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Existence of Positive Solution for Non Linear Singular Fractional Differential Equations

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ABSTRACT: In this dissertation, we discuss the existence of solutions for the singular nonlinear fractional boundary value problem

$$\begin{aligned}
 {}^c D^\alpha u(t) &= f\left(t, u(t), u'(t), {}^c D^\beta u(t)\right), \\
 u(0) &= au(1), \quad u'(0) = b {}^c D^\beta u(1), \\
 u''(0) &= u'''(0) = u^{(n-1)}(0) = 0,
 \end{aligned}$$

where $n \geq 3$ is an integer, $\alpha \in (n-1, n), 0 < \beta < 1, 0 < a < 1, 0 < b < \Gamma(2 - \beta)$, f is an L^q -Carathéodory function, $q > \frac{1}{\alpha-1}$ and $f(t, x, y, z)$ may be singular at value 0 in one dimension of its space variables x, y, z . here ${}^c D$ stands for the Caputo fractional derivative.

I. INTRODUCTION

A differential equation is a mathematical equation for an unknown function of one or several variables which relates the values of the function itself and of its derivatives of various order. It arises in many areas of science and technology.

Fractional differential equations are generalizations of ordinary differential equation to an arbitrary order. Recently fractional differential equations attracted many authors and it has been proved that the differential models involving nonlocal derivatives of fractional order arise in many engineering and scientific disciplines as the mathematical modeling of systems and process in many fields, for instance physics, chemistry, aerodynamics, electrodynamic of complex medium etc...

II. PRELIMINARIES

In this chapter, we give some definition and theorems which are very useful to prove our main result.

DEFINITION:

An equation involving one dependent variable and its derivatives with respect to one or more independent variables is called a **Differential Equation**.

DEFINITION:

A differential equation is called **Linear** if and only if, if the following conditions are satisfied.

(i) Every dependent variable and derivative involved occurs to the first degree only.

(ii) Product of any form of dependent variable is absent.

DEFINITION:

A differential equation is called **NonLinear** if it fails to be linear.

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DEFINITION:

Differential equation involving fractional derivatives are called **Fractional Differential Equation**.

THEOREM:

For each natural number n , problem

$${}^c D^\alpha u(t) = f\left(t, u(t), u'(t), {}^c D^\beta u(t)\right),$$

$$u(0) = au(1), \quad u'(0) = b {}^c D^\beta u(1),$$

$$u''(0) = u'''(0) = u^{(n-1)}(0) = 0,$$

has a solution $u_n \in P$ such that

$$u_n \geq \frac{mk_2}{\alpha - \beta}, \quad u'_n(t) \geq \frac{mt^{\alpha-1}}{\Gamma(\alpha)} \quad \text{and} \quad {}^c D^\beta u_n(t) = \frac{mt^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \quad \text{for all } t \in [0,1].$$

PROOF:

Let $n \geq 1$. It is sufficient to show that Q_n has a fixed point u_n in P with the desired conditions

$$\begin{aligned} (Q_n x)(t) &= \int_0^1 G(t,s) f_n(s, x(s), x'(s), {}^c D^\beta x(s)) ds \\ &\geq m \int_0^1 G(t,s) ds \\ &\geq m \int_0^1 k_2 (1-s)^{\alpha-\beta-1} ds \\ &\geq mk_2 \left[(-) \frac{(1-s)^{\alpha-\beta-1+1}}{\alpha-\beta-1+1} \right]_0^1 \\ &\geq \frac{mk_2}{\alpha-\beta} \end{aligned}$$

and so $\|Q_n x\|_* \geq \|Q_n x\| \geq \frac{mk_2}{\alpha-\beta}$

Put $\Omega_1 = \left\{ x \in X : \|x\|_* < \frac{mk_2}{\alpha-\beta} \right\}$

Then $\|Q_n x\|_* \geq \|x\|_*$ for all $x \in P \cap \partial\Omega_1$

If $v_n = h \binom{1}{n} + r \binom{1}{n} + k \binom{1}{n}$, then

$$\begin{aligned} |(Q_n x)(t)| &\leq \left| \int_0^1 G(t,s) f_n(s, u(s), u'(s), {}^c D^\beta u(s)) ds \right| \\ &\leq \int_0^1 |G(t,s)| (v_n + \gamma(s) w(1+|x(s)|, 1+|x'(s)|, 1+|{}^c D^\beta x(s)|)) ds \quad [\text{by 1.7}] \end{aligned}$$

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$$\begin{aligned} &\leq \left(\int_0^1 G(t,s)v_n ds \right) + \left(\int_0^1 G(t,s)\gamma(s) ds \right) \\ &\quad \left(\int_0^1 w(1 + |x(s)|, 1 + |x'(s)|, 1 + |{}^c D^\beta x(s)|) ds \right) \\ &\leq k_1 [v_n + w(1 + \|x\|, 1 + \|x'\|, 1 + \|{}^c D^\beta x\|)] \|\gamma\|_1 \end{aligned}$$

and

$$\begin{aligned} |(Q_n x)'(t)| &= \left| \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} f_n(s, x(s), x'(s), {}^c D^\beta x(s)) ds \right| \\ &\leq \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \left(v_n + w(1 + |x(s)|, 1 + |x'(s)|, 1 + |{}^c D^\beta x(s)|) \gamma(s) \right) ds \end{aligned}$$

[By 1.7]

$$\begin{aligned} &\leq \frac{1}{\Gamma(\alpha-1)} \left(\int_0^t (t-s)^{\alpha-2} v_n ds \right) \\ &+ \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \gamma(s) w(1 + \|x\|, 1 + \|x'\|, 1 + \|{}^c D^\beta x\|) ds \\ &\leq \frac{1}{\Gamma(\alpha-1)} \left(\left[(-) v_n \frac{(t-s)^{\alpha-2+1}}{\alpha-2+1} \right]_0^t \right. \\ &\quad \left. + w(1 + \|x\|, 1 + \|x'\|, 1 + \|{}^c D^\beta x\|) \int_0^t (t-s)^{\alpha-2} \gamma(s) ds \right) \\ &\leq \frac{1}{\Gamma(\alpha-1)} \left(v_n \frac{t^{\alpha-1}}{\alpha-1} \right. \\ &\quad \left. + w(1 + \|x\|, 1 + \|x'\|, 1 + \|{}^c D^\beta x\|) \int_0^t (t-s)^{\alpha-2} \gamma(s) ds \right) \end{aligned}$$

for all $x \in P$ and $t \in [0, 1]$. Because w is non-decreasing in all its variables.

Since $\|x\| \leq \|x\|_*$, $\|x'\| \leq \|x\|_*$,

$$\|{}^c D^\beta x\| \leq \frac{\|x'\|}{\Gamma(\beta)}$$

$$\leq \frac{\|x\|_*}{\Gamma(\beta)}$$

$$\text{and } \int_0^t (t-s)^{\alpha-2} \gamma(s) ds \leq \left(\frac{1}{d}\right)^{1/p} \|\gamma\|_q,$$

where $d = (\alpha - 2)p + 1$, we have

$$\|Q_n(x)\| \leq k_1 \left(v_n + w \left(1 + \|x\|_*, 1 + \|x\|_*, 1 + \frac{\|x\|_*}{\Gamma(\beta)} \right) \|\gamma\|_1 \right)$$

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and

$$\|(Q_n x)'\| \leq \frac{1}{\Gamma(\alpha-1)} \left(\frac{v_n}{\Gamma(\alpha-1)} + w \left(1 + \|x\|_*, 1 + \|x\|_*, 1 + \frac{\|x\|_*}{\Gamma(\beta)} \right) \left(\frac{1}{d}\right)^{1/p} \|\gamma\|_q \right)$$

Hence

$$\|Q_n x\|_* \leq M \left(\frac{v_n}{\alpha-1} + Nw \left(1 + \|x\|_*, 1 + \|x\|_*, 1 + \frac{\|x\|_*}{\Gamma(\beta)} \right) \right)$$

Where

$$N = \max \left\{ \|\gamma\|_1, \left(\frac{1}{d}\right)^{1/p} \times \|\gamma\|_q \right\} \text{ and}$$

$$M = \max \left\{ \frac{1}{\Gamma(\alpha-1)}, k_1 \right\}$$

Since $\lim_{v \rightarrow \infty} \frac{1}{v} w(1+v, 1+v, 1+v) = 0$ [by 1.5]

There exists a positive constant L such that

$$M \left(\frac{v_n}{\alpha-1} + Nw \left(1 + v, 1 + v, 1 + \frac{v}{\Gamma(\beta)} \right) \right) < v$$

for all $v \geq L$. Thus $\|Q_n x\|_* < \|x\|_*$ for all $x \in P$ with $\|x\|_* \geq L$

Put $\Omega_2 = \{x \in X: \|x\|_* < L\}$

Then $\|Q_n x\|_* < \|x\|_*$ for all $x \in P \cap \partial\Omega_2$

By using last result, Q_n has a fixed point u_n in $P \cap (\bar{\Omega}_2/\Omega_1)$

But $u_n = (Q_n u_n)(t) \geq \frac{mk_2}{\alpha-\beta}$ and

$$\begin{aligned} (Q_n x)'(t) &= \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} f_n(s, x(s), x'(s), x(s)) ds \\ &\geq \frac{m}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} ds \\ &\geq \frac{m}{\Gamma(\alpha-1)} \left[(-) \frac{(t-s)^{\alpha-2+1}}{\alpha-2+1} \right]_0^t \\ &\geq \frac{mt^{\alpha-1}}{\Gamma(\alpha)} \end{aligned}$$

for all $t \in [0,1]$ and $x \in P$

$$\text{Since } \int_0^t (t-s)^{-\beta} s^{\alpha-1} ds = \frac{\Gamma(\alpha)\Gamma(1-\beta)}{\Gamma(\alpha-\beta+1)} t^{\alpha-\beta} \quad (1.9)$$

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$$\begin{aligned}
 {}^c D^\beta u_n(t) &= \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} u_n'(s) ds \\
 &\geq \frac{m}{\Gamma(\alpha)\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} s^{\alpha-1} ds \\
 &\geq \frac{m}{\Gamma(\alpha)\Gamma(1-\beta)} \frac{\Gamma(\alpha)\Gamma(1-\beta)}{\Gamma(\alpha-\beta+1)} t^{\alpha-\beta} \quad [\text{By 1.9}] \\
 &\geq \frac{mt^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \text{ for all } t \in [0,1].
 \end{aligned}$$

Hence the proof.

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