

# **Reliability Modeling Techniques Encapsulated in Few Probability Distribution to Study Their Statistical Characteristics**

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**ABSTRACT:** Reliability theory and its characteristic is multidisciplinary in nature since problem handling requires methods of probability theory and mathematical statistics such as information theory, queuing theory, linear and non-linear programming, mathematical logic, the methods of statistical simulation on electronic computers, demography, etc. This paper deals with statistical reliability modeling of few probability distributions. Here, we assumed Weibull and Exponential probability distribution to study their statistical reliability characteristics. The results are illustrated by numerical example.

**KEYWORDS:** Weibull distribution; Exponential distribution; Failure time distribution; Likelihood function.

## **I. INTRODUCTION**

Reliability became a subject of great engineering interest in the 1950's due to the failure of American rockets as well as the failure of the first commercial jet aircraft; the British de Havilland comet. Since reliability is concerned with uncertainty questions about engineering systems, it would make sense to "derive" appropriate probability distributions based on the engineering physics of a given problem. This would seem to make more sense than picking few mathematically convenient probability distributions more or less at random.

In classical estimation theory, parameters of the population are estimated using the sample information drawn from the population. However, life testing experiments are very costly and time consuming. Many researches should appear expounding reliability theory from many point of view and gives some new concept of reliability theory and its characteristic. Papastavridis [1987] has developed a model for the reliability of a consecutive- $k$ -out-of- $n$ : F system. Rehman et al. [2000] studied a system with different strength and stress distribution, developed reliability model. Pajankar et al. [2006] studied the reliability of a system when a single strength acting on  $n$ -components in the study he assumed  $n$ -stress acted on a single strength component, which follows the same probability distribution. Since the parameters involved in the lifetime distribution also represent the reliability characteristics of the system, so these may not be regarded as constant because of continuous environmental stresses which bring a change in the reliability characteristics of the system. Thus an assumption that the parameters of the lifetime distribution are constant is unduly restricted and unrealistic. This variation in this parameter is represented by prior distribution. These variations are then updated in the Bayesian frame work, after getting experimental information on the system.

The present study deals with statistical classical reliability modeling of two probability distributions. Here, we assumed to very important probability distribution namely Weibull and Exponential probability distribution and make an attempt to study their characteristics. Generally, the Weibull distribution well describes the characteristics (e.g., life) of parts or

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components, while the exponential distribution is best suited to assemblies and systems. The results are illustrated by some numerical example.

## II. ESTIMATION IN WEIBULL DISTRIBUTION

The Weibull distribution is widely used in engineering practice because of its versatility. It was originally proposed for the interpretation of fatigue data, but now its use has extended to many other engineering problems. The real usefulness of Weibull distribution stems from the use of straight line plots to represent scattered data.

Consider the density function of the Weibull distribution is

$$f(x/p, \theta) = (p/\theta) x^{p-1} \exp(-x^p/\theta); \quad x, p > 0, \theta > 0 \quad (1)$$

where  $p$ , the shape parameter is known.

Let the prior distribution  $\theta$  be

$$g(\theta/a, c) = g(\theta) \propto \exp(-a/\theta)/\theta^c; \quad a, c \geq 0, \theta > 0 \quad (2)$$

Taking the likelihood function of Weibull distribution is

$$L(\theta/x) = (p/\theta)^n (\prod x_i)^{p-1} \exp \left[ - \left( \sum_{i=1}^n x_i^p \right) / \theta \right] \quad (3)$$

where  $x_i = (x_1, x_2, x_3, \dots, x_n)$

The posterior of  $\theta$  given  $x_1, x_2, x_3, \dots, x_n$  becomes

$$\begin{aligned} \Pi(\theta/x) &= \frac{L(\theta/x)g(\theta)}{\int_{\Omega} L(\theta/x)g(\theta)d\theta} \\ &= \frac{(p/\theta)^n (\prod x_i)^{p-1} \exp \left[ - \left( \sum_{i=1}^n x_i^p \right) / \theta \right] \exp(-a/\theta)/\theta^c}{\int_0^\infty (p/\theta)^n (\prod x_i)^{p-1} \exp \left[ - \left( \sum_{i=1}^n x_i^p \right) / \theta \right] \exp(-a/\theta)/\theta^c d\theta} \quad (4) \\ &= \frac{\exp \left[ - \left( \sum_{i=1}^n x_i^p + a \right) / \theta \right] \left( \sum_{i=1}^n x_i^p + a \right)^{n+c-1}}{\theta^{n+c} \Gamma(n+c-1)}; \quad \theta > 0 \end{aligned}$$

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The Bayes estimator of  $\theta$  is  $\theta^*$  (say) is given by

$$\begin{aligned}\theta^* &= E(\theta|x) \\ &= \int_0^\infty \theta \Pi(\theta|x) d\theta \\ &= \int_0^\infty \frac{\left( \sum_{i=1}^n x_i^p + a \right)^{n+c-1} \exp \left[ - \left( \sum_{i=1}^n x_i^p + a \right) / \theta \right]}{\theta^{n+c} \Gamma(n+c-1)} \theta d\theta \quad (5) \\ &= \frac{\sum_{i=1}^n x_i^p + a}{n+c-2}\end{aligned}$$

which for large sample (in relation to  $a$  and  $c$ ) will be quite close to the minimum mean square error (MSE)

$$\theta^* = \sum_{i=1}^n x_i^p / n \text{ for } a=0, c=2$$

The Bayes estimator of  $\theta$  is the same as the MLE of  $\theta$ .

The Bayes estimator of the reliability function  $R(t) = \exp(-t^p/\theta)$  say  $R^*(t)$  becomes,

$$\begin{aligned}R^*(t) &= E[R(t)/x] \\ &= E \left[ \exp(-t^p/\theta) / x \right]\end{aligned}$$

Therefore,

$$R^*(t) = \left[ t^p / \sum_{i=1}^n x_i^p + a \right]^{-n-c+1} ; t > 0 \quad (6)$$

**Example 1** A sample of 25 observations was generated from the Weibull distribution with  $p = \theta$ ,  $\theta = 4$ .

Sample				
2.84	2.88	2.79	2.65	2.31
1.37	2.57	1.53	2.71	1.77
1.75	2.77	1.59	2.93	2.88
4.05	2.35	1.62	2.05	5.15
2.35	4.04	1.64	2.61	2.88

$$\sum_{i=1}^{25} x_i^p = 182.85, p = 2, n = 25, \theta = 4$$

$$\text{Bayes estimates of } \theta, \theta^* = \frac{\sum_{i=1}^{25} x_i^p + a}{(n+c-2)}$$

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$c \backslash a$	0	2	4	6	8	$\left(\frac{Min}{Max}\right)/c$
0	7.95	8.03	8.12	8.21	8.29	<b>0.95</b>
1	7.61	7.70	7.78	7.86	7.95	<b>0.95</b>
2	7.31	7.39	7.74	7.55	7.63	<b>0.95</b>
3	7.03	7.10	7.18	7.26	7.34	<b>0.95</b>
$\left(\frac{Min}{Max}\right)/a$	<b>0.88</b>	<b>0.88</b>	<b>0.88</b>	<b>0.88</b>	<b>0.88</b>	-

## III. IMPORTANT INTERPRETATION

From the above table we observe that the (Min/Max) for fixed 'c' and varying 'a' is  $(n-2)/(n+1)$  and for large 'n' would be very close to 1. Similarly (Min/Max) for fixed 'a' and varying 'c' is  $(n+c-2)/(n+1)$  and, for large 'n' this would again be very near to one showing these by that, for large samples, choice of prior distribution is not crucial. Further we also observe that the Bayes' estimators are more robust to changes in 'a' than those in 'c'.

## IV. ESTIMATION IN EXPONENTIAL DISTRIBUTION

Exponential distribution is a special case of the Weibull distribution when  $p=1$  and  $x=t$ , where  $t$  is the time variable. This distribution is very useful in the analysis of failure rates of complete systems or assemblies. For example, in case of aircraft pumps, the probability of failure of a component, such as a shaft or a bearing can be described by a log normal or Weibull distribution. However, for the pump as a whole (as a system) the exponential distribution is a better choice. Consider the density function for an one-parameter exponential failure time distribution

$$f(t/\theta) = (1/\theta) \exp(-t/\theta); t > 0, \theta > 0 \quad (7)$$

where,  $\theta$  is treated as a random variable.

A straight forward computation gives the fisher information

$$I(\theta) = E [-\partial^2 / \partial \theta^2 (\log f(t/\theta))] \quad (8)$$

Therefore,

$$I(\theta) = 1/\theta^2 \quad (9)$$

Then the Jeffreys prior  $g(\theta) = \text{constant } \sqrt{I(\theta)}$

$$g(\theta) \propto 1/\theta$$

which is an improper (or quasi) prior, since

$$\int_0^\infty g(\theta) d\theta \neq 1$$

Let us consider a more general class of prior

$$g(\theta) \propto 1/\theta^c; c > 0 \quad (10)$$

Taking likelihood function of equation (1), then

$$L(t/\theta) = (1/\theta^n) \exp \left[ - \sum_{i=1}^n (t_i/\theta) \right] \quad (11)$$

where  $t_i = (t_1, t_2, t_3 \dots t_n)$

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The posterior distribution of  $\theta$  is given by,

$$\begin{aligned}\Pi(t/\theta) &= \frac{L(t/\theta)g(\theta)}{\int_{\Omega} L(t/\theta)g(\theta)d\theta} \\ &= \frac{(1/\theta^n) \exp(-n\bar{t}/\theta)(1/\theta^c)}{\int_0^{\infty} ((1/\theta^n) \exp(-n\bar{t}/\theta)(1/\theta^c))d\theta}\end{aligned}$$

Since

$$\int_0^{\infty} \frac{\exp(-a/t)}{t^{m+1}} dt = \frac{\Gamma m}{a^m}$$

Therefore

$$\Pi(\theta/t) = \frac{(n\bar{t})^{n+c-1} \exp(-n\bar{t}/\theta)}{\theta^{n+c} \Gamma(n+c-1)}; \theta > 0 \quad (12)$$

The Bayes' estimator of  $\theta$  say  $\theta^*$ , is given by

$$\begin{aligned}\theta^* &= E(\theta/t) = \int_0^{\infty} \theta \Pi(\theta/t) d\theta \\ E(\theta/t) &= \int_0^{\infty} \theta \frac{(n\bar{t})^{n+c-1} \exp(-n\bar{t}/\theta)}{\theta^{n+c} \Gamma(n+c-1)} d\theta\end{aligned}$$

Therefore,

$$\theta^* = n\bar{t}/(n+c-2) \quad (13)$$

## V. PARTICULAR CASES

- (i) For  $c = 1$ , Jeffrey prior  $g(\theta) = 1/\theta$  Substituting  $c = 1$ , from equation (7), then Bayes' estimator  $\theta^*$  becomes,  
 $\theta^* = n\bar{t}/(n-1)$
- (ii) Similarly for  $c = 2$ ,  $g(\theta) = 1/\theta^2$ , then Bayes' estimator,  $\theta^* = \bar{t}$  Here  $\bar{t}$  is a complete sufficient statistics for  $\theta$  and also unbiased estimator of  $\theta$ , therefore, by the Lehman Scheffe theorem  $\bar{t}$  is the UMVUE of  $\theta$ .
- (iii) Similarly for  $c = 3$ ,  $g(\theta) = 1/\theta^3$ , then Bayes' estimator,  $\theta^* = n\bar{t}/(n+1)$  the well know minimum mean square error (MSE) estimator of  $\theta$ . The Bayes estimator of the reliability function  $R(\tau) = \exp(-\tau/\theta)$  say  $R^*(\tau)$  becomes  
 $R^*(\tau) = E[R(\tau)/t]$   
 $R^*(\tau) = E[\exp(-\tau/\theta)/t]$

Therefore,

$$R^*(\tau) = [1 + (\tau/n\bar{t})]^{-n-c+1}; \tau > 0 \quad (14)$$

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**Example 1** The following is a sample of 20 observations from the exponential distribution with  $\theta = 4$ .

Sample				
4.41	4.56	4.22	3.74	2.73
3.61	3.46	3.34	3.94	1.59
1.56	4.13	1.34	4.72	4.56
10.32	2.84	1.39	10.28	18.22
4.41	4.56	4.22	3.74	2.73

$$\sum_{i=1}^{20} t_i = 94.93, \bar{t} = 4.74, n = 20, \theta = 4$$

Bayes estimates of  $\theta$ ,  $\theta^* = \frac{n\bar{t} + a}{(n + c - 2)}$

c \ a	0	1	2	3	$\left(\frac{Min}{Max}\right)/c$
0	5.27	5.32	5.38	5.44	<b>0.96</b>
1	4.99	5.04	5.10	5.15	<b>0.96</b>
2	4.74	4.79	4.84	4.89	<b>0.96</b>
3	4.52	4.56	4.61	4.66	<b>0.96</b>
$\left(\frac{Min}{Max}\right)/a$	<b>0.85</b>	<b>0.85</b>	<b>0.85</b>	<b>0.85</b>	-

## VI. IMPORTANT INTERPRETATION

From the above table we observe that the (Min/Max) for fixed 'c' and varying 'a' is  $n\bar{x}/(n\bar{x} + a)$  and for large 'n' would be very close to 1. Similarly (Min/Max) for fixed 'a' and varying 'c' is  $(n-2)/(n+1)$  and, for large 'n' this would again be very near to one showing these by that, for large samples, choice of prior distribution is not crucial. Further we also observe that the Bayes' estimators are more robust to changes in 'a' than those in 'c'.

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