

Upper Bound of an Achromatic Index of K_{15} is 48

S.D.Deo.¹, Ganesh Vishwas Joshi²Research Guide, Dept. of Mathematics, N.S.College, Bhadrawati, Dist. Chandrapur, Maharashtra, India¹Associate Professor, Dept. of Mathematics, Maharshi Dayanand College, Parel, Mumbai, Maharashtra, India²**ABSTRACT:** It is known that $41 \leq A(15) \leq 49$. In this paper we have tightened the upper bound of $A(15)$ to 48.**KEYWORDS:** Achromatic Index, Complete colouring, complete graphs

I. INTRODUCTION

A k -edge colouring of a simple graph G is assigning k colours to the edges of G so that any two adjacent edges receive different colours. If for each pair t_i & t_j of colours, there exist adjacent edges with these colours then the colouring is said to be complete. Let $G=K_n$ be a complete graph. The achromatic index $A(n)$ of the graph G is the maximum number of colours used in the edge colouring of G such that the colouring is complete. All though $A(n)$ is known for some complete graphs but in general it is not known for many complete graphs of order greater than 13. The number for the remaining complete graphs is still a challenge & in these cases only bounds for the index are known.

II. PROOF OF $A(15) \neq 49$

It is already known that $41 \leq A(15) \leq 49$.

It is very easy to see every colour in the optimal colouring has at least two edges.

If some colour has only one edge in the optimal colouring then this colour can be adjacent to at most 26 different colours at the extremities of this colour edge & that gives $A(15) \leq 26+1$ which gives contradiction.

Hence every colour in the optimal colouring of K_{15} has at least two edges.

Now suppose $A(15) = 49$.

Let C be an optimal edge complete colouring of K_{15} with 49 colours.

As $|E(K_{15})| = 105$ & $2.A(K_{15}) = 98$.

Therefore there exists at least one colour in C with at least 3 edges are coloured with it.

Suppose C has at least 3 edges of a colour k .

wlg $V_1, V_2, V_3, V_4, V_5, V_6$ belongs to $V(K_{15})$ be the vertices such that the edges V_1V_2, V_3V_4, V_5V_6 are coloured k .

Claim: If any edge V_iV_j has a colour t (where $1 \leq i < j \leq 6$) then t has at least 3 edges in C .

Suppose not,

Then there are exactly two edges of colour t in C .

Case i) t is adjacent to k exactly in two vertices, wlg say the edges V_2V_4 & $V_{14}V_{15}$ are coloured with t .

The number of distinct colours other than k that are adjacent to t are at most $12+12+11+11=46$ (namely the colours of the edges $V_2V_3, V_2V_5, V_2V_6, V_2V_7, V_2V_8, V_2V_9, V_2V_{10}, V_2V_{11}, V_2V_{12}, V_2V_{13}, V_2V_{14}, V_2V_{15}, V_4V_1, V_4V_5, V_4V_6, V_4V_7, V_4V_8, V_4V_9, V_4V_{10}, V_4V_{11}, V_4V_{12}, V_4V_{13}, V_4V_{14}, V_4V_{15}, V_{14}V_1, V_{14}V_3, V_{14}V_5, V_{14}V_6, V_{14}V_7, V_{14}V_8, V_{14}V_9, V_{14}V_{10}, V_{14}V_{11}, V_{14}V_{12}, V_{14}V_{13}, V_{15}V_1, V_{15}V_3, V_{15}V_5, V_{15}V_6, V_{15}V_7, V_{15}V_8, V_{15}V_9, V_{15}V_{10}, V_{15}V_{11}, V_{15}V_{12}, V_{15}V_{13}$)

$A(15) \leq 46+1+1$ (at most the above 46 colours in addition with colour k & colour t) which contradicts to the assumption.

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Case ii) if t is adjacent to k for more number of times, so wlg V_1V_2, V_3V_4, V_5V_6 are coloured k & V_2V_4 & V_1V_5 (or V_1V_3 or V_3V_5) are coloured with t , then counting in the similar manner as above it gives $A(15) \leq 48$ contradiction to the assumption.

Or

if t is adjacent to k for more number of times, so wlg V_1V_2, V_3V_4, V_5V_6 are coloured k & V_2V_4 & V_1V_i (or V_3V_i or V_5V_i) for some i where $7 \leq i$ are coloured with t then counting in the similar manner as above it gives $A(15) \leq 47$ contradiction to the assumption, hence the claim.

Let $S = \{V_i : \text{there exist some colour which appears maximum times in } C \text{ \& which is incident to } V_i\}$

The above claim shows that the subgraph H of K_{15} spanned by S must be $K_{|S|}$ & the number of edges in it is $|S| \cdot |S-1|/2$. Also it contains all the edges whose colours are appearing at least three times in C .

Let $|T|$ be the total number of edges of those colours which appear at least three times in C .

Therefore $|T| \geq |S| \cdot |S-1|/2$ (*)

Now C can be of one of the following combinations.

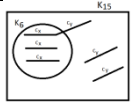
- a) If 47 colours with two edges each & 1 colour with six edges & 1 colour with five edges,
Then $|T| = 11$ & $|S| \geq 12$, hence $|T| = 11$ & $|S| \cdot |S-1|/2 \geq 66$. Therefore $11 \geq 66$ gives a contradiction.
- b) If 47 colours with two edges each & 1 colour with seven edges & 1 colour with four edges,
Then $|T| = 11$ & $|S| \geq 14$, hence $|T| = 11$ & $|S| \cdot |S-1|/2 \geq 91$. Therefore $11 \geq 91$ gives a contradiction.
- c) If 46 colours with two edges each & 2 colours with four edges each & 1 colour with five edges,
Then $|T| = 13$ & $|S| \geq 10$, hence $|T| = 13$ & $|S| \cdot |S-1|/2 \geq 45$. Therefore $13 \geq 45$ gives a contradiction.
- d) If 46 colours with two edges each & 2 colours with five edges each & 1 colour with three edges,
Then $|T| = 13$ & $|S| \geq 10$, hence $|T| = 13$ & $|S| \cdot |S-1|/2 \geq 45$. Therefore $13 \geq 45$ gives a contradiction.
- e) If 46 colours with two edges each & 1 colour with four edges & 1 colour with three edges, 1 colour with six edges, then $|T| = 13$ & $|S| \geq 12$ hence $|T| = 13$ & $|S| \cdot |S-1|/2 \geq 66$. Therefore $13 \geq 66$ gives a contradiction.
- f) If 46 colours with two edges each & 2 colours with three edges each & 1 colour with seven edges,
Then $|T| = 13$ & $|S| \geq 14$ hence $|T| = 13$ & $|S| \cdot |S-1|/2 \geq 91$. Therefore $13 \geq 91$ gives a contradiction.
- g) If 45 colours with two edges each & 2 colours with three edges each & 1 colour with four edges, 1 colour with five edges, then $|T| = 15$ & $|S| \geq 10$ hence $|T| = 15$ & $|S| \cdot |S-1|/2 \geq 45$. Therefore $15 \geq 45$ gives a contradiction.
- h) If 45 colours with two edges each & 1 colour with three edges & 3 colours with four edges
Then $|T| = 15$ & $|S| \geq 8$ hence $|T| = 15$ & $|S| \cdot |S-1|/2 \geq 28$. Therefore $15 \geq 28$ gives a contradiction.
- i) If 45 colours with two edges each & 3 colours with three edges & 1 colour with six edges,
Then $|T| = 15$ & $|S| \geq 12$ hence $|T| = 15$ & $|S| \cdot |S-1|/2 \geq 66$. Therefore $15 \geq 66$ gives a contradiction.
- j) If 44 colours with two edges each & 2 colours with four edges each & 3 colours with three edges each,
Then $|T| = 17$ & $|S| \geq 8$ hence $|T| = 17$ & $|S| \cdot |S-1|/2 \geq 28$. Therefore $17 \geq 28$ gives a contradiction.
- k) If 44 colours with two edges each & 4 colours with three edges each & 1 colour with five edges,
Then $|T| = 17$ & $|S| \geq 10$ hence $|T| = 17$ & $|S| \cdot |S-1|/2 \geq 45$. Therefore $17 \geq 45$ gives a contradiction.
- l) If 43 colours with two edges each & 5 colours with three edges each & 1 colour with four edges
Then $|T| = 19$ & $|S| \geq 8$ hence $|T| = 19$ & $|S| \cdot |S-1|/2 \geq 28$. Therefore $19 \geq 28$ gives a contradiction.
- m) If 42 colours with two edges each & 7 colours with three edges

Let $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ be the colours in the optimal colouring having each of them three edges in the colouring.

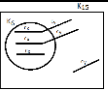
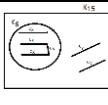
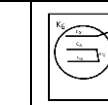
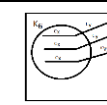
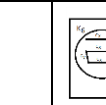
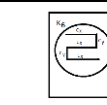
Let $S_i = \{V_j : \text{The colour } c_i \text{ is incident at } V_j\}$ where $i=1$ to 7 Let $a_{ij} = |S_i \cap S_j|$ $i \neq j$ & $\mu = \min\{a_{ij} : i \neq j \text{ \& } i < j\}$

Say $\mu = a_{xy}$ hence we may have $\mu = 1, 2, 3, 4, 5, 6$

if $\mu = 1$ we may have the following diagram.

	Out of 21 edges of the colours c_1 to c_7 , we have used 18 edges (edges in K_6 & three c_y edges) Due to (1), we need at least 12 edges of the colours c_1 to c_7 incident within the end nodes of the colour c_y , which is not possible
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If $\mu = 2, 3, 4$ then we may have the following diagrams respectively.

					
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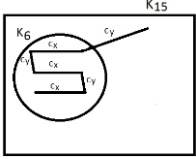
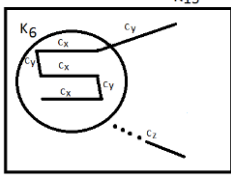
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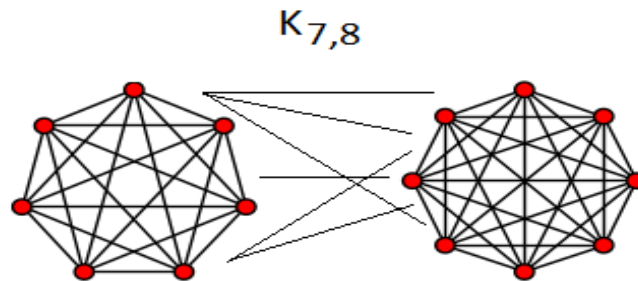
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The above can be rejected like $\mu = 1$

If $\mu = 5$ then we may have the following diagrams.

	<p>Out of 21 edges of the colours c_1 to c_7, we have used 16 edges (edges in K_6 & c_y edges) Due to (1), we need at least 4 edges of the colours c_1 to c_7 incident within the end nodes of the colour c_y The two sub-cases are discussed below.</p>
<p>So some c_z edge ($z \neq x, y$) from K_6 to one of outer 9 vertices</p> 	<p>Out of 21 edges of the colours c_1 to c_7, we have used 17 edges (edges in K_6 & c_y edges) Due to (1), we need at least 4 edges of the colours c_1 to c_7 incident within the end nodes of the colour c_y & now due to (1) we need at least 4 edges of the colours c_1 to c_7 incident from inner end nodes of c_z to one outer end node of c_z which is not possible.</p>
<p>Or the following case</p>	

Then $|T| = 21$ & $|S| \geq 7$ hence $|T|=21$ & $|S| \cdot |S-1|/2 \geq 21$ & we arrive at following diagram.
The left K_7 part is H & the whole diagram is considered being K_{15} with C.



Let t be a one of the colours from the 42 colours considered in case (m)

Now at every vertex of H there are only 6 colours are incident. Hence to have adjacency with the 7th colour used in H to form the complete colouring C, t must have 2 edges in $K_{7,8}$ which gives at most 28 colours in $K_{7,8}$ & K_8 can't involve any new colour as this colour can't have adjacency with any of the colours used in H. Hence edges of K_8 has to be colour by using either colours of K_7 , or $K_{7,8}$ which contradicts to the choice of case (m) itself.

If $\mu = 6$ then K_6 spanned by any c_i should contain all the edges of c_j ($i \neq j$)

Then K_6 should contain 21 edges which is contradiction.

Hence we conclude $A(15) \neq 49$. Therefore $41 \leq A(15) \leq 48$.

III. CONCLUSION

The upper bound of $A(15)$ is tightened to 48.

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