhala [23], Karamosil and Michalek [25], have introduced the concept of fuzzy metric spaces in different ways. Recently, many authors [1,6,11,17,20,21,22,27,28,31,32] have also studied the fixed point theory in the fuzzy metric spaces and [2,3,4,5,19,26,33] have studied for fuzzy mappings which opened an avenue for further development of analysis in such spaces and such mappings. Consequently in due course of time some metric fixed point results were generalized to fuzzy metric spaces by various authors.

Gahler in a series of papers [13, 14, and 15] investigated 2-metric spaces. Sharma, Sharma and Iseki [30] studied for the first time contraction type mappings in 2-metric space. We [34, 35] have also worked on 2-Metric spaces and 2-Banach spaces for rational expressions.

We know that 2 and 3-metric space is a real valued function of a point triples and four tuples respectively on a set X , which abstract properties were suggested by the area and volume function in Euclidean spaces. Fixed point theorems related to fuzzy 2 and fuzzy 3 metric spaces were established by Shrivastav and Singhal [37]. Now it is natural to expect the product of fuzzy 3-Metric space for further investigation.

II. SOME FIXED POINT THEOREMS IN FUZZY 3-METRIC SPACE

Definition (2 A): A binary operation $*$: $[0, 1] \times [0,1] \times [0,1] \times [0,1] \rightarrow [0,1]$ is called a continuous t-norm if $([0,1], *)$ is an abelian topological monoid with unit 1 such that $a_1 * b_1 * c_1 * d \geq a_2 * b_2 * c_2 * d_2$ whenever $a_1 \geq a_2$, $b_1 \geq b_2$, $c_1 \geq c_2$ and $d_1 \geq d_2$ for all $a_1, a_2, b_1, b_2, c_1, c_2$ and d_1, d_2 are in $[0,1]$.

Definition (2 B): The 3-tuple $(X, M, *)$ is called a fuzzy 3-metric space if X is an arbitrary set, $*$ is continuous t-norm and M is fuzzy set in $X^4 \times [0, \infty)$ satisfying the followings

$$(FM'' - 1): M(x, y, z, w, 0) = 0$$

$$(FM'' - 2): M(x, y, z, w, t) = 1, \forall t > 0$$

$$(FM'' - 3): M(x, y, z, w, t) = M(x, w, z, y, t) = M(z, w, x, y, t) = \dots (FM'' - 4): M(x, y, z, w, t_1 + t_2 + t_3) \geq M(x, y, z, u, t_1) * M(x, y, u, w, t_2) * M(x, u, z, w, t_3) * M(x, y, z, w, t_4)$$

$$(FM'' - 5): M(x, y, z, w): [0,1] \rightarrow [0,1] \text{ is left continuous, } \forall x, y, z, u \in X, t_1, t_2, t_3, t_4 > 0$$

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Definition (2 C): Let $(X, M, *)$ be a fuzzy 3-metric space. A sequence $\{x_n\}$ in fuzzy 3-metric space X is said to be convergent to a point $x \in X$,

$$\lim_{n \rightarrow \infty} M(x_n, x, a, b, t) = 1, \text{ for all } a, b \in X \text{ and } t > 0$$

A sequence $\{x_n\}$ in fuzzy 3-metric space X is called a Cauchy sequence, if

$$\lim_{n \rightarrow \infty} M(x_{n+p}, x_n, a, b, t) = 1, \text{ for all } a, b \in X \text{ and } t, p > 0$$

A fuzzy 3-metric space in which every Cauchy sequence is convergent is said to be complete.

Definition (2 D): A function M is continuous in fuzzy 3-metric space, iff whenever for all $a \in X$ and $t > 0$.

$$x_n \rightarrow x, y_n \rightarrow y, \text{ then } \lim_{n \rightarrow \infty} M(x_n, y_n, a, b, t) = M(x, y, a, t), \forall a, b \in X \text{ and } t > 0$$

Definition (2 E): Two mappings A and S on fuzzy 3-metric space X are weakly commuting iff $M(ASu, SAu, a, b, t) \geq M(Au, Su, a, t), \forall u, a, b \in X$ and $t > 0$.

Theorem 2.1. Let $(X, M, *)$ be a complete fuzzy 3-metric space. Let f and g be weakly compatible self maps of X satisfying

$$M(gx, gy, a, b, kt) \geq M(fx, fy, a, b, t) \text{ where } 0 < k < 1, a, b > 0 \quad (2.1)$$

$$g(X) \subseteq f(X) \quad (2.2)$$

If one of $g(X)$ or $f(X)$ is complete then f and g have a unique common fixed point.

Theorem 2.2. Let $(X, M, *)$ be a fuzzy 3-metric space. Let f and g be weakly compatible self maps on X satisfying (2.1) and (2.2). If one of $g(X)$ or $f(X)$ is complete then f and g have a unique common fixed point.

Theorem 2.3. Let $(X, M, *)$ be a fuzzy 3-metric space. Let f and g weakly compatible self maps of X satisfying condition (2.1) and (2.2). If one of $g(X)$ or $f(X)$ is complete then f and g have a unique common fixed point.

III. SOME FIXED POINT THEOREMS IN PRODUCT OF FUZZY 3-METRIC SPACES

Definition (3 A): Let $(X, M_X, *)$ and $(Y, M_Y, *)$ are two fuzzy 3- metric spaces defined with same continuous t- norms $*$. Let Δ be a continuous t- norm. The Δ product of $(X, M_X, *)$ and $(Y, M_Y, *)$ is the product space $(X \times Y, M_\Delta, *)$ where $X \times Y$ is the Cartesian product of the sets X and Y and M_Δ is the mapping from $(X \times Y \times (0,1))$ in to $[0,1]$ given by $M_\Delta(p, q, t + s) = M_1(x_1, x_2, t) \Delta M_1(y_1, y_2, s)$ for every $p = (x_1, y_1)$ and $q = (x_2, y_2)$ in $X \times Y$ and $t + s \in (0,1)$.

Theorem 3.1- If $(X, M_X, *)$ and $(Y, M_Y, *)$ are fuzzy 3-metric spaces under the same continuous t- norm $*$, then their $*$ -product $(X \times Y, M_\Delta, *)$ is a fuzzy metric space under $*$.

Definition (3 B): Let Δ and $*$ be continuous t-norms. We say that Δ is stronger than $*$, if for each $a_1, a_2, b_1, b_2 \in [0,1]$. $(a_1 * b_1) \Delta (a_2 * b_2) \geq (a_1 \Delta a_2) * (b_1 \Delta b_2)$

Lemma: Δ is stronger than $*$ then $\Delta \geq *$.

Theorem 3.2- If $(X, M_X, *)$ and $(Y, M_Y, *)$ are fuzzy metric spaces under the same continuous t- norm $*$. If there exist a continuous t- norm Δ stronger than $*$, then the Δ - product $(X \times Y, M_\Delta, *)$ is a fuzzy metric space under $*$.

Corollary: If $(X, M_X, *^1)$ and $(X, M_X, *^2)$ are fuzzy metric spaces and if there exist a continuous t- norm Δ stronger than $*^1$ and $*^2$ then their Δ product is a fuzzy metric space under Δ .

Theorem 3.3- $(X_1, M_1, *)$ and $(X_2, M_2, *)$ be fuzzy metric spaces under the same continuous t-norm $*$. Let U denote the neighbourhood system in $(X_1 \times X_2, M_*, *)$ consisting of Cartesian products $B_{x_1}(r, t) \times B_{x_2}(r, t)$ where $x_1 \in X_1, x_2 \in X_2, r \in (0,1)$ and $t > 0$. Then U and V induce the same fuzzy topology on $(X_1 \times X_2, M_*, *)$.

Now we prove the following theorem as generalized version of Theorem (2.1) and (2.2) for the product of fuzzy-3 metric spaces.

Theorem 3.4- Let $(X \times Y, M_\Delta, *)$ is a complete 3-fuzzy metric space which is the product of $(X, M_X, *)$ and $(X, M_Y, *)$ are two complete 3-fuzzy metric spaces. Let f and g be weakly compatible self maps on $X \times Y$ satisfying

$$M_\Delta(gp, gq, a, b, kt) \geq M_\Delta(fp, fq, a, b, t) \quad (3.1)$$

$$g(X \times Y) \subseteq f(X \times Y) \quad (3.2)$$

If one of $g(X \times Y)$ or $f(X \times Y)$ are complete then f and g have unique common fixed point.

Proof. Let $p_0 \in X \times Y$. Since $g(X \times Y) \subseteq f(X \times Y)$. Choose $p_1 \in X \times Y$ such that $g(p_0) = g(x_0, y_0) = f(x_1, y_1) = g(p_1)$. In general, choose p_{n+1} such that $q_n = f(x_{n+1}, y_{n+1}) = g(x_n, y_n) = g(p_n)$. Then by (3.1), we have

$$\begin{aligned} M_\Delta(fp_n, fp_{n+1}, a, b, t) &= M_\Delta(gp_{n-1}, gp_n, a, b, t) \geq M_\Delta(fp_{n-1}, fp_n, a, b, \frac{t}{k}) \\ &= M_\Delta(gp_{n-2}, gp_{n-1}, a, b, \frac{t}{k}) \geq \dots \dots \geq M_\Delta(fp_1, fx_0, a, b, \frac{t}{k^n}). \end{aligned}$$

Therefore, for any r ,

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$$M_{\Delta}(fp_n, fp_{n+r}, a, b, t) \geq M_{\Delta}\left(fp_n, fp_{n+1}, a, b, \frac{t}{r}\right) \geq \dots \geq M_{\Delta}\left(fp_{n+r-1}, fp_{n+r}, a, b, \frac{t}{r}\right) \geq M_{\Delta}\left(fp_0, fp_1, a, b, \frac{t}{rk^n}\right) \\ \geq \dots \geq M_{\Delta}\left(fp_0, fp_1, a, b, \frac{t}{rk^{n+r-1}}\right)$$

As $n \rightarrow \infty$. $\{fp_n\} = \{q_n\}$ is a Cauchy sequence in fuzzy 3-metric space and so, by completeness of $X \times Y$, $\{q_n\} = \{fp_n\}$ is convergent. We call the limit z , then $\lim_{n \rightarrow \infty} fp_n = \lim_{n \rightarrow \infty} gp_n = z$. As $f(X \times Y)$ is complete, so there exist a point z_1 in $X \times Y$ such that $fz_1 = z$. Now, from (1),

$$\text{As } n \rightarrow \infty. M(gz_1, gp_n, a, b, kt) \geq M(fz_1, fp_n, a, b, t),$$

$$M(gz_1, z, a, b, kt) \geq M(fz_1, z, a, b, t),$$

$$M(gz_1, z, a, b, kt) \geq M(z, z, a, b, t),$$

$$M(gz_1, z, a, b, kt) \geq 1,$$

$$M(gz_1, z, a, b, kt) = 1,$$

$$gz_1 = z = fz_1.$$

As f and g are weakly compatible. Therefore $fgz_1 = ggz_1$ i.e. $fz = gz$. Now, we show that z is fixed point of f and g . From (3.1),

$$\text{As } n \rightarrow \infty. M(gz, gp_n, a, b, kt) \geq M(fz, fp_n, a, b, t),$$

$$M(gz, z, a, b, kt) \geq M(fz, z, a, b, t),$$

$$M(gz, z, a, b, kt) \geq M(gz, z, a, b, t),$$

$$gz = z = fz.$$

Hence z is a common fixed point of f and g . For uniqueness, let w be another fixed point of f and g . Then by (3.1), $M(gz, gw, a, b, kt) \geq M(fz, fw, a, b, t)$, $M(z, w, a, b, kt) \geq M(z, w, a, b, t)$ and $z = w$.

Therefore z is unique common fixed point of f and g .

Theorem 3.5 Let $(X \times Y, M_{\Delta}, *)$ is a complete 3-fuzzy metric space which is the product of $(X, M_X, *)$ and $(X, M_Y, *)$ are two complete 3-fuzzy metric spaces. Let f and g be weakly compatible self maps on $X \times Y$ satisfying (3.1) and (3.2). If one of $g(X)$ or $f(X)$ is complete then f and g have a unique common fixed point.

Proof. From the proof of above theorem, we conclude that $\{fp_n\} = \{q_n\}$ is a Cauchy sequence in $X \times Y$. Now suppose that $f(X \times Y)$ is a complete subspace of $X \times Y$. Then the subsequence of $\{q_n\}$ must get a limit in $f(X \times Y)$. Call it be u and $f(v) = u$. As $\{q_n\}$ is a Cauchy sequence containing a convergent subsequence, therefore the sequence $\{q_n\}$ also converges implying thereby the convergence of subsequence of the convergent sequence. Now, from (3.1),

$$\text{As } n \rightarrow \infty. M_{\Delta}(gv, gp_n, a, b, kt) \geq M(fv, fp_n, a, b, t),$$

$$M_{\Delta}(gv, u, a, b, kt) \geq M_{\Delta}(fv, u, a, b, t),$$

$$M_{\Delta}(gv, u, a, b, kt) \geq M_{\Delta}(u, u, a, b, t),$$

$$M_{\Delta}(gv, u, a, b, kt) \geq 1,$$

$$M_{\Delta}(gv, u, a, b, kt) = 1,$$

$$gv = u = fv.$$

Which shows that pair (f, g) has a point of coincidence. Since, f and g are weakly compatible, $fgv = gfv$, i.e. $fu = gu$. Now, we show that u is a fixed point of f and g . From (3.1),

$$\text{As } n \rightarrow \infty. M_{\Delta}(gu, gp_n, a, b, kt) \geq M_{\Delta}(fu, fp_n, a, b, t),$$

$$M_{\Delta}(gu, u, a, b, kt) \geq M_{\Delta}(fu, u, a, b, t),$$

$$M_{\Delta}(gu, u, a, b, kt) \geq M_{\Delta}(fu, u, a, b, t),$$

$$gu = u = fu.$$

Hence u is a fixed point of f and g . For uniqueness, let w be another fixed point of f and g . Then by (3.1), $M_{\Delta}(gz, gw, a, b, kt) \geq M_{\Delta}(fz, fw, a, b, t)$, $M_{\Delta}(z, w, a, b, kt) \geq M_{\Delta}(z, w, a, b, t)$ and $z = w$.

Therefore z is unique common fixed point of f and g .

IV. CONCLUSION

Some generalizations of existing fixed point theorems are presented in product of 3-fuzzy metric spaces for weakly compatible mappings.

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