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# Dynamic Behaviour of a Cantilever Beam Subjected To Moving Mass and Elastic End Constraint

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**ABSTRACT:** Free vibration of an isotropic and linearly elastic beam is examined, based on Euler-Bernoulli beam theory. The beam is fixed on one end and subjected to an elastic end-constraint at the other. Newton's laws are used to formulate the differential equation of motion governing the system characteristics. A moving mass displaces from one end to the other. Dunkerley's method of natural frequency approximation is used to evaluate the effect of several system parameters on the natural frequency of the beam. The effect of the dynamics of the moving mass and the elastic end-constraint on the natural frequency of the beam is plotted and analysed using MATLAB.

**KEYWORDS:** Isotropic, Euler-Bernoulli beam, Elastic end-constraint, Dunkerley's method.

### I. INTRODUCTION

The moving load problem, evaluation of natural frequency and modes of vibration are some fundamental problems in structural dynamics. There have been investigations of the potential hazard caused by the moving masses on structures for several years. The dynamic response of structures carrying moving masses is a problem of practical significance. Beam type structures are extensively used in mechanical, civil, and aerospace engineering. The importance of moving mass lies in several applications in the field of transportation. Suspension bridges, railway and highway bridges, rails, roadways, tunnels and pipelines are example of structural elements that are needed to be designed to support moving masses. In the design of machining processes, many members can be modelled as beams acted upon by moving loads. The majority of the engineering structures are subjected to space and time varying loads. Moving loads have substantial effects on the dynamic behaviour of the engineering structures. The simplest case of a moving load investigation is the case of a simple beam over which a concentrated load is moving. While designing a mechanical, civil or aerospace system, the vibration aspect plays a very crucial role. Hence, the natural frequency estimation is of immense importance in the design of the structural components of such systems. Hence, the objective of this paper is to analyse the effect of spring rate of the elastic end-constraint, mass of the object and velocity of the object, on the natural frequency of the system.

### II. LITERATURE REVIEW

In the past, many researchers have investigated vibration and stability of various beam structures subjected to diverse loading and boundary conditions. The response of beams under the action of moving loads has attracted the attention of many scientists since the nineteenth century.

Mackertich [1] used the modal superposition method for the beam deflection and compared the response of a Timoshenko beam to a moving mass to that of a Bernoulli-Euler one. He approximated the total time derivative of the mass displacement by the partial derivative to put off the difficulty that arises from the existence of a coupling term in the mass acceleration expression.

Michaltsos et al. [2] followed the first approximation technique to derive a series solution for beam dynamic deflection in terms of beam normal modes without the effect of the mass inertia. He studied the linear dynamic

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response of a simply supported uniform beam under a moving load of constant magnitude and velocity by including the effect of its mass.

Ting et al. [3] formulated and solved the problem using the influence coefficients method. The inertial effects of the beam were considered as applied external forces, and again at each position of the mass, numerical integration had to be performed over the length of the beam. He developed an algorithm to solve the classical problem of the dynamic response of a finite elastic beam supporting a moving mass.

Lee [4] studied extensively the dynamic responses of a beam acted upon by moving forces or moving masses, in connection with the design of railway tracks and bridges and machining processes. The equation of motion in matrix form has been formulated for the dynamic response of a beam acted upon by a moving mass by using the Lagrangian approach.

Siddiqui et al. [5] developed a technique to obtain numerical solution over a long range of time for non-linear multi-body dynamic systems undergoing large amplitude motion. The system considered is an idealization of an important class of problems characterized by non-linear interaction between continuously distributed mass and stiffness and lumped mass and stiffness.

Thus, the governing differential equations of motion for various beam configurations and diverse loading conditions have already been developed. However, the literature on the approximation of the natural frequency of the system with Dunkerlay's approach is rare. This paper initiates a subtle approach to solve the problem using approximate methods.

### III. THEORY

Notations used in the text are below:

L = length of the cantilever beam

I = area moment of inertia

$\rho$  = density

A = cross-sectional area

E = Elastic Modulus of the cantilever material

k = spring rate of the elastic end-constraint

m = mass of the object over the beam

v = velocity of the mass

t = time

g = gravitational acceleration ( $9.8 \text{ ms}^{-2}$ )

#### Differential equation of motion of the cantilever beam (Newtonian Approach):

Newton's second law of motion is used to obtain the governing differential equation of motion of a cantilever beam. This is the well-known ordinary differential equation of a beam, undergoing transverse vibration. A schematic of the system is shown in Figure 1.

$$EI \frac{\partial^4 y}{\partial x^4} + \rho A \frac{\partial^2 y}{\partial t^2} = 0 \quad \text{----- (i)}$$

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The solution to this equation is  $y = y(x, t) = f(x) g(t)$

$$f(x) = A \sin(px) + B \cos(px) + C \sinh(px) + D \cosh(px)$$

$$g(t) = E \cos(\omega t) + F \sin(\omega t)$$



Figure 1: Beam with a moving mass and elastic end constraint

Figure 1 shows the schematic representation of the model. At any instant of time, say  $t$ , the mass has moved a distance  $x$ . There is an elastic end-constraint at the free end of the cantilever beam. The material and geometric properties of the beam is mentioned in the figure.

The following characteristic equation relates the circular frequency  $\alpha$  to the wave number  $p$ :

$$\alpha^2 = \frac{EI}{\rho A} p^4 \quad \text{----- (ii)}$$

Boundary Conditions:

$$\text{At } x=0, \quad y = 0 \quad \frac{\partial y}{\partial x} = 0$$

$$\text{At } x = L, \quad \frac{\partial^2 y}{\partial x^2} = 0 \quad EI \frac{\partial^3 y}{\partial x^3} = ky$$

Using the boundary conditions, the following equations are obtained:

$$-A \sin(pL) - B \cos(pL) - A \sinh(pL) - B \cosh(pL) = 0$$

$$-A \left\{ p^3 \cos(pL) + \frac{k}{EI} \sin(pL) \right\} + B \left\{ p^3 \sin(pL) - \frac{k}{EI} \cos(pL) \right\} = A \left\{ p^3 \cosh(pL) - \frac{k}{EI} \sinh(pL) \right\} + B \left\{ p^3 \sinh(pL) - \frac{k}{EI} \cosh(pL) \right\}$$

Rewriting the above equations in matrix form,

$$\begin{bmatrix} \sin(pL) + \sinh(pL) & \cos(pL) + \cosh(pL) \\ -p^3 \cos(pL) - p^3 \cosh(pL) - \frac{k}{EI} \sin(pL) + \frac{k}{EI} \sinh(pL) & p^3 \sin(pL) - p^3 \sinh(pL) - \frac{k}{EI} \cos(pL) + \frac{k}{EI} \cosh(pL) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

In order to have non-trivial solution, the determinant of the leftmost matrix has to be equated to zero. And this gives the values of  $pL$ . For different values of  $pL$ , the corresponding values of natural frequencies can be obtained, from equation (ii).

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Dunkerley’s method of natural frequency approximation:

Dunkerley’s method of natural frequency approximation is used to combine different natural frequencies of several components of a system. For the problem in hand, the natural frequency modes of the cantilever beam can be evaluated using equation (ii). For the natural frequency of the other component of the system, the elementary concepts of Strength of Materials are used. In this case, it is considered that, the beam does not have any mass, but it has a stiffness value. The mass is acted upon by the gravitational field. As a result, during its movement, it deflects the beam. At different points of time, it has a specific distance moved and correspondingly deforms the beam. E.g., the beam deflection is zero and maximum, when the mass is at the fixed end and at free end, respectively. A subtle approach is used to find the natural frequency of the system. The deflection of the beam due to the mass is calculated first. Then, by laws of static equilibrium, the natural frequency  $\beta$  is obtained. Finally, the method of reciprocals is used to approximate the natural frequency. The mathematical formulations are explained below.

$$\delta = \frac{P}{6EI} (3x^2L - x^3)$$

$$x = vt$$

$$P = mg$$

$$\therefore \delta = \frac{mg}{6EI} (3v^2t^2L - v^3t^3)$$

$$\beta^2 = \frac{k}{m} = \frac{g}{\delta} = \frac{6EI}{m(3v^2t^2L - v^3t^3)}$$

Dunkerley’s method of natural frequency approximation states that the reciprocal of the square of the natural frequency of the system is equal to the sum of the individual reciprocals of the natural frequencies of each components of the system. Hence,

$$\frac{1}{\omega^2} = \frac{1}{\alpha^2} + \frac{1}{\beta^2} \quad \text{----- (iii)}$$

$$\omega^2 = \frac{6EI k^4}{6\rho A + mk^4(3x^2L - x^3)}$$

$$\omega^2 = \frac{6EI k^4}{6\rho A + mk^4(3v^2t^2L - v^3t^3)}$$

## IV. RESULTS

For the simulation and modelling of the cantilever beam system, the value of the spring rate,  $k$  is taken  $10^5 \text{ N/m}^2$ . The elastic end-constraint at the free end of the cantilever beam increases the stiffness of the system, thus increasing the natural frequency. In fact, this is a subtle technique to increase the system natural frequency. Also, intermediate implementation of elastic constraints increases the stiffness of the system, thereby increasing the natural frequency.

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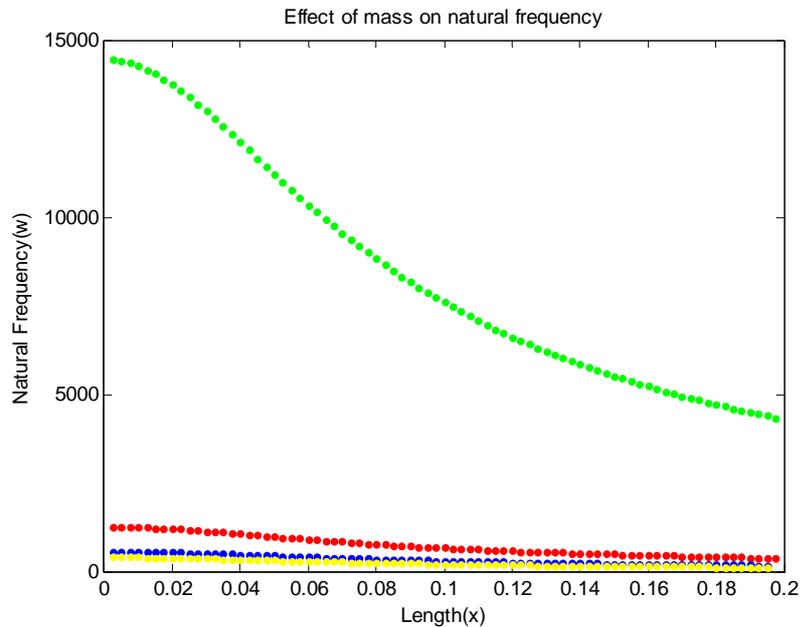


Figure 2: Effect of mass,  $m$  on  $\omega$

In Figure 2, the effect of mass of the moving object on the natural frequency is shown, where we find that with the increase in the mass of the object, the natural frequency of the system falls drastically. For a velocity of  $v = 0.25$  m/s, and beam length = 2.5 m, mass1 (marked green) = 1 kg, mass2 (marked red) = 10 kg, mass3 (marked blue) = 50 kg, mass4 (marked yellow) = 100 kg, the effect of mass of the object on system behaviour is shown in Figure 2.

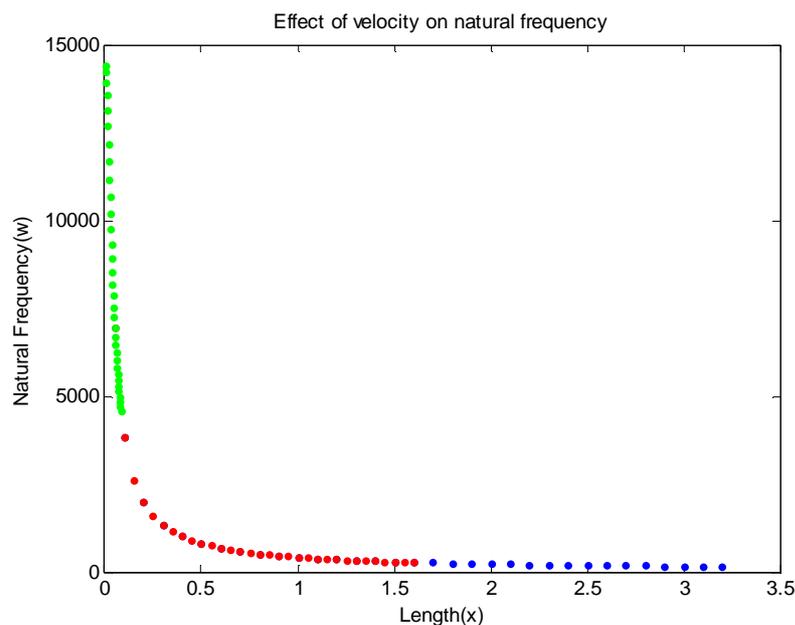


Figure 3: Effect of velocity,  $v$  on  $\omega$

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In Figure 3, for an object of mass,  $m = 5$  kg, the effect of different velocities on the system natural frequency is shown, where,  $v_1$  (marked green) = 0.25 m/s,  $v_2$  (marked red) = 5 m/s,  $v_3$  (marked blue) = 10 m/s. Here, too, the spring rate of the elastic end-constraint is taken to be of the same value as in the previous case. The slope of the curve is more when the velocity is in the lower range, although, the natural frequency is at a higher value.

### V. CONCLUSION

In Figure 2, the transition of natural frequency with increasing time and velocity is seen. No explicit solution could be found due to the intricacies in the evaluation of the solution to the transcendental equation, equation (iv), obtained during mode shape evaluation. MATLAB was used to solve the equation, but the solution couldn't be obtained. The following code was executed in MATLAB to estimate the wave number, so that the natural frequency of the beam could be evaluated. But complexity arose because of the presence of the end constraint.

$$\text{solve}(((p^3)*\sin(p*L)-(p^3)*\sinh(p*L)-(k/(E*I))*\cos(p*L)+(k/(E*I))*\cosh(p*L))*(\sin(p*L)+\sinh(p*L)) \\ = -(p^3)*\cos(p*L)-(p^3)*\cosh(p*L)-(k/(E*I))*\sin(p*L)+(k/(E*I))*\sinh(p*L)); \quad \text{----- (iv)}$$

To obtain an approximate solution, the value of the stiffness was not taken into account for the elastic end-constraint at the free end. Using high computing facilities and numerical methods, the solution can be reached. After having obtained the wave number, the natural frequency of the cantilever beam can be calculated. And, from the elementary strength of materials, the natural frequency due to the presence of the additional mass can be evaluated. The two natural frequencies can be used in equation (iii), to obtain the natural frequency of the entire system. In the past, many researchers have analysed the dynamic behaviour of a cantilever beam, under the action of moving loads, using techniques like double Laplace transform, Green's function, influence coefficients method, modal superposition method and so on. But, in this paper, a subtle approach leads us to analyse the effects of the moving mass on the system natural frequency. In real-life situations, a lot of problems similar to one analysed here appears and there is a need of designing the structural members. Such a simple approach can lead us to satisfactory results.

### REFERENCES

- [1]Mackertich S., "Response of a beam to a moving mass", The Journal of the Acoustical Society of America, Volume 92, Issue 3, pp.1766–1769, 1992.
- [2]Michaltsos G. et al., "The effect of moving mass and other parameters on the dynamic response of a simply supported beam", Journal of Sound and Vibration, Volume 191, Issue 3, pp. 357–362, 1996.
- [3]Ting E.C. et al., "A general algorithm for moving mass problems", Journal of Sound and Vibration, Volume 33, Issue 1, pp. 49–58, 1974.
- [4]Lee H. P., "Dynamic response of a beam with a moving mass", Journal of Sound and Vibration, Volume 198, Issue 2, pp. 249–256, 1996.
- [5]Siddiqui S.A.Q. et al., "Large free vibrations of a beam carrying a moving mass", International Journal of Non-Linear Mechanics, Volume 38, Issue 10, pp. 1481–1493, 2003.
- [6]Dehestani M. et al., "Investigation of critical influential speed for moving mass problems on beams", Elsevier, Volume 33, Issue 10, pp. 3885–3895, 2009.
- [7]Uzzal et al., "Dynamic response of a beam subjected to moving load and moving mass supported by Pasternak Foundation", Shock and Vibration, Volume 19, Issue 2, pp. 205–220, 2012.
- [8]Kelesoglu O. et al., "Comparing numerical methods for response of beams with moving mass", Elsevier, Volume 41, Issues 7–8, pp. 976–980, 2010.
- [9]Mehmood A. et al., "Vibration Analysis Of Beam Subjected To Moving Loads Using Finite Element Method", IOSR Journal of Engineering, Vol. 04, Issue 05, pp. 07–17, 2014.