

Positive Solutions for Systems of Nonlinear Singular Differential Equations

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ABSTRACT: By constructing a special cone and using the fixed point theorem of cone expansion and compression, this paper shows the existence of positive solutions for two-point boundary-value problems of nonlinear singular differential systems.

I. INTRODUCTION

The Necessary and sufficient conditions for the existence of

$C^2[0,1] \times C[0,1]$ and $C^3[0,1] \times C^1[0,1]$ positive solutions for the coupled systems. We investigate the BVP $u^{(4)} =$

$$f(t,u,v); -v'' = g(t,u,v);$$

$$u(0) = u(1) = u''(0) = u''(1) = 0; v(0) = v(1) = 0 \quad \rightarrow (1.1)$$

where $t \in (0,1)$, $f, g \in C[(0,1) \times [0,\infty) \times [0,\infty), [0,\infty)]$

i.e. f, g may be singular at $t=0$ and $t=1$.

II. PRELIMINARIES

2.1 Lebesgue Dominated Convergence Theorem:

Let $\{f_n\}$ be a sequence of Lebesgue integrable function on an interval I . Assume that, a) $\{f_n\}$ converges almost everywhere on I to a limit function f , and,

b) there is a non-negative function $f \in L(I)$ the sequence $\left\{ \int_I f_n \right\}$ converges and $\int_I f = \lim_{n \rightarrow \infty} \int_I f_n$

Arzela Ascoli Theorem:

Let A be a compact metric space, then a non-empty subset of $C_k(X)$ is relatively compact iff it is bounded and equicontinuous.

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III. POSITIVE SOLUTIONS FOR SYSTEMS OF NONLINEAR SINGULAR DIFFERENTIAL EQUATIONS

Let us list some conditions to be used later.

(H1) $g \in C[(0,1) \times [0,\infty) \times [0,\infty), [0,\infty)]$ and satisfy

$$\int_0^1 t(1-t) f(t, t(1-t), 1) dt < \infty, \int_0^1 t(1-t) g(t, t(1-t), 1) dt < \infty$$

$$(H2) \int_0^1 f(t, t(1-t), t(1-t)) dt < \infty, \int_0^1 g(t, t(1-t), t(1-t)) dt < \infty$$

(H3) f is quasi-homogenous with respect to the last two variables,

that is, there are constants $\lambda_1, \mu_1, \alpha_1, \beta_1, N_1, M_1, N_2, M_2$ with

$$0 \leq \lambda_1 \leq \mu_1 < +\infty, 0 \leq \alpha_1 \leq \beta_1 \leq 1, \mu_1 + \beta_1 < 1, 0 < N_1 \leq 1 \leq M_1,$$

$$0 < N_2 \leq 1 \leq M_2 \text{ such that for all } 0 < t < 1, u \geq 0, v \geq 0$$

$$(a) c^{\mu_1} f(t, u, v) \leq f(t, cu, v) \leq c^{\lambda_1} f(t, u, v), 0 < c \leq N_1;$$

satisfying $c^{\lambda_1} f(t, u, v) \leq f(t, cu, v) \leq c^{\mu_1} f(t, u, v), c \geq M_1;$

$$(b) c^{\beta_1} f(t, u, v) \leq f(t, u, cv) \leq c^{\alpha_1} f(t, u, v), 0 < c \leq N_2;$$

$$c^{\alpha_1} f(t, u, v) \leq f(t, u, cv) \leq c^{\beta_1} f(t, u, v), c \geq M_2.$$

(H4) g is quasi-homogenous with respect to the last two variables,

that is, there are constants $\lambda_2, \mu_2, \alpha_2, \beta_2, N_3, M_3, N_4, M_4$ with

$$0 \leq \lambda_2 \leq \mu_2 < +\infty, 0 \leq \alpha_2 \leq \beta_2 \leq 1, \mu_2 + \beta_2 < 1,$$

$$0 < N_3 \leq 1 \leq M_3, 0 < N_4 \leq 1 \leq M_4 \text{ such that for all}$$

$$0 < t < 1, u \geq 0, v \geq 0 \text{ satisfying,}$$

$$(a) c^{\mu_2} g(t, u, v) \leq g(t, cu, v) \leq c^{\lambda_2} g(t, u, v), 0 < c \leq N_3;$$

$$c^{\lambda_2} g(t, u, v) \leq g(t, cu, v) \leq c^{\mu_2} g(t, u, v), c \geq M_3.$$

$$(b) c^{\beta_2} g(t, u, v) \leq g(t, u, cv) \leq c^{\alpha_2} g(t, u, v), 0 < c \leq N_4;$$

$$c^{\alpha_2} g(t, u, v) \leq g(t, u, cv) \leq c^{\beta_2} g(t, u, v), c \geq M_4.$$

(H5) There exist $0 < \gamma_i < 1, k_i \geq 0 (i = 1, 2)$ such that

$$f(t, u, v) \geq k_1 (u + v)^{\gamma_1}, g(t, u, v) \geq k_2 (u + v)^{\gamma_2}$$

for any $t \in J, (u, v) \in P$.

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Result-3.1:

The functions $u \in C^2[0,1] \cap C^{(4)}(0,1), v \in C[0,1] \cap C^2(0,1)$ form a solution to (1.1) iff (u,v) is a fixed point of the integral operator

$A(u, v) = (A_1(u, v), A_2(u, v))$ in $C^2[0,1] \times C[0,1]$ where

$$\begin{aligned}
 A_1(u, v)(t) &= \int_0^1 G(t, s) \int_0^1 G(s, \tau) f(\tau, u(\tau), v(\tau)) d\tau ds \\
 A_2(u, v)(t) &= \int_0^1 G(t, s) g(s, u(s), v(s)) ds \\
 G(t, s) &= \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1; \\ s(1-t), & 0 \leq s \leq t \leq 1, \end{cases}
 \end{aligned} \rightarrow (3.1)$$

Result-3.2:

Suppose $(u, v) \in P$ and $\mu \in \left(0, \frac{1}{2}\right)$.

Then $u(t)+v(t) \geq \mu(1-\mu)(\|u\|_0 + \|v\|_0), t \in [\mu, 1-\mu]$.

Lemma-3.3:

Assume that (H1),(H3),(H4) hold. Then $A:P \rightarrow P$ is a completely continuous operator.

Theorem-3.4:

Suppose (H3) –(H5) hold. Then (1.1) has a $C^2[0,1] \times C[0,1]$ positive solution (u,v) , iff (H1) holds.

Theorem-3.5:

Suppose (H3) –(H5) hold. Then (1.1) has a $C^3[0,1] \times C^1[0,1]$ positive solution (u,v) , iff (H2) holds

Proof :

Sufficiency: First we prove that $\int_0^1 f(t, t(1-t), t(1-t)) dt < +\infty$

Implies $\int_0^1 t(1-t) f(t, t(1-t), 1) dt < +\infty$

Choosing positive number $c \geq \max \left\{ M_2, \frac{1}{4N_2} \right\}$.

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Then

$$\begin{aligned} f(t, t(1-t), t(1-t)) &= f(t, t(1-t), c \frac{t(1-t)}{c}) \\ &\geq c^{\alpha_1 - \beta_1} (t(1-t))^{\beta_1} f(t, t(1-t), 1) \\ &\geq c^{\alpha_1 - \beta_1} t(1-t) f(t, t(1-t), 1). \end{aligned}$$

Consequently, we can get

$$\int_0^1 f(t, t(1-t), t(1-t)) dt \geq c^{\alpha_1 - \beta_1} \int_0^1 t(1-t) f(t, t(1-t), 1) dt,$$

Namely, we can get, $\int_0^1 t(1-t) f(t, t(1-t), 1) dt < +\infty$

we can prove that $\int_0^1 g(t, t(1-t), t(1-t)) dt < +\infty$

This implies $\int_0^1 t(1-t) g(t, t(1-t), 1) dt < +\infty$

From above inequalities,

we know that (1.1) exists a $C^2[0, 1] \times C[0, 1]$ positive solution (u, v) .

Therefore it suffices to show that $u'''(0^+), u'''(1^-), v'(0^+)$ and $v'(1^-)$ exists.

The same reason as the proof of Theorem-3.4 of necessity asserts that there exists $0 < m_1 < 1 < m_2$ and $0 < n_1 < 1 < n_2$ satisfying $m_1 t(1-t) \leq u(t) \leq m_2 t(1-t)$ and $n_1 t(1-t) \leq v(t) \leq n_2 t(1-t), t \in J$.

Let $c_1 \geq \max \left\{ M_1, \frac{m_2}{N_1} \right\}$ and $c_1 \geq \max \left\{ M_2, \frac{n_2}{N_2} \right\}$

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$$\begin{aligned} \int_0^1 |u^{(4)}(t)| dt &= \int_0^1 f(t, u(t), v(t)) dt \\ &= \int_0^1 f\left(t, c_1 \frac{u(t)}{c_1 t(1-t)}, c_2 \frac{v(t)}{c_2 t(1-t)}\right) dt \\ &\leq c_1^{\mu_1} \left(\frac{u(t)}{c_1 t(1-t)}\right)^{\lambda_1} c_2^{\beta_1} \left(\frac{v(t)}{c_2 t(1-t)}\right)^{\alpha_1} \int_0^1 f(t, t(1-t), t(1-t)) dt \\ &\leq c_1^{\mu_1 - \lambda_1} c_2^{\beta_1 - \alpha_1} m_2^{\lambda_1} n_2^{\alpha_1} \int_0^1 f(t, t(1-t), t(1-t)) dt \\ &< +\infty \end{aligned}$$

This guarantees $u'''(0^+)$ and $u'''(1^-)$ exists.

On the other hand, let $c_3 \geq \max\left\{M_3, \frac{m_2}{N_3}\right\}$ and $c_4 \geq \max\left\{M_4, \frac{n_2}{N_4}\right\}$ Then

$$\begin{aligned} \int_0^1 |-v''(t)| dt &= \int_0^1 g(t, u(t), v(t)) dt \\ &= \int_0^1 g\left(t, c_3 \frac{u(t)}{c_3 t(1-t)}, c_4 \frac{v(t)}{c_4 t(1-t)}\right) dt \\ &= c_3^{\mu_2} \left(\frac{u(t)}{c_3 t(1-t)}\right)^{\lambda_2} c_4^{\beta_2} \left(\frac{v(t)}{c_4 t(1-t)}\right)^{\alpha_2} \int_0^1 g(t, t(1-t), t(1-t)) dt \\ &\leq c_3^{\mu_2 - \lambda_2} c_4^{\beta_2 - \alpha_2} m_2^{\lambda_2} n_2^{\alpha_2} \int_0^1 g(t, t(1-t), t(1-t)) dt \\ &< +\infty \end{aligned}$$

This means that $v'(0^+)$ and $v'(1^-)$ exists

Necessity:

Let (u, v) be a $C^3[0, 1] \times C^1[0, 1]$ positive solution of (1.1).

The same reason as the beginning of the proof of sufficiency asserts that there exists $0 < m_1 < 1 < m_2$ and $0 < n_1 < 1 < n_2$ satisfying $m_1 t(1-t) \leq u(t) \leq m_2 t(1-t)$ and $n_1 t(1-t) \leq v(t) \leq n_2 t(1-t), t \in J$.

Suppose $c_1 \leq \min\left\{N_1, \frac{1}{M_1 m_2}\right\}, c_2 \leq \min\left\{N_2, \frac{1}{M_2 n_2}\right\}, c_3 \leq \min\left\{N_3, \frac{1}{M_3 m_2}\right\}$

and $c_4 \leq \min\left\{N_4, \frac{1}{M_4 n_2}\right\}$

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$$\begin{aligned}
 f(t, t(1-t), t(1-t)) &= f\left(t, c_1 \frac{t(1-t)}{c_1 u(t)} u(t), c_2 \frac{t(1-t)}{c_2 v(t)} v(t)\right) \\
 &\leq c_1^{\lambda_1} \left(\frac{t(1-t)}{c_1 u(t)}\right)^{\mu_1} c_2^{\alpha_1} \left(\frac{t(1-t)}{c_2 v(t)}\right)^{\beta_1} f(t, u(t), v(t)) \\
 &\leq c_1^{\lambda_1 - \mu_1} c_2^{\alpha_1 - \beta_1} m_1^{-\mu_1} n_1^{-\beta_1} f(t, u(t), v(t))
 \end{aligned}$$

Consequently,

$$\begin{aligned}
 \int_0^1 f(t, t(1-t), t(1-t)) dt &\leq c_1^{\lambda_1 - \mu_1} c_2^{\alpha_1 - \beta_1} m_1^{-\mu_1} n_1^{-\beta_1} [u'''(1^-) - u'''(0^+)] \\
 &< +\infty
 \end{aligned}$$

On the other hand, we can also prove

$$\begin{aligned}
 g(t, t(1-t), t(1-t)) &= g\left(t, c_3 \frac{t(1-t)}{c_3 u(t)} u(t), c_4 \frac{t(1-t)}{c_4 v(t)} v(t)\right) \\
 &\leq c_3^{\lambda_2} \left(\frac{t(1-t)}{c_3 u(t)}\right)^{\mu_2} c_4^{\alpha_2} \left(\frac{t(1-t)}{c_4 v(t)}\right)^{\beta_2} g(t, u(t), v(t)) \\
 &\leq c_3^{\lambda_2 - \mu_2} c_4^{\alpha_2 - \beta_2} m_1^{-\mu_2} n_1^{-\beta_2} g(t, u(t), v(t))
 \end{aligned}$$

Consequently,

$$\begin{aligned}
 \int_0^1 g(t, t(1-t), t(1-t)) dt &\leq c_3^{\lambda_2 - \mu_2} c_4^{\alpha_2 - \beta_2} m_1^{-\mu_2} n_1^{-\beta_2} [v'(1^-) - v'(0^+)] \\
 &< +\infty
 \end{aligned}$$

Therefore, our conclusion follows.

IV. CONCLUSION

In this work, using the fixed point theorem of Cone expansion and compression, the existence of positive solutions for two-point boundary-value problems of nonlinear singular differential systems are discussed.