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Numerical Solution of the Navier-Stokes Equations at High Reynolds Numbers

Blanca Bermúdez[1], W. Fermín Guerrero S., [2]A. Rangel-Huerta[1]

Benemérita Universidad Autónoma de Puebla, Facultad de Ciencias de la Computación, Puebla, México[1]

Benemérita Universidad Autónoma de Puebla, Facultad de Ciencias Físico- Matemáticas, Puebla, México[2]

ABSTRACT: In this work, we are working with the unsteady Navier-Stokes equations in the stream function-vorticity formulation. In order to show that thenumerical schemes used are able to handle highReynolds numbers, we are reporting results for the well known problem of the un-regularized driven cavity. We are dealing with Reynolds numbers in the range of $7500 \leq Re \leq 50000$. The results shown here are obtained using two numerical schemes, the first one, is based on a fixed point iterative process (see [1]) applied to the elliptic nonlinear system that results after time discretization. The second scheme (see [2], [3]) which, as we are going to show in the results, is faster than the first one, solves the transport type equation appearing in the Stream function-vorticity formulation of the Navier-Stokes equations using matrixes A and B; the first one resulting from the discretization of the Laplacian term appearing in the equation, and the second one resulting from the discretization of the advective term. Both schemes, for this problem, have been robust enough to handle such high Reynolds numbers, but the second one has proved to be much faster, especially for high Reynolds numbers. In [4] it has already been said that even though turbulence is a tri-dimensional phenomenon, two-dimensional flows at high Reynolds numbers give some clues of transition to real turbulence.

KEYWORDS: Navier-Stokes equations, Stream Function-vorticity formulation, high Reynolds numbers, fixed point iterative process.

I. INTRODUCTION

In this paper we are presenting numerical results for Reynolds numbers in the range of $7500 \leq Re \leq 50000$. Results for high Reynolds numbers have already been presented in the bibliography. Just to mention some of them: In [5] the range $10000 \leq Re \leq 20000$ is presented; in [6], with primitive variables, the range $25000 \leq Re \leq 40000$, and in [7] the range $400 \leq Re \leq 5000$ to capture the steady state flow and the range $10000 \leq Re \leq 20000$ for time-dependent flows.

In the case of this work, we are using the stream function-vorticity formulation of the Navier-Stokes equations, and, as already mentioned in the abstract, results are obtained using two numerical schemes. The first one [1] is a simple numerical scheme for the unsteady Navier-Stokes equations in stream function and vorticity variables, based on a fixed point iterative process already used in the bibliography. This scheme worked very well, as shown in [7], [8], [9], [10], but the processing time was, in general, very large, especially for high Reynolds numbers. The second scheme discussed in [2], [3], works not only with the symmetric and positive matrix A resulting from the discretization of the Laplacian, but also with the matrix B resulting from the discretization of the advective term.

In the case of moderate Reynolds numbers, for instance $Re \leq 7500$, the flow approaches to an asymptotic steady state as t tends to ∞, but in the case of high Reynolds numbers, such as the ones presented here, the solution seems to be time-dependent.

As the Reynolds numberincreases the mesh has to be refined and a smaller time step has to be used: numerically, by stability matters andphysically, to capture the fast dynamics of the flow.It hasalready been pointed out in [4] that to get the right vorticitycontours is more difficult than to get the rightstreamlines of the stream function, dueto oscillations appearing on the top right corner of the cavity because of insufficient mesh refining. So, in this work, we are going to present results for the vorticity contours.

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II. THE MATHEMATICAL MODEL OF THE PROBLEM

Be $D = \Omega \times (0,T)$, $T > 0, \Omega \subset R^2$, the region of the flow of aviscous incompressible fluid, and Γ its boundary. As already known, this kind of flow is governed by the well-knownNavier-Stokes equations. The Navier-Stokes equations in primitive variables are given by:

$$\mathbf{u}_t - \frac{1}{Re}\nabla^2\mathbf{u} + \nabla p + \mathbf{u}\cdot\nabla\mathbf{u} = f \tag{1a}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{1b}$$

whereu is the velocity, p is the pressure and the dimensionless parameter Re is the Reynolds number. This system must be supplemented with appropriate initial and boundary conditions: $u(x,0) = u_0(x)$ in Ω and $u = g$ on Γ. respectively. Restricting ourselves to the two dimensional case, and taking the curl on both sides of (1a) and taking into account the following relations:

$$u_1 = \frac{\partial\psi}{\partial y}, \quad u_2 = \frac{\partial\psi}{\partial x}, \tag{2}$$

whereψ is the Stream function, (u_1, u_2) the two components of the velocity, we get the following system of equations:

$$\nabla^2\psi = -\omega \tag{3a}$$

$$\omega_t - \frac{1}{Re}\nabla^2\omega + \mathbf{u}\cdot\nabla\omega = 0 \tag{3b}$$

whereω is the vorticity given by:

$$\omega = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}. \tag{4}$$

In this formulation, the incompressibility condition (1b) is automatically satisfied in Ω, but no boundary condition is given for the vorticity. In [11], for example, a procedure is given to get the boundary condition for ω in general domains.

III. THE NUMERICAL SCHEMES

The time derivative is approximated by the second-order scheme:

$$\omega_t(x,(n+1)\Delta t) \approx \frac{3\omega^{n+1} - 4\omega^n + \omega^{n-1}}{2\Delta t}, \tag{5}$$

for $n \geq 1, x \in \Omega$, and Δt is the time step.

So the following nonlinear elliptic system has to be solved at each time level:

$$\nabla^2\psi = -\omega, \qquad \psi|_\Gamma = 0, \tag{6a}$$

$$\alpha\omega - \frac{1}{Re}\nabla^2\omega + \mathbf{u}\cdot\nabla\omega = f_\omega, \qquad \omega|_\Gamma = \omega_{bc} \tag{6b}$$

where $\alpha = \frac{3}{2\Delta t}$, and $f_\omega = \frac{4\omega^n - \omega^{n-1}}{2\Delta t}$. To obtain (ψ^1, ω^1) the first subinterval is subdivided into M subintervals, and a first order scheme, such as Euler, is applied at each of the M subintervals.

Let R_ω be defined by:

$$R_\omega(\omega,\psi) = \alpha\omega - \frac{1}{Re}\nabla^2\omega + \mathbf{u}\cdot\nabla\omega - f_\omega, \text{ in } \Omega, \tag{7}$$

Now, system (6) is equivalent to:

$$\nabla^2\psi = -\omega, \qquad \psi|_\Gamma = 0, \tag{8a}$$

$$R_\omega(\omega,\psi) = 0, \qquad \omega|_\Gamma = \omega_{bc}. \tag{8b}$$

Then system (8) is solved at time level (n+1) by the following fixed point iterative process (see [1]):

Given$\omega^{n,0} = \omega^n$, and $\psi^{n,0} = \psi^{n,0}$, solve until convergence in ω and Ψ:

$$\nabla^2\psi^{n,m+1} = -\omega^{n,m} \quad , \quad \text{in } \Omega, \quad \psi^{n,m+1}|_\Gamma = 0, \tag{9a}$$

$$\left(\alpha I - \frac{1}{Re}\nabla^2\right)\omega^{n,m+1} = \omega^m - \rho R_\omega(\omega^{n,m}, \psi^{n,m+1}), \text{ in } \Omega, \tag{9b}$$

$$\omega^{n,m+1} = \omega_{bc}^{n,m+1}, \text{ on } \Gamma, \ \rho > 0;$$

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and then take $(\omega^{n+1}, \psi^{n+1}) = (\omega^{m+1}, \psi^{m+1})$.

In order to reduce computing time, we worked on solving the aboved mentioned system by the following method at each time step:

$$\nabla^2\psi^{n+1} = -\omega^n, \qquad \psi^{n+1}|_\Gamma = 0, \tag{10a}$$
$$\left(\alpha I - \frac{1}{Re}A\right)\omega^{n+1} + B\omega^{n+1} = f_\omega, \qquad \omega^{n+1}|_\Gamma = \omega_{bc}^{n+1} \tag{10b}$$

Here A and B are the matrixes associated with the discretization of the diffusive (Laplacian) and the advective term respectively. Equation (10b) is solved using Gauss-Seidel.

IV. Numerical Experiments

The experiments we are showing are for the un-regularized driven cavity problem. For this problem, $\Omega=(0,1)\times(0,1)$, and the boundary is given by the four walls of the cavity; the top wall is moving with a velocity given by (1,0) and the other walls of the cavity are solid and fixed. For this walls, the velocity is given by (0, 0). In this work, the range considered for the Reynolds numbers is $7500 \le Re \le 50000$. Results are reported through the isovorticity contours; themesh size and the time step are denoted by h and Δt respectively. Results obtained by both methods agree very well, so we just present results obtained by second method already described.

Now, we have to translate the boundary conditions in terms of the velocity primitive variable u to the Stream function-vorticity variables. In the case of this work, we follow [12]. Ψ is a constant function on solid and fixed walls. At the moving wall $\Psi = 0$ is chosen in Γ. By Taylor expansion of (3a) on the boundary, and denoting by $h_x = h_y$ the space steps, the following equations are obtained:

$$\omega(0,y,t) = -\frac{1}{2h_x^2}[8\psi(h_x,y,t) - \psi(2h_x,y,t)] + O(h_x^2), \tag{11a}$$
$$\omega(a,y,t) = -\frac{1}{2h_x^2}[8\psi(a-h_x,y,t) - \psi(a-2h_x,y,t)] + O(h_x^2), \tag{11b}$$
$$\omega(x,0,t) == -\frac{1}{2h_y^2}\left[8\psi(x,h_y,t) - \psi(x,2h_y,t)\right] + O\left(h_y^2\right), \tag{11c}$$
$$\omega(x,b,t) = -\frac{1}{2h_y^2}\left[8\psi(x,b-h_y,t) - \psi(x,b-h_y,t)\right] - \frac{3}{h_y} + O\left(h_y^2\right), \tag{11d}$$

In Figure 1 we show the isovorticity contours for$Re = 7500$, in a) with $h_x = h_y = 1/128$ and in b) with $h_x = h_y = h = 1/384$, and $T_{final} = 100$ for both of them. We show the isovorticity contours, just to show the oscillations occurring on the top right corner of the wall. The mesh used in a) is a very coarse one, and that is why oscillations are occurring. In b), with a finer mesh, oscillations do not occur. It can be observed that the vorticity is accumulated around the walls.

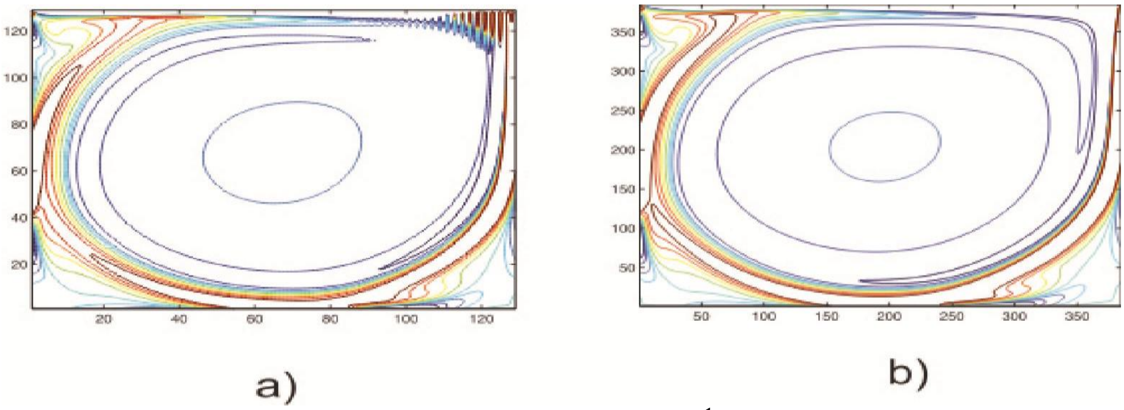

Figure 1.Isovorticity contours for Re=7500, with a) $h_x = h_y = \frac{1}{128}, \Delta t = .001, T_{final} = 100$, and with b) $h_x = h_y = \frac{1}{384}, \Delta t = .00025, T_{final} = 5$.

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In Figure 2, we show the isovorticity contours for $Re = 10000$, with $h_x = h_y = h = 1/384$, $T_{final} = 100$, and $\Delta t = .0025$.

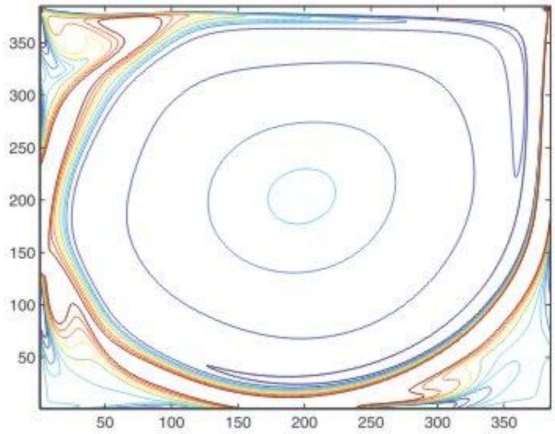

Figure 2.Isovorticity contours for Re=10000, with $h_x = h_y = \frac{1}{384}, \Delta t = .0025, T_{final} = 100$.

In Figure 3 we show the isovorticity contours for $Re = 25000$, with a mesh size given by $h = \frac{1}{768}$, the time step $\Delta t = .0025$, and $T_{final} = 5$.

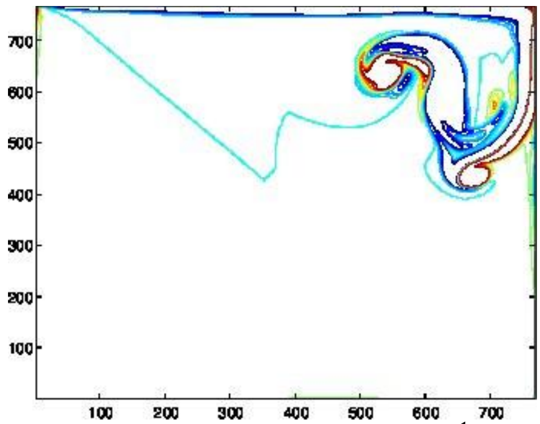

Figure 3.Isovorticity contours for Re=25000, with $h_x = h_y = \frac{1}{768}, \Delta t = .00025, T_{final} = 5$.

In Figure 4 we show the isovorticity contours for $Re = 31000$, with a mesh size given by $h = \frac{1}{512}$, the time step $\Delta t = .0025$, and $T_{final} = 5$.

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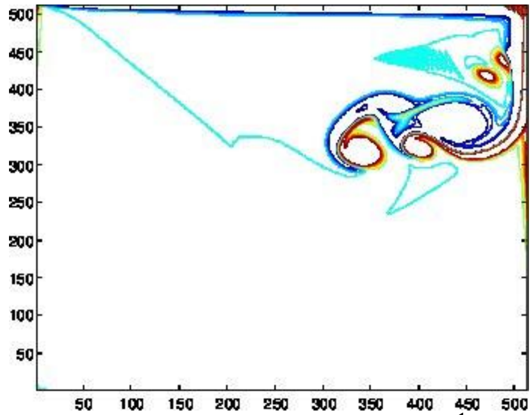

Figure 4.Isovorticity contours for Re=31000, with $h_x = h_y = \frac{1}{512}, \Delta t = .00025, T_{final} = 5.$

In Figure 5, we show the isovorticity contours for $Re = 50000$, with a mesh size given by $h = \frac{1}{1024}$, time step $\Delta t = 0.00025$, and $T_{final} = 5$.

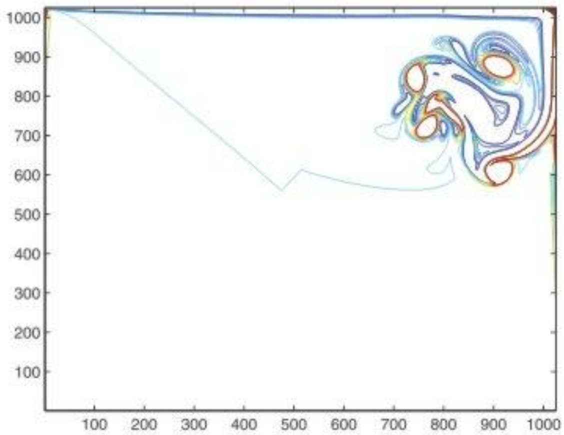

Figure 5.Isovorticity contours for Re=50000, with $h_x = h_y = \frac{1}{1024}, \Delta t = .00025, T_{final} = 5.$

As can be seen, the mesh size and the time step have to be reduced as the Reynolds number increases, in order to avoid oscillations. In Figure 2, for $Re = 10000$, it can be observed that the vorticity tends to spread through all the cavity. It can also be observed the for $Re = 31000$and$Re = 50000$, a turbulent part appears. This Reynolds numbers are very high, and it is not an easy task to deal with them.

High Reynolds numbers on the un-regularized driven cavity deserve some additional comments:
1) In Figure 1 a) and b) we see the isovorticity contours dependence on the h parameter for a Re= 7500. Figures 2 to 5 show that the actual numerical method can be successfully applied to describe the time dependent flows at high Reynolds numbers such as Re=50 000.

2) As the Reynolds number increases, going from Re=25 000 to 31 000, the structure of the turbulent flow changes drastically (according to the literature [4]) as in Figures 3 and 4. However in this work it has been forced the numerical calculation to describe driving flows at Re= 50 000. In the same reference it is found that in a range of Re= 25 000 to Re= 31 000 the end of the turbulent component goes down as long the Re increases. In this work we have found an opposite behavior in the extended interval of Re= 31 000 to Re= 50 000. The end of the turbulent flow component reduces in size to approximately the early form in Re=25 000 with a larger turbulent structure.

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3) This last circumstance justify the extension of the actual numerical procedure instead of the fixed point iterative method used before.

In Table 1, we show the times used by both methods. As can be seen, the second method resulted much faster than the fixed point iterative method, especially when increasing the Reynolds number. It is almost 3 times faster.

Re	H	Δt	T_{final}	F.P.I.M (time secs.)	With A and B (time secs.)
10000	1/384	.0025	12.5	46370	15400
10000	1/384	.0025	1000	362182	116814
31000	1/512	.00025	5	18864	10013
50000	1/1024	.0025	5	149370	55535

Table 1.Comparison of the two methods.

V. CONCLUSIONS

We are presenting two efficient numerical schemes for solving the Navier-Stokes in the Stream function-vorticity formulation. The idea of the fixed point iterative method was to work with a symmetric positive definite matrix (matrix A resulting from the discretization of the Laplacian term). This method showed to be robust enough to handle high Reynolds numbers, but computing time was, in general, very large. That is why we seek to reduce computing time, and implemented the second method.

Results with both methods, agree very well with those reported in the bibliography [4], [5], [6], but with the second method we were able to reduce computing time by almost three times, especially when increasing the Reynolds number. In this case, as we have said, smaller values of h have to be used, numerically for stability matters and physically, to capture the fast dynamics of the flow.

As can be seen from the results, fluid flows, at high Reynolds numbers approach two dimensional turbulence, caused by the recirculation in the driven cavity problem.The numerical results shows that the two dimensional flow, at high Reynolds numbers, presents great vorticity structure coming from recirculation of cavity. Under this circumstance, we still look forward to reduce the computing time of calculation. Results for even higher Reynolds numbers are also being obtained but these results will be reported in a future work.

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