

Mixed Convection Stagnation Point Flow of Casson Fluid with Convective Boundary Conditions

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ABSTRACT: Casson fluid flow has many practical applications food processing, metallurgy, drilling operations and bio-engineering operations. In the present paper we investigate the mixed convection stagnation-point flow of an incompressible non-Newtonian fluid over a stretching sheet with magnetic field under convective boundary condition. The resulting partial differential equations are converted into the ordinary differential equations by the suitable transformations. The velocity, temperature and concentration profiles are computed by employing the homotopy analysis method. The plotted graphs illustrate the flow and heat transfer characteristics and their dependence upon the embedded parameters. Numerical values of skin-friction coefficient, Nusselt number and Sherwood number are given and examined. Comparison of the present results with the existing solution is also given.

KEYWORDS: Casson fluid, Magnetic field, convective boundary condition, Stagnation point.

I. INTRODUCTION

The stagnation point flow and heat transfer over a stretching sheet are commonly seen in paper production, the spinning of fibers, glass blowing, continuous casting, manufacture of sheeting material through extrusion process especially in the polymer extrusion in a melt spinning process, aerodynamic extrusion of plastic sheets etc. Chiam[1] studied a two-dimensional stagnation point flow of viscous fluid towards a linear stretching surface. Mahapatra and Gupta[2] analyzed the heat transfer in the stagnation point flow towards a stretching surface. The steady stagnation point flow of an incompressible micropolar fluid over a stretching surface is addressed by Nazar et al.[3] The stagnation point flow of an upper convected Maxwell fluid has been numerically reported by Sadeghy et al.[4] The mixed convection stagnation point flow of an incompressible viscous fluid towards a vertical permeable stretching sheet is studied by Ishak et al.[5]. The effect of radiation on the mixed convection MHD stagnation point flow in a porous space is examined by Hayat et al.[6] Mahapatra et al.[7] carried out the analysis for the steady two-dimensional oblique stagnation point flow of an incompressible viscoelastic fluid towards a stretching sheet. Grosan et al.[8] carried out the analysis Magneto hydrodynamic oblique stagnation-point flow

Robert et al. carried out the analysis Hydromagnetic stagnation point flow of a viscous fluid over a stretching or shrinking sheet[9] Lok et al.[10] analysis Mixed convection flow of a micropolar fluid near a non-orthogonal stagnation-point on a stretching vertical sheet. Ali et al.[11] carried out the analysis MHD stagnation point flow and heat transfer towards stretching sheet with induced magnetic field. Ming et al.[12] presented the fractal analysis of Casson fluid flow through porous space. Attia, [13] carried out the analysis, Hydromagnetic stagnation point flow with heat transfer over a permeable surface. Garg[14] carried out the analysis, Heat transfer due to stagnation point flow of a non-newtonian fluid. Nakayama et al.[15] studied the analysis of, Free convection heat transfer over a non-isothermal body of arbitrary shape embedded in a fluid saturated porous media.. Hassanien, et al.[16] Effect of thermal dispersion and stratification on non-darcy mixed convection from a vertical plate in a porous medium. Dutta et al.[17] studied the, Temperature field in the flow over a stretching surface with uniform heat flux. Chiam[18] carried out the analysis, the Stagnation point flow towards a stretching plate.

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II.MATHEMATICAL FORMULATION

We consider a two-dimensional steady stagnation point flow of an incompressible Casson fluid over a stretching surface of uniform ambient temperature T_∞ . A Cartesian coordinate system is considered such that x -axis is along the stretching surface and the y -axis perpendicular to it. The velocity distribution is assumed to be in the forms $u_e(x) = ax$ and $v_e(y) = -ay$ and the velocity of the stretching surface is $u_w(x) = cx$, where a and c are positive constants. The flow is taken to be steady and the magnetic Reynolds number is considering to be very small so that the induced magnetic field can be neglected in comparison to the applied magnetic field. The rheological equation of state for an isotropic flow of Casson fluid is

$$\tau_{ij} = \begin{cases} 2\left(\mu_B + \frac{p_y}{\sqrt{2\pi}}\right)e_{ij}, \pi > \pi_c, \\ 2\left(\mu_B + \frac{p_y}{\sqrt{2\pi}}\right)e_{ij}, \pi < \pi_c \end{cases} \quad (1)$$

Where $\pi = e_{ij}e_{ij}$, and e_{ij} is the (i, j) th component of deformation rate, π is the product of the component of deformation rate with itself, π_c is the critical value of this product based on the non-Newtonian model, μ_B is the plastic dynamic viscosity of non-Newtonian fluid and p_y is the yield stress of fluid. The boundary layer equations for the considered problem are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (2)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial y} = u_e \frac{du_e}{dx} + v \left(1 + \frac{1}{\beta}\right) \frac{\partial^2 u}{\partial y^2} + g\beta_T (T - T_\infty) + g\beta_c (C - C_\infty) - \left(\frac{\sigma B_0^2}{\rho} u\right)$$

(3)

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} + \frac{v}{C_p} \left(1 + \frac{1}{\beta}\right) \left(\frac{\partial u}{\partial y}\right)^2$$

$$u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = D \frac{\partial^2 C}{\partial y^2} \quad (4)$$

The boundary conditions of the problems are

$$u = u_w(x) = cx, v = 0,$$

$$-k \frac{\partial T}{\partial y} = h(T_f - T), C = C_w \quad \text{at } y = 0 \quad (5)$$

$$u \rightarrow u_e = ax, T \rightarrow T_\infty, C \rightarrow C_\infty \quad \text{as } y \rightarrow \infty \quad (6)$$

where u and v are the velocity components in the x - and y -directions $\beta = \mu_B \sqrt{2\pi} / p_y$ is the non-Newtonian Casson parameter, g is the gravitational acceleration, β_T is the thermal expansion coefficient, β_c is the volumetric solute expansion coefficient, T is the fluid temperature, $\nu = (\mu_B/\rho)$ is the kinematic viscosity, C_p is the specific heat at constant pressure, α is the thermal diffusivity, k is the thermal conductivity of fluid, h is the convective heat transfer coefficient, and T_f is the convective fluid temperature.

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Equations (2)–(6) can be reduced to the dimensionless form by introducing the following new variables

$$u = cx f'(\eta), v = -\sqrt{cv} f(\eta) \tag{7}$$

$$\eta = y \sqrt{\frac{c}{v}}, \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \phi(\eta) = \frac{C - C_\infty}{C_w - C_\infty}$$

Now Eq. (2) is automatically satisfied and Eqs. (3)–(6) turn into the following forms

$$\left(1 + \frac{1}{\beta}\right) f''' + ff'' - f'^2 + a^2/c^2 + \lambda_1 \theta + \lambda_2 \phi - Mf' = 0 \tag{8}$$

$$\theta'' + Pr f\theta' + Pr \left(1 + \frac{1}{\beta}\right) Ecf'' = 0 \tag{9}$$

$$\phi'' + Scf\phi' = 0 \tag{10}$$

The corresponding boundary conditions are

$$f = 0, f' = 1, \theta' = -Bi(1 - \theta(0)), \phi = 1 \text{ at } \eta = 0, \tag{12}$$

$$f' = a/c, \theta = 0, \phi = 0 \text{ as } \eta \rightarrow \infty \tag{13}$$

Where $\lambda_1 = \frac{Gr_x}{Re_x^2} = \frac{g\beta_T(T_w - T_\infty)x}{u_e^2}$ is the local buoyancy parameter, $\lambda_2 = \frac{g\beta_C(C_w - C_\infty)x}{u_e^2}$ is the modified

local buoyancy parameter, $Re_x = \frac{u_e(x)x}{\nu}$ is the Reynolds number, $Pr = \nu/\alpha$ is the Prandtl number,

$Ec = \frac{u_w^2}{C_p(T_f - T_\infty)}$ is the Eckert number and $Bi = \frac{h}{k} \sqrt{\frac{\nu}{c}}$ is the Biot number, $M = \frac{\sigma B_0^2}{\rho D}$ is the magnetic number,

$Sc = \frac{\nu}{D}$ is the Schmidt number.

III. THE SKIN FRICTION COEFFICIENT, THE LOCAL NUSSOLT NUMBER AND LOCAL SHERWOOD NUMBER

$$C_f = \frac{\tau_w}{\rho u_w^2(x)}, Nu = \frac{xq_w}{k(T_f - T_\infty)}, Sh_x = \frac{xq_m}{D(C_w - C_\infty)}, \tag{14}$$

The shear stress at the surface of the wall τ_w is defined as

$$\tau_w = \left(\mu_B + \frac{\tau_y}{\sqrt{2\pi c}}\right) \frac{\partial u}{\partial y} \Big|_{y=0} = \left(\mu_B + \frac{\tau_y}{\sqrt{2\pi c}}\right) \left(\frac{a^3}{\nu}\right)^{1/2} xf''(0) \tag{15}$$

Using the above equation, the skin friction is written as

$$Re_x^{1/2} C_f = \left(1 + \frac{1}{\beta}\right) f''(0) \tag{16}$$

The surface flux at the wall is defined as

$$q_w = -k \frac{\partial T}{\partial y} \Big|_{y=0} = -k(T_w - T_\infty) \left(\frac{a}{\nu}\right)^{1/2} \theta'(0) \tag{17}$$

Using the above equation, the Local Nusselt number Nu_x is written as

$$Re_x^{1/2} Nu_x = -\theta'(0) \tag{18}$$

The mass flux at the surface is written as

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$$q_m = -D_s \left. \frac{\partial C}{\partial y} \right|_{y=0} = -D_s (C_w - C_\infty) \left(\frac{a}{\nu} \right) \phi'(0) \tag{19}$$

Using the above equation, the local Sherwood number Sh_x written as

$$Re_x^{-1/2} Sh_x = -\phi'(0) \tag{20}$$

IV.RESULTS AND DISCUSSIONS

In the present study, we solved the system of differential equations (8) –(11) with boundary condition Eqn. **Error! Reference source not found.** to quantify the effects of various physical parameters. We used similarity variables to transform the governing partial differential equations into ordinary differential equations and then solved the later using the ‘bvp4c’ Matlab package.

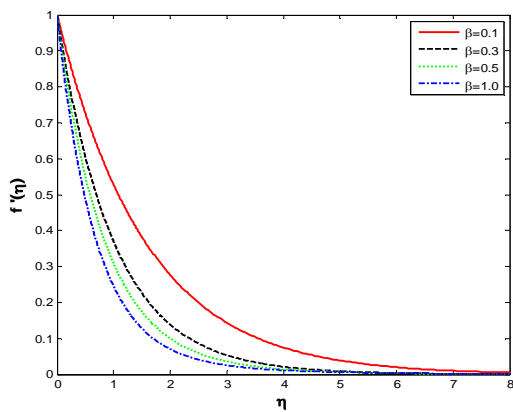


Fig.1 influence of β on (a) $f'(\eta)$

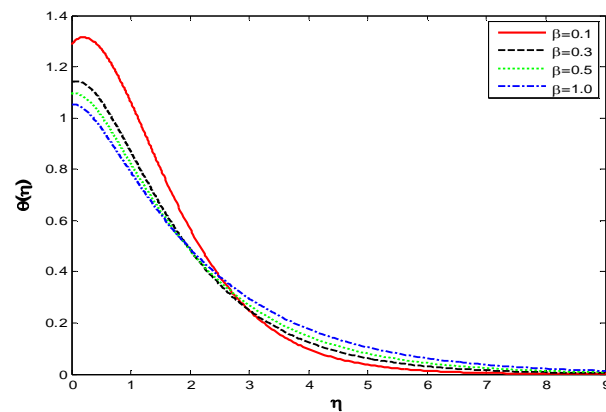


Fig.1 influence of β on (b) $\theta(\eta)$

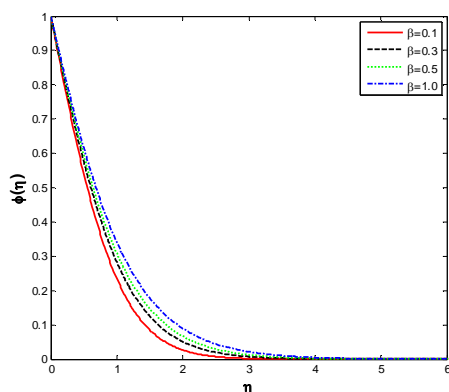


Fig.1 influence of β on (c) $\phi(\eta)$

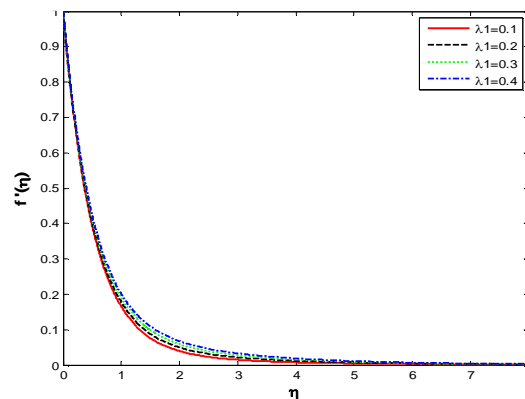


Fig.2 Influence λ_1 on (a) $f'(\eta)$

Figures 1(a) and 1(b) examine the influence of Casson fluid parameter β on the velocity and temperature profiles respectively. Both the velocity and temperature fields are decreasing function of β , where as concentration is a increasing function of Casson parameter of 1(c). The decreasing nature of the momentum of boundary layer thickness

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with increasing β is noted of velocity and temperature. However the thermal boundary layer thickness increases with increasing β of concentration.

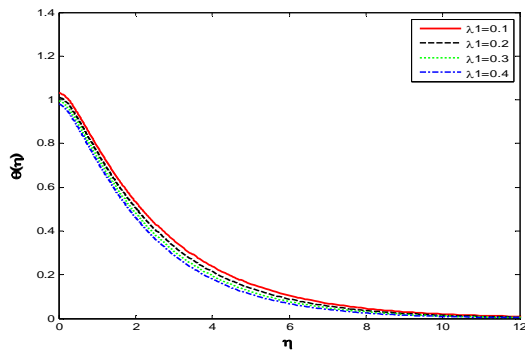


Fig.2 Influence λ_1 on (b) $\theta(\eta)$.

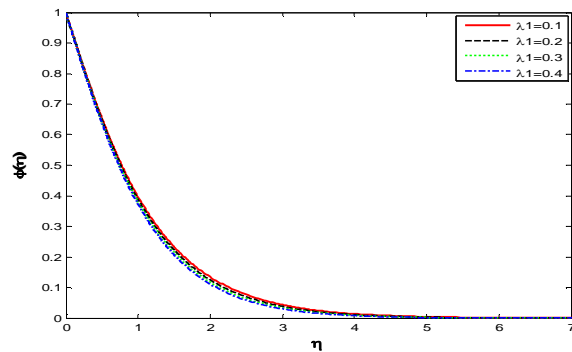


Fig.2 Influence λ_1 on (c) $\phi(\eta)$.

Fig.2 (a) visualizes the effect of local buoyancy parameter λ_1 on the velocity field $f'(\eta)$. There is a prominent increase in the velocity and boundary layer thickness when λ_1 increases. However, the temperature and concentration profile decreases by increasing λ_1 (see Fig. 2(b) and 2(c)).

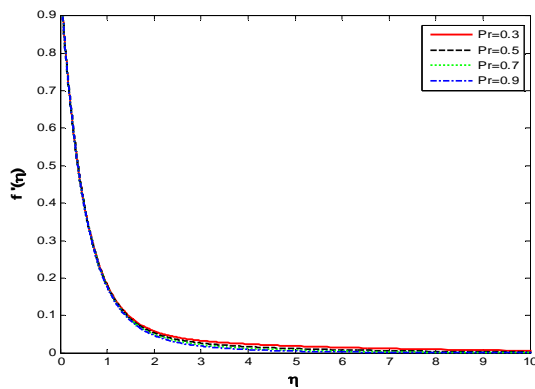


Fig.3 Influence Pr on (a) $f'(\eta)$

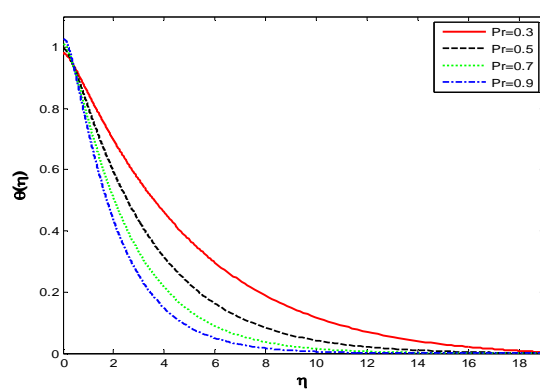


Fig.3 Influence Pr on (b) $\theta(\eta)$

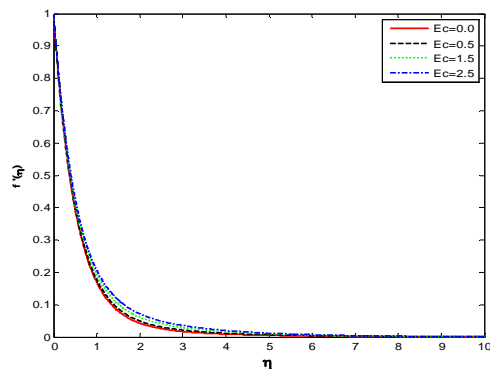


Fig.4 Influence Ec on (a) $f'(\eta)$

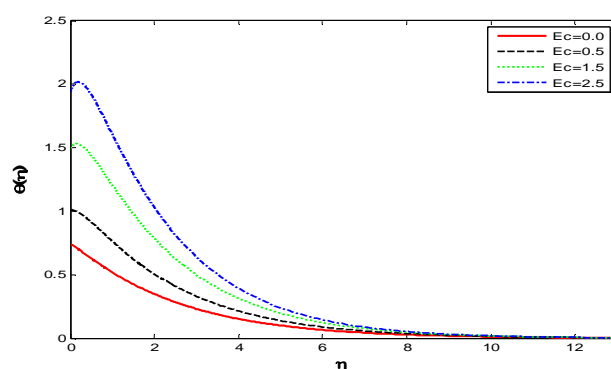


Fig.4 Influence Ec on (b) $\theta(\eta)$

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Figures 3(a) and 3(b) illustrate that an increase in the Prandtl number Pr decreases the velocity and temperature profiles. An increase in the Prandtl number also corresponds to the weaker thermal diffusivity which therefore reduces the thermal boundary layer thickness.

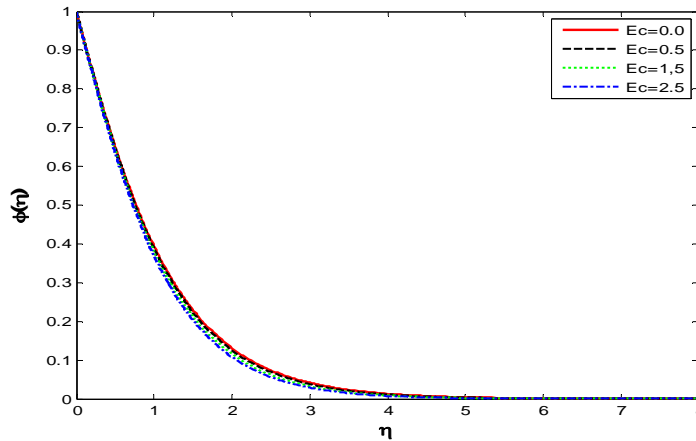


Fig.4 Influence Ec on (c) $\phi(\eta)$

The influence of Eckert number Ec on velocity and temperature fields is shown in Figs. 4(a) and 4(b). An increase in the Eckert number increases the thermal boundary thickness. Whereas the concentration and the momentum boundary layer thickness decrease with an increase in the Eckert number. It is important to note that $Ec = 0$ corresponds to the case of no viscous dissipation. When viscous dissipation effects increase, the temperature profile $\theta(\eta)$ shows a rapid increase. From Figure 5(a), we observe that first the velocity decreases rapidly with the increase of M . In addition, Figure 5(b) and 5(c) show that the temperature field as well as concentration increases with the increase of magnetic field. The magnetic field lines act like a string and tend to retard the motion of the fluid. The consequence of which is to increase the rate of heat transfer; temperature variation is prominent as seen in.

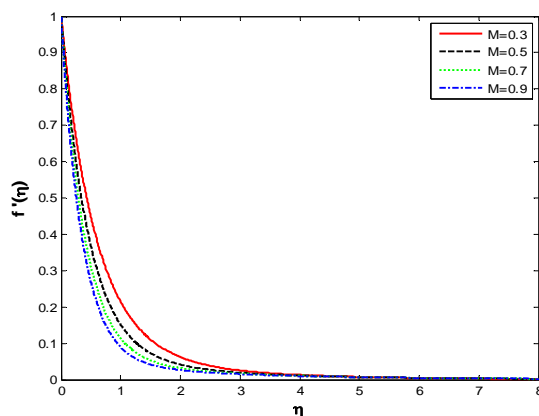


Fig.5 Influence M on (a) $f'(\eta)$

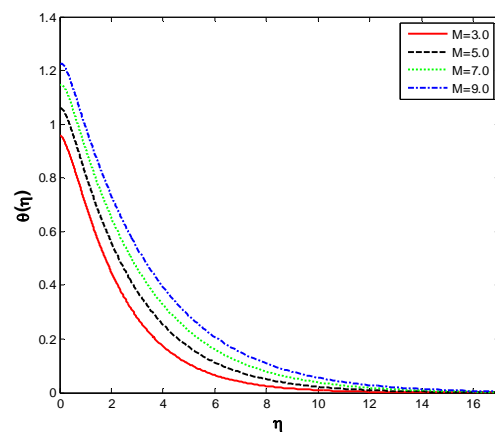


Fig.5 Influence M on (b) $\theta(\eta)$

Figure 6 shows the behavior of velocity for different values of Biot number Bi . The numerical results show that the effect of increasing values of Biot number results in a decreasing velocity. It is observed that an increase in the Biot number results in an increase of the thermal boundary layer thickness within the boundary.

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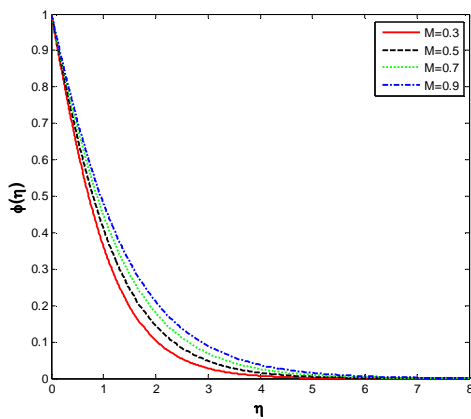


Fig.5 Influence M on (c) $\phi(\eta)$

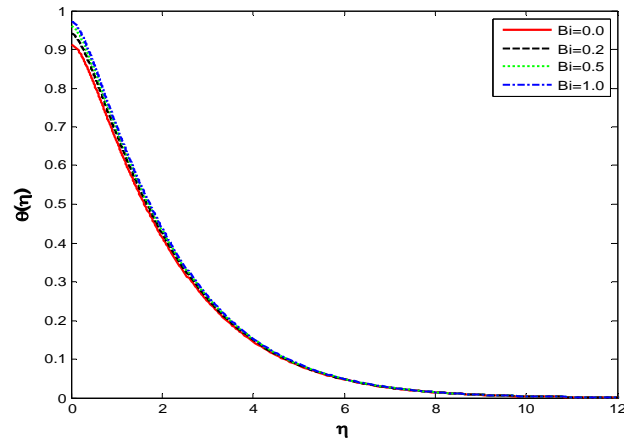


Fig.6 Influence of Pr on $f'(\eta)$

. The effect of stretching ratio a/c on the velocity field $f'(\eta)$ is observed in Fig. 7(a). Figure 7(a) exhibits that velocity $f'(\eta)$ is an increasing function of a/c . Figure 7(b) & (c) indicates that an increase in a/c decreases the dimensionless temperature and concentration profile. Fig. 8(a)&8(b) illustrates the effect of the Schmidt number (Sc) on the velocity. The Schmidt number (Sc) embodies the ratio of the momentum diffusivity to the mass (species) diffusivity. It is noticed that as Schmidt number (Sc) increases the velocity field decreases and temperature field increases with increase Sc . Fig. 8(c) illustrates the effect of Schmidt number (Sc) on the concentration. It is noticed that as the Schmidt number increases, there is a decreasing trend in the concentration field. Fig. 9 (a). There is a prominent increase in the velocity and boundary layer thickness increases when λ_2 increases. However, the temperature profile decreases by increasing λ_2 (see Fig. 9(b)).

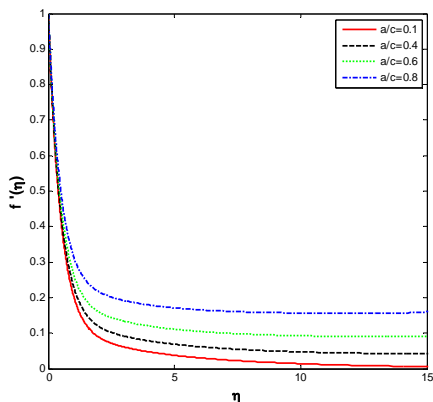


Fig.7 Influence of a/c on a) $f'(\eta)$

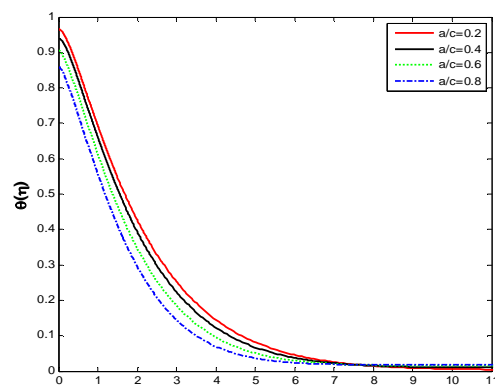


Fig.7 Influence of a/c on b) $\theta(\eta)$

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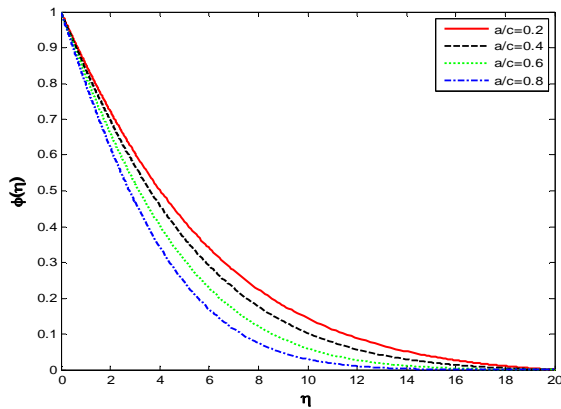


Fig.7 Influence of a/c on a) $\phi(\eta)$

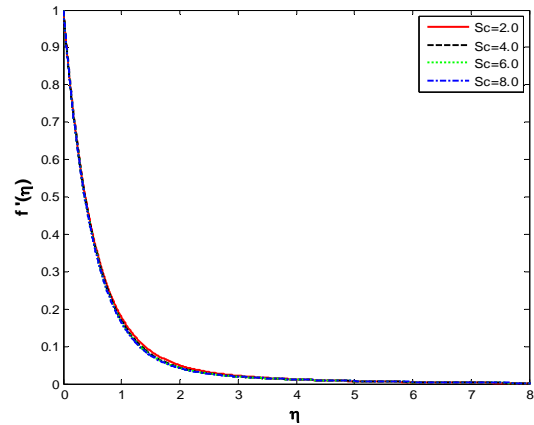


Fig.8 Influence of Sc on a) $f'(\eta)$

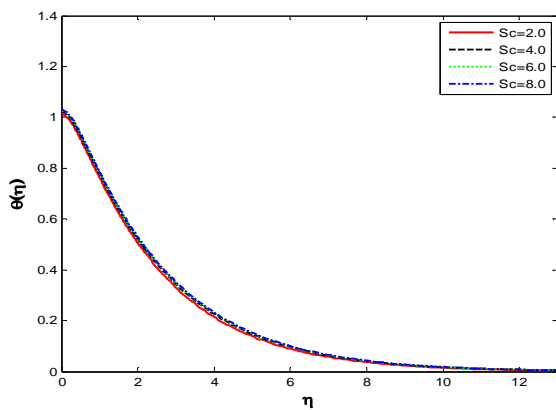


Fig.8 Influence of Sc on b) $\theta(\eta)$

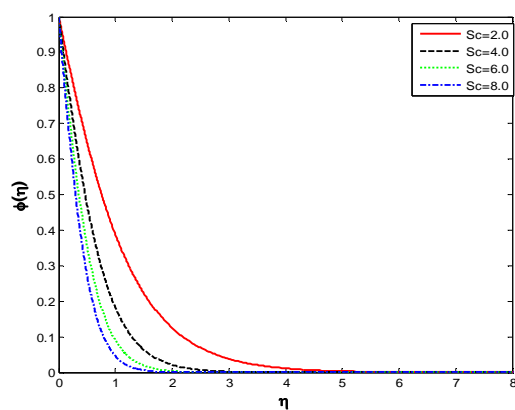


Fig.8 Influence of Sc on c) $\phi(\eta)$

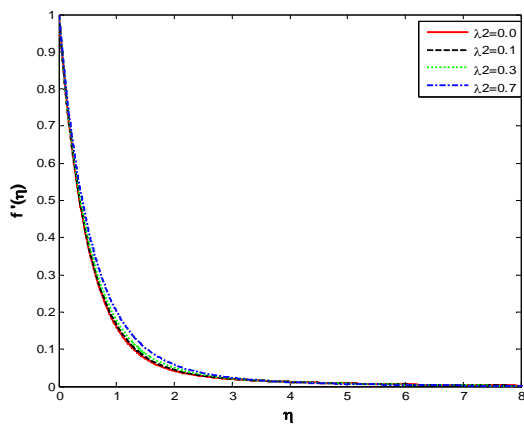


Fig.9 Influence of Sc on a) $f'(\eta)$

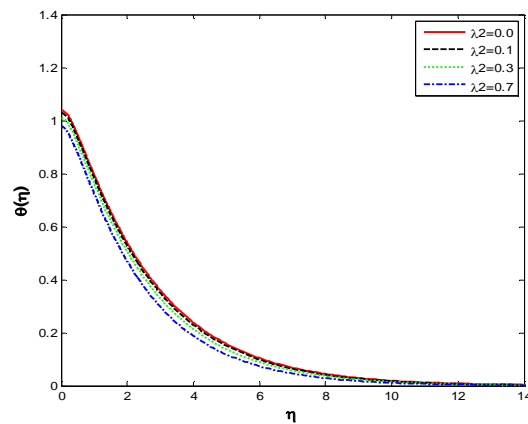


Fig.9 Influence of Sc on b) $\theta(\eta)$

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Table1: Comparison for values of $f''(0)$ with Mahapatra and Gupta[2] and Ishak et al.[5] T. Hayat et.al for different values of a/c when $\beta \rightarrow \infty$

a/c	Mohapatra&Gupta	Ishak et.al	Hayat et.al	Present Result
0.01		-0.9980	-0.998024	-0.998024
0.10	-0.9694	-0.9694	-0.969386	-0.969386
0.20	-0.9181	-0.9181	-0.918107	-0.918107
0.50	-0.9181	-0.6673	-0.667263	-0.667263
2.00	2.0175	2.0175	2.01756	2.017502
3.00	4.7294	4.7294	4.72963	4.729280

Table 2: Numerical values of skin-friction coefficient $\left(1 + \frac{1}{\beta}\right)f''(0)$ and local Nusselt number $-\theta'(0)$ and Sherwood number $-\phi'(0)$ for different values of $\beta, a/c, Pr, Ec, \lambda_1, \lambda_2, M, Sc$ and Bi .

β	a/c	Pr	Ec	λ_1	λ_2	M	Sc	Bi	$f''(0)$	$-\theta'(0)$	$-\phi'(0)$
0.1	0.02	0.7	0.5	0.2	0.3	4.0	1.0	1.0	0.632948	-0.278650	0.660350
0.3									0.996894	-0.126107	0.586322
0.5									1.192402	-0.081059	0.551211
1.0									1.453107	-0.040014	0.510483
	0.1								1.450182	-0.037690	0.511904
	0.4								1.404225	-0.003493	0.532061
	0.6								1.342487	0.037270	0.554848
	0.8								1.255658	0.087784	0.581901
	0.3								1.451360	0.005990	0.515136
	0.5								1.452442	-0.016077	0.512469
	0.7								1.453107	-0.040014	0.510483
	0.9								1.453496	-0.064857	0.508971
		0.0							1.473742	0.277664	0.503891
		0.5							1.453107	-0.040014	0.510483
		1.5							1.415326	-0.624839	0.521850
		2.5							1.422014	-1.227488	0.523019
			0.1						1.46240	-1.33137	0.509937
			0.2						1.401606	-1.190618	0.527299
			0.3						1.347685	-1.07730	0.541266
			0.4						1.298879	-0.98251	0.553028
			0.0						1.442481	-1.265558	0.518613
			0.1						1.422014	-1.227488	0.523019
			0.3						1.381255	-1.154890	0.531462
			0.7						1.300396	-1.022394	0.547090
				3.0					1.188807	0.057120	0.557699
					5.0				1.524504	-0.053033	0.506036
						7.0			1.806179	-0.149688	0.467031
						9.0			2.053134	-0.236489	0.436137
							2.0		1.389974	-0.025278	0.836826
							4.0		1.415934	-0.045213	1.290652
							6.0		1.430330	-0.053632	1.644529
							8.0		1.439855	-0.058286	1.944422
								0.0	1.944422	-0.000000	0.529862
								0.2	1.365280	0.000020	0.529862
								0.5	1.365278	0.000033	0.529862
								1.0	1.365277	0.000044	0.529863

Table 1, gives the comparison of present results with already existing numerical solutions. An excellent agreement between the presented solutions is noticed. Table 2, analyzes the variations of physical parameters on skin friction

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coefficient and the local Nusselt number and local Sherwood number for some values of β , a/c , Pr , Ec , λ_1 , Bi , λ_2 , M , Sc .

V.CONCLUSION

The nonlinear problem of stagnation point flow of Casson fluid over a stretching surface subject to convective boundary conditions with magnetic field. The main points of the present study are listed as follows:

1. The velocity and temperature profiles decrease when the non-Newtonian Casson parameter increases β and the concentration profiles increases.
2. There is a prominent increase in the velocity and boundary layer thickness when λ_1 increases. However, the temperature and concentration profile decreases by increasing λ_1 .
3. An increase in the Prandtl number also corresponds to the weaker thermal diffusivity which therefore reduces the thermal boundary layer thickness.
4. An increase the Eckert number thermal boundary thickness increases. Whereas the concentration and the momentum boundary layer thickness decreases with increase of Eckert number.
5. The temperature field as well as concentration increases with the increase of magnetic field.
6. The influence of the Biot number Bi on velocity and temperature profiles are qualitatively similar.
7. The velocity $f'(\eta)$ is an increasing function of a/c . An increase in a/c decreases the dimensionless temperature and concentration profile.

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