

# Numerical Simulation on Unsteady Heat Transfer of a Sphere

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**ABSTRACT:** In the present numerical study, a well-known analytical model for transient, one-dimensional heat transfer is considered. A sphere of uniform temperature  $T_i$  at time  $t = 0$  is immersed in a stagnant fluid at different temperature  $T_\infty$ . Heat is transferred within the sphere by conduction and from the outer surface of the sphere to the fluid by convection. Analytical model is discussed first and then a detailed numerical analysis has been done for accurate solution. The computer codes are developed for this numerical analysis in Matlab environment. In the second part of the present study, the computer codes developed are applied to a model problem. The results are then compared with those published in literature and find a good agreement.

**KEYWORDS:** Biot number, Conduction, Convection, Matlab, Numerical Solution, Unsteady.

## Symbols

a, b coefficients of Eq (32)

Bi Biot number,  $hr_0/k$ , dimensionless

$C_1, C_2$  constants in Eq (21)

$C_{v,s}$  specific heat capacity of solid sphere,  $J/Kg.^{\circ}C$

D diameter of the sphere, m

g gravitational acceleration,  $9.81 m/s^2$

G function defined in Eq (12)

$Gr_D$  Grashof number for sphere,  $\frac{g\beta}{\nu^2}(T_s - T_\infty)D^3$ , dimensionless

$\bar{h}$  mean heat transfer coefficient,  $W/m^2.^{\circ}C$

H function defined in Eq (19)

$k_f$  thermal conductivity of water,  $W/m.^{\circ}C$

$Nu_D$  mean Nusselt number for sphere,  $\frac{\bar{h}D}{k_f}$ , dimensionless

Pr Prandtl number, dimensionless

r radial distance, m

$r_0$  radius of sphere, m

$Ra_D$  Rayleigh number for sphere,  $Gr_D Pr$ , dimensionless

S surface area of sphere,  $m^2$

t time, s

T temperature,  $^{\circ}C$

$T_i$  initial temperature,  $^{\circ}C$

$T_s$  surface temperature,  $^{\circ}C$

$T_\infty$  surface temperature,  $^{\circ}C$

V volume of sphere,  $m^3$

Greek

- $\alpha$  thermal diffusivity of sphere,  $m^2/s$
- $\beta$  coefficient of volumetric thermal expansion,  $1/^\circ C$
- $\theta$  dimensionless temperature defined in Eq (5)
- $\xi$  dimensionless coordinate defined in Eq (5)
- $\tau$  dimensionless time defined in Eq (5)
- $\nu$  kinematic viscosity,  $m^2/s$
- $\rho_s$  density of sphere,  $Kg/m^3$
- $\Phi$  function defined in Eq (12)
- $\lambda$  root of Eq (24)

Subscript

- f water
- s solid

## I.INTRODUCTION

Many heat conduction problems encountered in engineering applications involve time as an independent variable. The goal of analysis is to determine the variation of the temperature as a function of time and position  $T(r, t)$  within the heat conducting body. In general, we deal with conducting bodies in a three dimensional space and the goal is to predict the evolution of the temperature field for future times ( $t > 0$ ) [1-7].

Heat is transferred when a hot solid sphere is immersed in a cold stagnant fluid by either forced convection or natural convection [8-15]. Heat is transferred within the sphere by conduction and from the outer surface of the sphere to the fluid by convection.

This study deals with the transient heat transferred of a sphere. The objective of the present study is two-fold. The first is to develop the analytical model of the problem and Matlab codes for the numerical techniques for the accurate solution. The second is the validation of the codes by comparing its results applied to a model problem with the empirical correlation available in literature [16].

We consider the transient heat conduction in a sphere of radius  $r_0$ . Initially, the temperature of the sphere is  $T_i$ . At time  $t = 0$ , the sphere is immersed into a well mixed cooling bath at temperature  $T_\infty$ . The heat transfer takes place by conduction within the sphere and by convection between the surface of the sphere and the liquid. The physical experiment depicted in the sketch below.

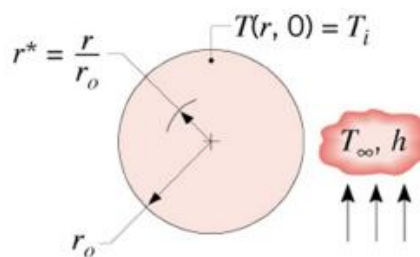


Fig1. A heated solid sphere immersed in a cold stagnant fluid

The temperature distribution must obey the governing equation

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$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T \quad 0 < r < r_0 \quad 1$$

subject to the following initial and boundary conditions

$$\text{IC: } T(r, t) = T_i \text{ at } t=0 \quad 2$$

$$\text{BC: } -k \frac{\partial T}{\partial r} = h(T - T_\infty), r = r_0 \quad 3$$

Eq (1) can be written in spherical coordinates

$$\frac{\partial T}{\partial t} = \frac{\alpha}{r^2} \frac{\partial}{\partial r} \left( \frac{1}{r^2} \frac{\partial T}{\partial r} \right) \quad 0 < r < r_0 \quad 4$$

**Analytical Solution**

The analytical solution for equation (4), subject to equations (2) and (3) is available in several textbooks [1-7]. We introduce the following dimensionless variables

$$\theta = \frac{T - T_\infty}{T_i - T_\infty}$$

$$\xi = \frac{r}{r_0}$$

$$\tau = \frac{\alpha t}{r_0^2} \quad 5$$

So that the dimensionless temperature  $\theta$  is a function of dimensionless coordinate  $\xi$  and time  $\tau$ :

$$\theta = \theta(\xi, \tau) \quad 6$$

The initial and boundary conditions (2) and (3) becomes

$$\text{IC: } \theta(\xi, 0) = 1 \text{ at } \tau = 0 \quad 7$$

$$\text{BC: } \frac{\partial \theta}{\partial \xi} + Bi \theta_s = 0 \text{ at } \xi = 1 \quad 8$$

where Bi the Biot number

$$Bi = \frac{hr_0}{k} \quad 9$$

and

$$\theta_s = \frac{T_s - T_\infty}{T_i - T_\infty} \quad 10$$

The PDE (4) becomes

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$$\frac{\partial \theta}{\partial \tau} = \frac{1}{\xi^2} \frac{\partial}{\partial \xi} \left( \xi^2 \frac{\partial \theta}{\partial \xi} \right) \quad 0 < \xi < 1 \quad 11$$

We look for a solution using the method of separation of variables:

$$\theta(\xi, \tau) = \Phi(\xi)G(\tau) \quad 12$$

Substituting the above expression in the PDE (11), we get

$$\Phi \frac{dG}{d\tau} = G \frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\Phi}{d\xi} \right)$$

$$\frac{1}{G} \frac{dG}{d\tau} = \frac{1}{\Phi} \frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\Phi}{d\xi} \right) \quad 13$$

The left side depends on  $\tau$  whereas the right side depends on  $\xi$  and both are independent of each other. Therefore

both sides should be equal to a negative constant  $-\lambda^2$  (negative sign as the temperature of the sphere will decrease with time as it cools).

$$\frac{1}{G} \frac{dG}{d\tau} = -\lambda^2 \quad 14$$

and

$$\frac{1}{\Phi} \frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\Phi}{d\xi} \right) = -\lambda^2 \quad 15$$

Solution of (14) is

$$G(\tau) \sim e^{-\lambda^2 \tau} \quad 16$$

Eq (15) can be written as

$$\frac{d}{d\xi} \left( \xi^2 \frac{d\Phi}{d\xi} \right) + \lambda^2 \xi^2 \Phi = 0$$

$$\xi^2 \frac{d^2 \Phi}{d\xi^2} + 2\xi \frac{d\Phi}{d\xi} + \lambda^2 \xi^2 \Phi = 0 \quad 17$$

with boundary condition as

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$$\frac{d\Phi}{d\xi} + Bi\Phi = 0 \quad \text{at } \xi = 1 \quad 18$$

Now we introduce a new variable:

$$H(\xi) = \xi\Phi(\xi) \quad 19$$

or

$$\Phi(\xi) = \frac{H(\xi)}{\xi}$$

$$\frac{d\Phi}{d\xi} = -\frac{1}{\xi^2}H(\xi) + \frac{1}{\xi} \frac{dH}{d\xi}$$

$$\frac{d}{d\xi} \left( \frac{d\Phi}{d\xi} \right) = \frac{2}{\xi^3}H(\xi) - \frac{1}{\xi^2} \frac{dH}{d\xi} - \frac{1}{\xi^2} \frac{dH}{d\xi} + \frac{1}{\xi} \frac{d^2H}{d\xi^2}$$

Substituting these expressions in Eq(17), we get

$$\frac{d^2H}{d^2\xi} + \lambda^2 H = 0 \quad 20$$

Its general solution is

$$H(\xi) = C_1 \cos(\lambda\xi) + C_2 \sin(\lambda\xi) \quad 21$$

So

$$\Phi(\xi) = \frac{H(\xi)}{\xi} = C_1 \frac{1}{\xi} \cos(\lambda\xi) + C_2 \frac{1}{\xi} \sin(\lambda\xi) \quad 22$$

To find the integration constants  $C_1$  and  $C_2$ , we consider the boundary conditions for  $\Phi$ .

At  $\xi = 0$ ,  $\Phi(0)$  must be finite.

But  $\cos(\lambda\xi) \rightarrow 1$  and  $\sin(\lambda\xi) \rightarrow 0$  as  $\xi \rightarrow 0$

So

$$\frac{1}{\xi} \cos(\lambda\xi) \rightarrow \infty \quad \text{and} \quad \frac{1}{\xi} \sin(\lambda\xi) \rightarrow \frac{0}{0} \quad \text{as } \xi \rightarrow 0$$

Applying L'Hospital's rule to the second term, we find

$$\lim_{\xi \rightarrow 0} \left( \frac{1}{\xi} \sin(\lambda\xi) \right) = \lim_{\xi \rightarrow 0} \left( \lambda \frac{1}{\lambda\xi} \sin(\lambda\xi) \right) = \lambda$$

Hence for  $\Phi$  to be finite at  $\xi = 0$ , we require that  $C_1 = 0$ . Therefore,

$$\Phi(\xi) = C_2 \frac{1}{\xi} \sin(\lambda \xi) \tag{23}$$

And the BC Eq (18) gives

$$-C_2 \frac{1}{\xi^2} \sin(\lambda \xi) + C_2 \frac{1}{\xi} \lambda \cos(\lambda \xi) + Bi C_2 \frac{1}{\xi} \sin(\lambda \xi) = 0$$

At  $\xi = 1$ , it gives the following characteristic equation for the eigenvalues of  $\lambda$ :

$$1 - \lambda \cot(\lambda) = Bi \tag{24}$$

It is a non linear transcendental equation, for a given Bi, the roots  $\lambda_n$  must be found out numerically.

Finally, the solution to our PDE is (from (16) and (23))

$$\theta(\xi, \tau) = \Phi(\xi)G(\tau) = \sum_{n=1}^{\infty} c_n e^{-\lambda^2 \tau} \frac{1}{\xi} \sin(\lambda_n \xi) \tag{25}$$

From the orthogonally condition, the values of  $D_n$  can be written as

$$C_n = \frac{4[\sin(\lambda_n) - \lambda_n \cos(\lambda_n)]}{2\lambda_n - \sin(2\lambda_n)} \tag{26}$$

Eqs (24) through (26) provide a complete description of the solution.

## II. NUMERICAL TECHNIQUES USING MATLAB

i) Finding the roots of Eq (24)

The first task is to find the roots,  $\lambda_n$ , of Eq (24), for a given Bi. This is a root-finding problem. There are an infinite number of  $\lambda_n$  on the positive real line. Matlab codes myfun.m and roots.m were developed to do this job.

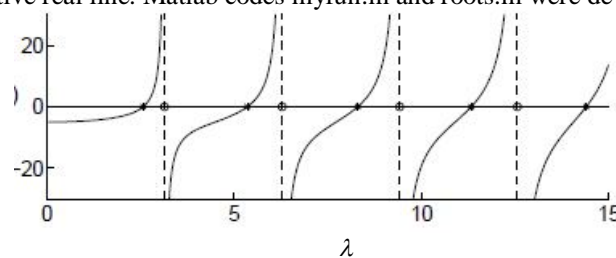


Fig2 Plots of  $f(\lambda) = 1 - \lambda \cot(\lambda) - Bi$  for Bi = 5 showing roots (\*) and singularities (0) of Eq (24) in the range  $0 \leq \lambda \leq 15$ .

ii) Evaluating  $\theta(\xi, \tau)$

Once the  $\lambda_n$  are known for a given Bi,  $C_n$  values are computed from Eq(26) and then the non-dimensional temperature distribution  $\theta(\xi, \tau)$  using Eq (25). Finally, the temperature distribution  $T(r, t)$  from the definition of  $\theta$ , (Eq 5) viz.,  $T(r, t) = T_\infty + (T_i - T_\infty) \times \theta$  is computed. Related matlab codes are nondtem.m and example.m.

### III.RELEVANT EMPIRICAL CORRELATION

The empirical correlation [16] for natural convection of a heated solid sphere in a cold stagnant fluid is

$$\overline{Nu_D} = 2 + 0.589 \frac{Ra_D^{1/4}}{[1 + (\frac{0.469}{Pr})^{9/16}]^{4/9}} \quad \text{for } Ra_D < 10^{11} \quad 27$$

From the relation  $\overline{Nu_D} = \frac{\bar{h}}{k_f} D$ , we can write

$$\begin{aligned} \bar{h} &= 2 \frac{k_f}{D} + 0.589 \frac{k_f}{D} \frac{Ra_D^{1/4}}{[1 + (\frac{0.469}{Pr})^{9/16}]^{4/9}} \\ &= 2 \frac{k_f}{D} + 0.589 \frac{k_f}{D} \frac{(\frac{g\beta}{v^2})^{1/4} (T_s - T_\infty)^{1/4} D^{3/4} Pr^{1/4}}{[1 + (\frac{0.469}{Pr})^{9/16}]^{4/9}} \quad [Ra_D = Gr_D \times Pr = \frac{g\beta(T_s - T_\infty)D^3}{v^2} \times Pr] \\ &= 2 \frac{k_f}{D} + 0.589 \frac{k_f}{D^{1/4}} \frac{(\frac{g\beta}{v^2})^{1/4}}{[1 + (\frac{0.469}{Pr})^{9/16}]^{4/9}} Pr^{1/4} (T_s - T_\infty)^{1/4} \end{aligned} \quad 28$$

The applicable lumped heat equation is

$$\rho_s c_{v,s} V \frac{dT_s}{dt} = -\bar{h} S (T_s - T_\infty) \quad T(0) = T_i \quad 29$$

$$\begin{aligned} \frac{dT_s}{dt} &= -\bar{h} \frac{S}{V} \left( \frac{1}{\rho_s c_{v,s}} \right) (T_s - T_\infty) \\ &= -\bar{h} \frac{6}{D} \left( \frac{1}{\rho_s c_{v,s}} \right) (T_s - T_\infty) \end{aligned} \quad 30$$

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Putting the value of  $\bar{h}$  from Eq (28), we get

$$\frac{dT_s}{dt} = -12 \frac{k_f}{D^2} \left(\frac{1}{\rho_s c_{v,s}}\right) (T_s - T_\infty) - 0.353 \frac{1}{D^{5/4}} \left(\frac{1}{\rho_s c_{v,s}}\right) \left[k_f \left(\frac{g\beta}{\nu^2}\right)^{1/4} \frac{\text{Pr}^{1/4}}{\left[1 + \left(\frac{0.469}{\text{Pr}}\right)^{9/16}\right]^{4/9}}\right] (T_s - T_\infty)^{5/4}$$

31

The above eqn can be written as

$$\frac{dT_s}{dt} + a(T_s - T_\infty) + b(T_s - T_\infty)^{5/4} = 0$$

32

where

$$a = 12 \frac{k_f}{D^2} \left(\frac{1}{\rho_s c_{v,s}}\right)$$

$$b = 0.353 \frac{1}{D^{5/4}} \left(\frac{1}{\rho_s c_{v,s}}\right) \left[k_f \left(\frac{g\beta}{\nu^2}\right)^{1/4} \frac{\text{Pr}^{1/4}}{\left[1 + \left(\frac{0.469}{\text{Pr}}\right)^{9/16}\right]^{4/9}}\right]$$

The exact analytical solution of Eq (32) [17] is

$$T_s(t) = T_\infty + (T_i - T_\infty) \left[1 + \frac{b}{a} (e^{at/4} - 1)\right]^{-4}$$

33

Thermo-physical property values are evaluated at the film temperature  $\frac{1}{2}(T_s + T_\infty)$

**Validation**

We consider the following model problem: A 5 mm diameter metal sphere initially at 90°C is plunged into a water bath at 30°C. Assume that the heat transfer coefficient is 1000 W/(m<sup>2</sup>.°C), the thermal conductivity of the metal is k<sub>s</sub> = 20

W/m.°C and the thermal diffusivity of the metal is  $\alpha = 6.66 \times 10^{-6}$  m<sup>2</sup>/s.

**IV.RESULTS AND DISCUSSION**

The temperature distribution of the outer surface of the sphere is obtained using the matlab codes developed. This is then compared with that obtained from Eq (33). Figure 3 demonstrates the comparison.



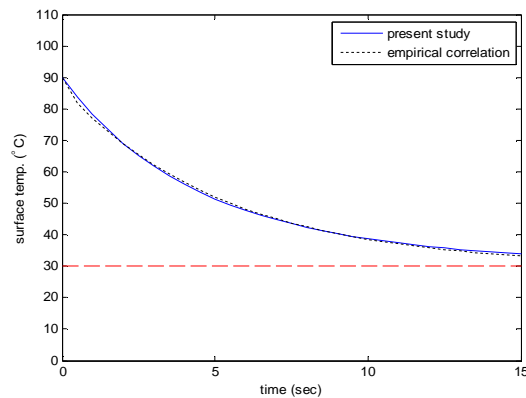


Fig 3 Variation of Temperature of the outer surface of the sphere at different times

## V.CONCLUSION

In this numerical research work, the transient heat transfer is considered when a solid sphere at a uniform temperature is immersed in cold stagnant fluid. In the solution procedure, first analytical model and then numerical techniques have been discussed for the solution. Computer codes in Matlab are developed accordingly. Applying these codes to a model problem, temperature distributions are obtained. These are compared with those obtained from empirical correlation available in literature and find a good agreement.

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## BIOGRAPHY



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