

Standardized CUSUM Control Chart for Process Mean with MDSS

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ABSTRACT: Many authors proposed the CUSUM control chart for single observation in each group (sample) to detect even small shift in μ . In this paper a standardized CUSUM control chart is proposed for more than one observation in each sample. The variable sample size (VSS), and Markov dependent sample sizes (MDSS) of three types are considered to implement the standardized CUSUM control chart for process variability and illustrated in the tabular form.

KEYWORDS: Statistical Process Control (SPC), Cumulative Sum (CUSUM), Average Run Length (ARL), Variable Sample Size (VSS), Markov Dependent Sample Size (MDSS), Computer Simulation.

I. INTRODUCTION

Cumulative sum (CUSUM) control charts were first proposed by Page [6] and subsequently studied by many authors, like Arnold Jesse and Reynolds Marion [1]. The conventional Shewhart \bar{X} control chart may not give signal for smaller shifts in mean. The cumulative sum (CUSUM) \bar{X} control chart is a good alternative to identify the small shifts in μ and to check the target process mean $\mu_{0[11,1,13]}$.

The CUSUM \bar{X} is defined as:

$$S_i = (\bar{X}_i - \mu_0) + S_{i-1}; \text{ for } i = 1, 2, 3, \dots \text{ with } S_0 = 0$$

In order to use the CUSUM \bar{X} chart and check for any shift in mean, V-mask is used. But there are some disadvantages in developing a V-mask [12]. An alternative to this method is the tabular form. This can be easily implemented with the help of a computer.

II. A TABULAR FORM OF THE CUSUM \bar{X} CONTROL CHARTS

Now a days it becomes a modern methodology to monitor the quality of product or process without drawing actual graphs. Montgomery [5], Pandurangan and Varadharajan [8] have proposed an algorithmic tabular CUSUM as well as standardized CUSUM to monitor the process mean [2]. Let $S_H(i)$ be an upper one sided tabular CUSUM for period i and $S_L(i)$ be a lower one sided tabular CUSUM for sample i . Assuming X as iid normal with mean μ and variance σ^2 , the quantities $S_H(i)$ and $S_L(i)$ are defined for each sample as [3]

$$S_H(i) = \max[0, \bar{X}_i - (\mu_0 + K) + S_H(i-1)]$$

and

$$S_L(i) = \max[0, (\mu_0 + K) - \bar{X}_i + S_L(i-1)]$$

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Starting with $S_H(0)$ or $S_L(0)$, when S_H or S_L becomes negative reset to zero. If either $S_H(i)$ or $S_L(i)$ exceeds the decision interval $H/4,5$, it must be concluded that the process is out of control. In that case the new process mean is estimated using the formula

$$\hat{\mu}_1 = \begin{cases} (\mu_0 + K) + \frac{S_H(i)}{N_H} & \text{if } S_H(i) > H \\ (\mu_0 - K) + \frac{S_L(i)}{N_L} & \text{if } S_L(i) > H \end{cases}$$

Where, the quantities N_H and N_L be the number of consecutive periods that the CUSUMs $S_H(i)$ or $S_L(i)$ have been non-zero[7].

The tabular CUSUM is designed by using the reference value of K and the decision interval H . It is usually recommended that these parameters are selected to provide a good run length value[6]. Montgomery[5] has recommended $K=0.5$ and $H=4$ or 5 to have a good ARL when the shift in the process mean is about 1σ [8].

III. FIRST INITIAL RESPONSE (OR) HEAD START

A procedure was introduced by Lucas and Crosier [4] to improve the sensitivity of a CUSUM at process start-up. Increased sensitivity at process start-up would be desirable if the corrective action did not reset the mean to the target value[10]. The first initial response (FIR) or head start essentially just sets the starting value $S_H(0)$ and $S_L(0)$ equal to non zero value typically $H/2$.

STANDARDIZED CUSUM

A drawback in the conventional CUSUM control chart is that the reference value of H and K must be chosen by satisfying certain conditions. To overcome this difficulty, instead of considering the actual average for each sample (sub group), it can be standardized as follows[9]:

$$Z_i = \frac{(\bar{X} - \mu_0)}{S_i}; \quad \text{for } i = 1, 2, 3, \dots$$

Where $S_i = \sqrt{\frac{1}{n-1} \sum_{i=1}^n (\bar{x} - \bar{x}_i)^2}$, (S_i is the unbiased standard deviation for the i^{th} sample)

Then the standardized CUSUM is calculated using

$$S_H(i) = \text{Max}[0, Z_i - K + S_H(i-1)]$$

and

$$S_L(i) = \text{Max}[0, K - Z_i + S_L(i-1)]$$

Then Z_i becomes identically independently distributed standard normal variable[11].

Start with $S_H(0) = 0 = S_L(0)$. Further, the head start value may be taken as $H/2$ for effective detection of any shift in the mean. It is also customary to take $K = 0.5$ and $H = 4$ or 5 as decision parameters for optimum ARL.

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IV. NUMERICAL ILLUSTRATION

(i) Variable Sample Size (VSS)

Charts with variable sample sizes can be used for certain advantages. Costa[3] has proposed a chart with variable sample sizes to study the process mean. The same concept is extended for CUSUM control chart. Consider a production process of forged piston rings with outer diameter (OD) as 100mm. The quality characteristics of the piston ring OD can be checked. In this case, the in-control production process is modeled as a normal process whose mean is equal to 100 mm with standard deviation 5 mm[13,5,7]. Box and Mullern[2] have suggested the method to simulate normal variable. Sample observations are simulated taking $\mu = 100$, $\sigma = 5$ for the first 10 samples (subgroups) and $\mu = 101$, $\sigma = 5$ from 11th sample (subgroups) onwards. The sample sizes are taken as 9 or 4.

The Shewhart control limits for \bar{X} chart with variable sample size are:
 $UCL = 105.0$; $LCL = 95.0$ for sample size $n_1 = 9$ and
 $UCL = 107.5$; $LCL = 92.5$ for sample size $n_2 = 4$.

The simulated sample observations and subgroup averages are shown below.

Table 4

Sl. No.	Simulated Observations										n	Average	SDs
1	98.13	101.9	97.26	96.91	103	103.9	99.88	102.8	104.6		9	100.873	2.872
2	98.14	101.47	97.85	99.46							4	99.230	1.650
3	101.54	100.56	96.44	96.84	99.66	98.47	104.8	97.01	100.41		9	99.526	2.691
4	99.93	100.1	103.87	103.05	99.77	99.76	99.76	102.7	96.95		9	100.654	2.156
5	102.85	101.21	100.42	102.72							4	101.800	1.183
6	99.62	99.18	103.92	100.35	99.96	98.54	101.4	100.2	97.54		9	100.134	1.830
7	102.65	102.9	103.68	104.33	101.3	96.14	103.7	99.19	100.26		9	101.567	2.658
8	96.28	104.56	100.52	103.15							4	101.128	3.640
9	100.85	100.26	101.25	98.71							4	100.268	1.115
10	103.84	101.35	101.65	102.62							4	102.365	1.123
11	100.15	105.77	105.67	102.32	102.7	102.3	104.8	105.7	104.25		9	103.738	1.973
12	104.27	99.8	101.66	105.32	104	102.1	104.6	102.6	103.69		9	103.112	1.731
13	100.25	105.1	105.33	105.6							4	104.070	2.555
14	101.65	103.58	99.94	102.95							4	102.030	1.608
15	104.7	103.54	103.64	99.47	104.6	105.6	105.4	98.54	104.65		9	103.353	2.569
16	105.99	105.65	103.55	104.98	105.8	105.8	105.8	104.9	104.96		9	105.272	0.775

The simulated inspection is terminated since the average goes beyond the UCL of the conventional Control Chart. By using the subgroup averages \bar{X}_i the standardized CUSUM for each sample is calculated. For example, the initial values are: $S_H(0) = 0 = S_L(0)$

$$Z_1 = \frac{(\bar{X}_1 - \mu_0)}{S_1} = \frac{100.8722 - 100}{12.871501} = 0.303751$$

$$S_H(1) = \text{Max}[0, Z_1 - 0.5 + S_H(0)] = \text{Max}[0, 0.035598 - 0.5 + 0] = 0$$

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and

$$S_L(1) = \text{Max}[0, 0.5 - Z_1 + S_L(0)] = \text{Max}[0, 0.5 - 0.303751 + 0] = 0.196249$$

And so on. The computations are given in the Table 5 below:

Table 5

Computation of Standardized CUSUMs for the \bar{X}

Sl. No.	Z_i	$S_n(i)$	N_n	$S_i(i)$	N_i	Size n	Cum n
1	0.3037	0	0	0.1963	1	9	9
2	-0.4405	0	0	1.1368	2	4	13
3	-0.1746	0	0	1.814	3	9	22
4	0.3048	0	0	2.0065	4	9	31
5	1.5210	1.0210	1	0.9855	5	4	35
6	0.0711	0.5921	2	1.4144	6	9	44
7	0.5885	0.605	3	1.3259	7	9	53
8	0.3098	0.4904	4	1.5162	8	4	57
9	0.2399	0.2303	5	1.7762	9	4	61
10	2.1063	1.8366	6	0.1699	10	4	65
11	1.8962	3.2329	7	0	11	9	74
12	1.7943	4.5271	8	0	12	9	83
13	1.5931	5.6202	9	0	13	4	87
14	1.2621	6.3822	10	0	14	4	91
15	1.3052	7.1876	11	0	15	9	100
16	6.8622	13.5498	12	0	16	9	109

The standardized CUSUM for \bar{X} from the above Table 5 indicates that the process is out of control at sample 13 since S_H is greater than 5.

Now we form the standardized CUSUM with the first initial value $S_H(0) = 2.5 = S_L(0)$.

The computations are as follows:

$$S_H(1) = \text{Max}[0, Z_1 - 0.5 + S_H(0)] = \text{Max}[0, 0.303751 - 0.5 + 2.5] = 2.3038$$

$$S_L(1) = \text{Max}[0, 0.5 - Z_1 + S_L(0)] = \text{Max}[0, 0.5 - 0.303751 + 2.5] = 2.6962$$

$$S_H(2) = \text{Max}[0, Z_2 - 0.5 + S_H(1)] = \text{Max}[0, -0.440525 - 0.5 + 2.3038] = 1.3632$$

$$S_L(2) = \text{Max}[0, 0.5 - Z_2 + S_L(1)] = \text{Max}[0, 0.5 - (-0.440525) + 2.6962] = 3.66377$$

And so on. The other values are computed in the same way and given in Table-6.

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Table 6
Computation of Standardized CUSUMs with FIR

S. No.	Z_i	$S_H(i)$	N_H	$S_L(i)$	N_L	Size	Cum n
1	0.3037	2.038	1	2.6963	1	9	9
2	-0.4405	1.3632	2	3.6368	2	4	13
3	-0.1746	0.6886	3	4.3114	3	9	22
4	0.3048	0.4935	4	4.5065	4	9	31
5	1.5210	1.5145	5	3.4855	5	4	35
6	0.0711	1.0856	6	3.9144	6	9	44
7	0.5885	1.1741	7	3.8259	7	9	53
8	0.3098	0.9839	8	4.0161	8	4	57
9	0.2399	0.7238	9	4.2762	9	4	61
10	2.1063	2.3301	10	2.6699	10	4	64
11	1.8962	3.7264	11	1.2736	11	9	74
12	1.7943	5.0207	12	0	0	9	83
13	1.5931	6.1137	13	0	0	4	87
14	1.2621	6.8758	14	0	0	4	91
15	1.3052	7.681	15	0	0	9	100
16	6.8622	14.043	16	0	0	9	109

The standardized CUSUM chart in the above table indicates the process is out of control at sample 12, since S_H is greater than 5 the total sample inspection is only 83 when head start is $H/2$. The standardized CUSUM control chart with variable sample size(VSS) takes 87 sample observations to signal when the initial value is 0.

(ii) Markov Dependent Sample Size

Markov chain rule plays an important role in sample selection. Pandurangan[7] has suggested various control charts under Markovian environment. Consider the illustration given in his work for Markov Dependent Sample Size (MDSS). The specified outer diameter (OD) of the piston rings was taken as 100 mm. In this case, the in-control production process whose mean is equal to 100mm with standard deviation 5 mm. To check the quality, sample observations were simulated for each subgroup of size 6 or 4 or 5.[7,5,7] The size of each subgroup is decided on Markov Chain rule.

To decide the size of an MDSS, the TPM was taken with weights 8, 7 and 5 respectively for the sample sizes 6,4 and 5.

$$P = \begin{bmatrix} 3/8 & 3/8 & 2/8 \\ 3/7 & 2/7 & 2/7 \\ 1/5 & 3/5 & 1/5 \end{bmatrix}$$

Then the Markov distribution $[\Pi P = \Pi, \Pi e = 1]$

gives $\pi_1 = 0.352201$, $\pi_2 = 0.396226$ and $\pi_3 = 0.25157$

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By a random technique (lottery method) an MDSS sequence is obtained as:

$$1 \rightarrow 3 \rightarrow 1 \rightarrow 3 \rightarrow 2 \rightarrow 1 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 1 \rightarrow 3 \rightarrow 1 \rightarrow 2 \rightarrow 2 \rightarrow 1$$

Where 1 denotes type-1 sample of size $n_1 = 6$, 2 denotes type-2 sample of size $n_2 = 4$ and 3 denotes type-3 sample of size $n_3 = 5$. Then the control limits for chart under MDSS are:

$$UCL = \mu_0 + 3\sigma_0 \sqrt{\frac{\pi_1^2}{n_1} + \frac{\pi_2^2}{n_2} + \frac{\pi_3^2}{n_3}} = 104.0411$$

$$LCL = \mu_0 - 3\sigma_0 \sqrt{\frac{\pi_1^2}{n_1} + \frac{\pi_2^2}{n_2} + \frac{\pi_3^2}{n_3}} = 95.9589$$

Sample observations were simulated with $\mu = 100$ mm and $\sigma = 5$ mm for the first 10 samples and then $\mu = 101$ mm and $\sigma = 5$ mm from the 11th sample onwards. The subgroup averages are shown in Table 7.

Table-7

S.No	Simulated Observations						n	Mean	SDs
1	101.35	98.50	100.85	99.65	103.00	103.92	6	101.212	2.024
2	100.35	102.45	102.85	101.65	100.55		5	101.570	1.112
3	100.25	98.85	99.85	100.20	99.66	100.35	6	99.860	0.560
4	102.05	100.25	103.87	103.05	99.77		5	101.798	1.763
5	102.85	98.20	100.42	99.15			4	100.155	2.014
6	100.55	99.18	98.48	102.42	99.96	99.16	6	99.958	1.403
7	102.35	102.90	99.58	104.33	101.25	99.14	6	101.592	1.998
8	101.45	98.54	100.52	99.15			4	99.915	1.316
9	101.52	100.26	102.32	102.71	101.25		5	101.612	0.958
10	100.15	101.66	99.45	98.25	102.65	100.25	6	100.402	1.566
11	104.27	102.25	103.66	104.33	104.55		5	103.812	0.933
12	102.85	105.45	103.65	103.40	102.85	103.65	6	103.642	0.957
13	103.25	104.35	105.65	102.38			4	103.908	1.414
14	104.17	104.32	104.35	102.85			4	103.923	0.719
15	105.92	104.94	105.40	105.20	104.35	105.65	6	105.243	0.555

The above table has issued an alarm at the 15th subgroup, since the mean value exceeds the UCL with $n = 6$. But actually the shift has occurred at the 11th sample. Total items inspected was 78.

By using \bar{X}_i a tabular CUSUM is formed by standardizing the variable.

Then the calculation of $S_H(i)$ and $S_L(i)$ are as follows. The first initial value $S_H(0) = 0$, $S_L(0) = 0$ to be used

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$$Z_1 = \frac{(\bar{X}_1 - \mu_0)}{S_1} = \frac{101.212 - 100}{2.024} = 0.5988$$

$$S_H(1) = \text{Max}[0, Z_1 - 0.5 + S_H(0)] = \text{Max}[0, 0.5988 - 0.5 + 0] = 0.0988$$

$$S_L(1) = \text{Max}[0, 0.5 - Z_1 + S_L(0)] = \text{Max}[0, 0.5 - 0.5988 + 0] = 0$$

$$S_H(2) = \text{Max}[0, Z_2 - 0.5 + S_H(1)] = \text{Max}[0, 1.4119 - 0.5 + 0.0988] = 1.0107$$

$$S_L(2) = \text{Max}[0, 0.5 - Z_2 + S_L(1)] = \text{Max}[0, 0.5 - (1.4119) + 0] = 0$$

And so on. The computations are given in the following Table 8.

Table-8

S.No.	Z_i	$S_H(i)$	N_H	$S_L(i)$	N_L	n	cum n
1	0.5988	0.0988	1	0	0	6	6
2	1.4116	1.0104	2	0	0	5	11
3	-0.2500	0.2604	3	0.750	1	6	17
4	1.0198	0.7802	4	0.230	2	5	22
5	0.0770	0.3572	5	0.653	3	4	26
6	-0.0297	0.0	0	1.183	4	6	32
7	0.7966	0.2966	1	0.886	5	6	38
8	-0.0646	0.0	0	1.451	6	4	42
9	1.6819	1.1819	1	0.269	7	5	47
10	0.2565	0.9384	2	0.512	8	6	53
11	4.0836	4.5220	3	0	0	5	58
12	3.8041	7.8261	4	0	0	6	64
13	2.7636	10.0897	5	0	0	4	68
14	5.4530	15.0427	6	0	0	4	72
15	9.4486	23.9914	7	0	0	6	78

The above Table 8 indicates the process is out of control at sample 12, since S_H is greater than 5. But in the case of conventional MDSS control chart a signal was obtained in the 15th sample. The sample inspection in this case is 64 whereas it was 78 in the case of conventional MDSS system. Thus, the standardized CUSUM control chart system for average is more economical than the conventional Shewhart control chart for VSS or MDSS.

V. CONCLUSION

The Standardized CUSUM control chart is used to monitor the process mean. The Standardized CUSUM system can easily identify even a small shift in the process mean. This method is compared with the conventional Shewhart control chart for Fixed Sample Size (FSS), Variable Sample Size (VSS) and Markov Dependent Sample Size (MDSS). Among all the three methods, the Standardized CUSUM control chart with MDSS proposed in this paper is shown to be the most economical chart. This procedure can be extended to detect the small shift in the process mean by using the sample median.

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