

Theoretical analysis of thermoelastic response of a thick circular plate

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ABSTRACT: We apply integral transformation techniques to study thermoelastic response of a thick circular plate, in general in which sources are generated according to the linear function of the temperature, with boundary conditions of the radiation type. The results are obtained as series of Bessel functions. Numerical calculations are carried out for a particular case of a thick plate made of Aluminum metal and the results are depicted in figures.

KEYWORDS: Transient response, thick plate, temperature distribution, thermal stress, integral transform

I. INTRODUCTION

As a result of the increased usage of industrial and construction materials the interest in isotropic thermal stress problems has grown considerably. However there are only few studies concerned with the two-dimensional steady state thermal stress. Nowacki [5] has determined steady-state thermal stresses in a thick circular plate subjected to an axisymmetric temperature distribution on the upper face with zero temperature on the lower face and circular edge. Wankhede [8] has determined the quasi-static thermal stresses in circular plate subjected to arbitrary initial temperature on the upper face with lower face at zero temperature. However, there aren't many investigations on transient state. Roy Choudhuri [7] has succeeded in determining the quasi-static thermal stresses in a circular plate subjected to transient temperature along the circumference of circular upper face with lower face at zero temperature and the fixed circular edge thermally insulated. Noda et al. [6] have considered a circular plate and discussed the transient thermoelastic-plastic bending problem, making use of the strain increment theorem. In a recent work, some problems have been solved by Noda et al. [6] and Deshmukh et al. [1]. In all aforementioned investigations an axisymmetrically heated plate has been considered. Recently, Nasser [2,3] proposed the concept of heat sources in generalized thermoelasticity and applied to a thick plate problem. They have not however considered any thermoelastic problems with boundary conditions of radiation type, in which sources are generated according to the linear function of the temperatures, which satisfies the time-dependent heat conduction equation. The success of this novel research is mainly lies with the new mathematical procedures with a much simpler approach for optimization for the design in terms of material usage and performance in engineering problem, particularly in the determination of thermoelastic behaviour in thick plate engaged as the foundation of pressure vessels, furnaces, etc. This paper is concerned with the transient thermoelastic problem in a thick circular plate in which sources are generated according to the linear function of temperature, occupying the space $D = \{(x,y,z) \in R^3 : 0 \leq (x^2 + y^2)^{1/2} \leq b, -h \leq z \leq h\}$, where $r = (x^2 + y^2)^{1/2}$ with radiation type boundary conditions.

II. STATEMENT OF THE PROBLEM

In the first instance, we consider a thick circular plate. The material of the plate is isotropic, homogenous and all properties are assumed to be constant. Heat conduction and the prescribed boundary conditions of the radiation type is considered, the thermal stresses are required to be determined. The equation for heat conduction is $T(r,z,t)$, the temperature, in cylindrical coordinates, is:

$$\kappa \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{\partial^2 T}{\partial z^2} \right] = \frac{\partial T}{\partial t} \quad (1)$$

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where κ is the thermal diffusivity of the material of the cylinder (which is assumed to be constant), subject to the initial and boundary conditions

$$M_t(T, 1, 0, 0) = 0 \quad \text{for all } 0 \leq r \leq a, -h \leq z \leq h \quad (2)$$

$$M_r(T, 1, 0, a) = 0 \quad \text{for all } -h \leq z \leq h, t > 0 \quad (3)$$

$$M_z(T, 1, k_1, h) = \exp(-\omega t) \delta(r - r_0), \quad M_z(T, 1, k_2, -h) = 0 \quad \text{for all } 0 \leq r \leq a, t > 0 \quad (4)$$

The most general expression for these conditions can be given by

$$M_g(f, \bar{k}, \bar{k}, s) = (\bar{k} f + \bar{k} \hat{f})_g = s$$

where the prime (^) denotes differentiation with respect to \mathcal{G} ; $\delta(r - r_0)$ is the Dirac Delta function having $0 \leq r_0 \leq a$; $\omega > 0$ is a constants; $\exp(-\omega t) \delta(r - r_0)$ is the additional sectional heat available on its surface at $z = h$; \bar{k} and \bar{k} are radiation constants on the upper and lower surfaces of the plate respectively.

The displacement components in the cylindrical coordinate system are represented by the Goodier's thermoelastic displacement potential $\phi(r, z, t)$ and Michell's function M as [7]

$$u_r = \frac{\partial \phi}{\partial r} - \frac{\partial^2 M}{\partial r \partial z}, \quad (5)$$

$$u_z = \frac{\partial \phi}{\partial z} + 2(1 - \nu) \nabla^2 M - \frac{\partial^2 M}{\partial z^2} \quad (6)$$

where u_r and u_z are the displacement components in the radial and axial directions, respectively in which Goodier's thermoelastic potential must satisfy the equation

$$\nabla^2 \phi = \frac{1 + \nu}{1 - \nu} \alpha_t \theta \quad (7)$$

and the Michell's function M must satisfy the equation

$$\nabla^2 (\nabla^2 M) = 0 \quad (8)$$

where

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{\partial^2}{\partial z^2}$$

The stress functions are represented by the use of the potential ϕ and Michell's function M as

$$\sigma_{rr} = 2G \left\{ \left(\frac{\partial^2 \phi}{\partial r^2} - \nabla^2 \phi \right) + \frac{\partial}{\partial z} \left(\nu \nabla^2 M - \frac{\partial^2 M}{\partial r^2} \right) \right\}, \quad (9)$$

$$\sigma_{\theta\theta} = 2G \left\{ \left(\frac{1}{r} \frac{\partial \phi}{\partial r} - \nabla^2 \phi \right) + \frac{\partial}{\partial z} \left(\nu \nabla^2 M - \frac{1}{r} \frac{\partial M}{\partial r} \right) \right\}, \quad (10)$$

$$\sigma_{zz} = 2G \left\{ \left(\frac{\partial^2 \phi}{\partial r^2} - \nabla^2 \phi \right) + \frac{\partial}{\partial z} \left((2 - \nu) \nabla^2 M - \frac{\partial^2 M}{\partial z^2} \right) \right\}, \quad (11)$$

and

$$\sigma_{rz} = 2G \left\{ \frac{\partial^2 \phi}{\partial r \partial z} + \frac{\partial}{\partial r} \left((1 - \nu) \nabla^2 M - \frac{\partial^2 M}{\partial z^2} \right) \right\} \quad (12)$$

where G and ν are the shear modulus and Poisson's ratio respectively.

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The equations (1) to (12) constitute the mathematical formulation of the problem under consideration.

III. SOLUTION OF THE PROBLEM

In order to solve fundamental differential equation (1) under the boundary condition (3), we firstly introduce the method of Hankel transform of order n over the variable r . Let n be the parameter of the Hankel transform, then the Hankel transform and its inversion theorem are written as

$$g^*(\beta_n, z, t) = \int_0^a g(r, z, t) r k_0(\beta_n, r) dr, \quad g(r, z, t) = \sum_{n=1}^{\infty} g^*(\beta_n, z, t) k_0(\beta_n, r) \quad (13)$$

where the symbol (*) means a function in the transformed domain, and the nucleus for the finite Hankel transform defined by

$$k_0(\beta_n, r) = \frac{\sqrt{2}}{a} \begin{pmatrix} J_0(\beta_n r) \\ J_1(\beta_n a) \end{pmatrix}$$

The eigenvalues β_n are the positive roots of the characteristic equation $J_0(\beta_n a) = 0$, and J_n is the Bessel function of the first kind of order n .

We introduce another integral transform [4] that responds to the boundary conditions of type (4) as

$$\bar{f}(m, t) = \int_{-h}^h f(z, t) P_m(z) dz, \quad f(z, t) = \sum_{m=1}^{\infty} \frac{\bar{f}(m, t)}{\lambda_m} P_m(z) \quad (14)$$

where the symbol ($\bar{}$) means a function in the transformed domain, and the nucleus is given by the orthogonal functions in the interval $-h \leq z \leq h$ as

$$P_m(z) = Q_m \cos(a_m z) - W_m \sin(a_m z),$$

where

$$\begin{aligned} Q_m &= a_m (k_1 + k_2) \cos(a_m h), \\ W_m &= 2 \cos(a_m h) + (k_2 - k_1) a_m \sin(a_m h), \\ \lambda_m &= \int_{-h}^h P_m^2(z) dz = h [Q_m^2 + W_m^2] + \frac{\sin(2a_m h)}{2a_m} [Q_m^2 - W_m^2] \end{aligned}$$

The eigenvalues a_m are the positive roots of the characteristic equation

$$\begin{aligned} &[k_1 a \cos(ah) + \sin(ah)] [\cos(ah) + k_2 a \sin(ah)] \\ &= [k_2 a \cos(ah) - \sin(ah)] [\cos(ah) - k_1 a \sin(ah)] \end{aligned}$$

Applying the transformation rules (13) and (14) to equation (1) under the condition (3)-(4), the following reduction is made

$$\frac{d\bar{T}^*}{dt} + \kappa \Lambda_{m,n} \bar{T}^* = H(a_m, \beta_n) \quad (15)$$

where

$$\Lambda_{m,n} = a_m^2 + \beta_n^2$$

and

$$H(a_m, \beta_n) = \left(\frac{P_m(h) \kappa r_0}{k_1} \right) k_0(\beta_n, r_0) \exp(-\omega t)$$

Then, the transformed temperature solution of equation (15) is given by

$$\bar{T}^* = \frac{H(a_m, \beta_n)}{\kappa \Lambda_{m,n} - \omega} [\exp(-\omega t) - \exp(-\kappa \Lambda_{m,n} t)] \quad (16)$$

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Accomplishing the inversion theorems of transformation rules defined in equations (13) and (14) on equation (16), one obtains

$$T(r, z, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \wp_{m,n} [\exp(-\omega t) - \exp(-\kappa \Lambda_{m,n} t)] P_m(z) k_0(\beta_n, r) \quad (17)$$

where

$$\wp_{m,n} = \frac{H(a_m, \beta_n)}{\lambda_m (\kappa \Lambda_{m,n} - \omega)}$$

The function given in equation (17) represents the temperature at every instant and at all points of thick circular plate of finite height when there are conditions of radiation type.

Referring to the fundamental equation (1) and its solution (17) the solution for the displacement components are represented by the Goodier's thermoelastic displacement potential ϕ governed by equation (7) are represented by

$$\phi(r, z, t) = -\left(\frac{1+\nu}{1-\nu}\right) a_t \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\wp_{m,n}}{\Lambda_{m,n}} [\exp(-\omega t) - \exp(-\kappa \Lambda_{m,n} t)] P_m(z) k_0(\beta_n, r) \quad (18)$$

Similarly, the solution for Michell's function M are assumed so as to satisfy the governed condition of equation (8) as

$$M(r, z, t) = -\left(\frac{1+\nu}{1-\nu}\right) a_t \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\wp_{m,n}}{\Lambda_{m,n}} [\exp(-\omega t) - \exp(-\kappa \Lambda_{m,n} t)] k_0(\beta_n, r) \times [\cosh(\beta_n z) + z \sinh(\beta_n z)] \quad (19)$$

Using thermoelastic displacement potential ϕ and Michell's function M in equations (5) and (6) one obtains

$$u_r = -\left(\frac{1+\nu}{1-\nu}\right) a_t \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\wp_{m,n}}{\Lambda_{m,n}} [\exp(-\omega t) - \exp(-\kappa \Lambda_{m,n} t)] \left[\frac{\sqrt{2} J_0(\beta_n r)}{a J_1(\beta_n a)} \right] \times \{p_m(z) - [(\beta_n + 1)] \sinh(\beta_n z) + z \beta_n \cosh(\beta_n z)\} \quad (20)$$

$$u_z = -\left(\frac{1+\nu}{1-\nu}\right) a_t \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\wp_{m,n}}{\Lambda_{m,n}} [\exp(-\omega t) - \exp(-\kappa \Lambda_{m,n} t)] k_0(\beta_n, r) \times \{-a_m [Q_m \sin(a_m z) + W_m \cos(a_m z)] + \beta_n [z - \beta_n - 2\nu] \cosh(\beta_n z) + [(2\nu - 1)\beta_n - 2\nu] z \beta_n \sinh(\beta_n z)\} \quad (21)$$

Then, the stress function can be evaluated by substituting the values of thermoelastic displacement potential ϕ from equation (18) and Michell's function M from equation (19) in equations (9), (10), (11) and (12), one obtains

$$\sigma_{rr} = 2\sqrt{2}G \left(-\frac{1+\nu}{1-\nu} \right) a_t \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{\wp_{m,n} [\exp(-\omega t) - \exp(-\kappa \Lambda_{m,n} t)]}{\Lambda_{m,n} a J_1(a\beta_n)} \right) \times \left\{ \begin{aligned} &\beta_n J_0(r\beta_n) \left[-(\beta_n + 2\nu + 1) + \frac{J_1(\beta_n r)}{r} (\beta_n + 1) \right] \sinh(\beta_n z) \\ &- \beta_n \left[\beta_n J_0(\beta_n r) - \frac{J_1(\beta_n r)}{r} \right] z \cosh(\beta_n z) - (a_m^2 \frac{J_0(\beta_n r)}{\beta_n} - \frac{J_1(\beta_n r)}{r}) p_m(z) \end{aligned} \right\} \quad (22)$$

$$\sigma_{\theta\theta} = 2\sqrt{2}G \left(-\frac{1+\nu}{1-\nu} \right) a_t \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{\wp_{m,n} [\exp(-\omega t) - \exp(-\kappa \Lambda_{m,n} t)]}{\Lambda_{m,n} a J_1(a\beta_n)} \right)$$

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$$\times \left\{ \left[\beta_n J_0(r\beta_n)[-2\nu] - \frac{J_1(\beta_n r)}{r}(\beta_n + 1) \right] \sinh(\beta_n z) \right. \\ \left. - \frac{J_1(\beta_n r)}{r} z \cosh(\beta_n z) - \left[(a_m^2 + \beta_n^2) \frac{J_0(\beta_n r)}{\beta_n} - \frac{J_1(\beta_n r)}{r} \right] p_m(z) \right\} \quad (23)$$

$$\sigma_{zz} = 2\sqrt{2}G \left(-\frac{1+\nu}{1-\nu} \right) a_t \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{\varphi_{m,n} [\exp(-\omega t) - \exp(-\kappa \Lambda_{m,n} t)]}{\Lambda_{m,n} a J_1(a\beta_n)} \right) \\ \times \{ \beta_n J_0(r\beta_n) [-(1-2\nu-\beta_n)] \sinh(\beta_n z) + (\beta_n z) \cosh(\beta_n z) - p_m(z) \} \quad (24)$$

and

$$\sigma_{rz} = 2\sqrt{2}G \left(-\frac{1+\nu}{1-\nu} \right) a_t \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{\varphi_{m,n} [\exp(-\omega t) - \exp(-\kappa \Lambda_{m,n} t)]}{\Lambda_{m,n} a J_1(a\beta_n)} \right) \\ \times \{ -a_m [W_m \cos(a_m z) + Q_m \sin(a_m z)] - \beta_n^2 [(1-\nu) \cosh(\beta_n z) + z \sinh(\beta_n z)] \} \quad (25)$$

IV. NUMERICAL RESULTS, DISCUSSION AND REMARKS

To interpret the numerical computations, we consider material properties of Aluminum metal, which can be commonly used in both, wrought and cast forms. The low density of aluminum results in its extensive use in the aerospace industry, and in other transportation fields. Its resistance to corrosion leads to its use in food and chemical handling (cookware, pressure vessels, etc.) and to architectural uses.

Modulus of Elasticity, E	$7.0 \times 10^{10} \text{ N/m}^2$
Shear modulus, G	$2.5 \times 10^{10} \text{ N/m}^2$
Poisson ratio, ν	0.281
Thermal diffusivity, κ	0.86
Thickness, h (cm)	1

Table 1: Material properties and parameters used in this study.

In the foregoing analysis are performed by setting the radiation coefficients constants, $k_1 = 0.86$ and $k_2 = 1$, so as to obtain considerable mathematical simplicities. The other parameters considered are $r_0 = 1.5$, $z_0 = 0.5$ and $\omega = 0.5$. The derived numerical results from equation (17) and (22) to (25) has been illustrated graphically.

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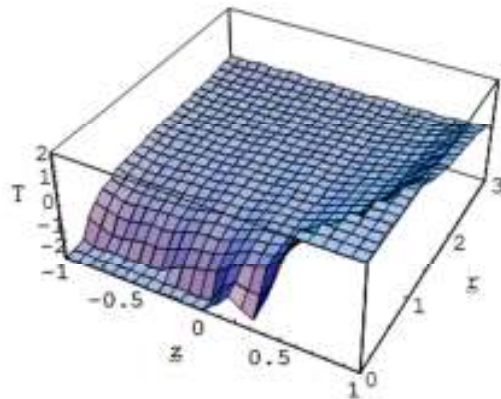


Figure 1: Temperature distribution along r - and z -direction for $t=0.25$

Fig. 1 shows the temperature distribution along the radial and thickness direction of the thick circular plate at $t = 0.25$. It is observed that due to the thickness of the plate, a steep increase in temperature was found at the beginning of the transient period. As expected, temperature drop becomes more and more gradually along thickness direction.

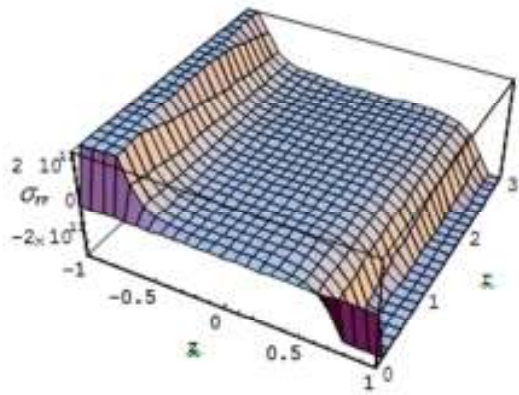


Figure 2: Radial stress distribution along r and z -axis for $t=0.25$

Fig. 2 shows the radial stress distribution σ_{rr} along the radial and thickness direction of the thick circular plate at $t = 0.25$. From the figure, the location of points of minimum stress occurs at the end points through-the-thickness direction, while the thermal stress response are maximum at the inner surface leading inner part being under tensile stress.

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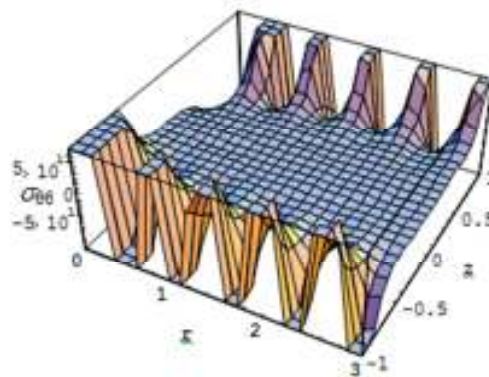


Figure 3: Tangential stress distribution along r and z -axis for $t=0.25$

Fig. 3 shows the tangential stress distribution $\sigma_{\theta\theta}$ along the radial and thickness direction of the thick circular plate at $t=0.25$. The tangential stress follows a sinusoidal nature with high crest and troughs at both end.

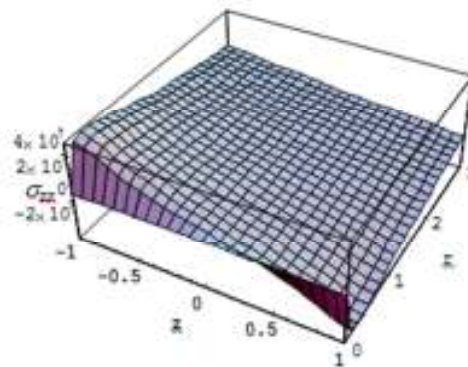


Figure 4: Axial stress distribution along r and z -axis for $t=0.25$

Fig. 4 shows the axial stress distribution σ_{zz} , which is similar in nature, but small in magnitude as compared to radial stress

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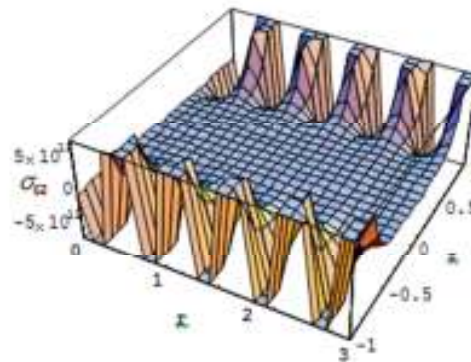


Figure 5: Shear stress distribution along r and z -axis for $t=0.25$

Fig. 5 shows the shear stress distribution σ_{rz} , along the radial and thickness direction. Steep increase in stress was found at the two ends due to the thickness of the plate.

V. CONCLUSION

We have implemented an automatic In this problem, the temperature distributions, displacement and stress functions on the upper plane surface of a thick circular plate have been obtained where the plate is subjected to known heat source function $\exp(-\omega t)\delta(r-r_0)$. We develop the analysis for the temperature field by introducing the transformation techniques. The results are obtained in terms of Bessels functions in the form of infinite series. Any particular case of special interest can be derived by assigning suitable values to the parameters and functions in the expressions.

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