

Positive Solutions of the Non Local Boundary Value Problem for a Singular Fractional Differential Equation

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ABSTRACT: In this dissertation, we study on positive solutions of the nonlocal boundary value problem for a class of singular fractional differential equations. The existence of positive solutions is obtained by the method of upper and lower solutions together with the Schauder fixed point theorem.

I. INTRODUCTION

GENERAL STUDY:

Differential Equations arise in all branches of mathematics. A Differential Equation is an equation that involves one or more derivatives of an unknown function. Solving the Differential equation means finding the function that satisfies the Differential equation. Differential Equations are classified as Ordinary Differential Equations and Partial Differential Equations.

Fractional calculus is a generalization of Ordinary differentiation and integration to arbitrary order. The idea of fractional calculus has been the subject of interest not only among mathematics, but also among physicists and engineers.

The study of Fractional Differential Equations (FDE) has become a very important and useful area of mathematics over the last few decades due to its numerous applications in various areas of science and engineering.

The Caputo fractional derivative satisfies these demands, while the Riemann-Liouville fractional derivative is not suitable for mixed boundary conditions. Recently, the theory on the existence and uniqueness of solutions of linear and nonlinear Fractional differential equations has attracted the attention of many authors.

Consider the fractional differential equation with an integral boundary condition:

$$\left. \begin{aligned} & {}^c D_0^\alpha u(t) + f(t, u(t)) = 0, \\ u'(0) = \dots = u^{(n-1)}(0) &= 0, \\ u(1) &= \int_0^1 u(s) d\mu(s), \end{aligned} \right\} \text{--- (1.2)}$$

Where $n \geq 2$, $\alpha \in (n - 1, n)$ and $\mu(s)$ is a function of bounded variation.

We assume that f may have a singularity at $t = 1$ and

$$\int_0^1 d\mu(s) < 1. \text{--- (1.3)}$$

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II. PRELIMINARIES

In this chapter, we give some basic definitions and lemma which are very useful to prove the main results of the dissertation.

DEFINITION:

The Differential equations together with specified values of the dependent variable y and its derivative at two or more given points. Such specified values are called **Boundary conditions** and the differential equation along with these boundary conditions is called a **Boundary value problem**.

DEFINITION: 1

An operator $A : D \rightarrow X$ where D is a subset of Banach space X is said to be **Completely continuous** if it is continuous and compact.

DEFINITION: 2

Let X be a Banach space and F be an operator such that $F : X \rightarrow X$. We say that $x \in X$ is a **Fixed point** of F if $Fx = x$.

DEFINITION: 3

A set G of functions $[a, b]$ is called **Equicontinuous** if for every $\epsilon > 0$, there exists some $\delta > 0, g \in G$ and all $x_1, x_2 \in [a, b]$ with $|x_1 - x_2| < \delta$ we have $|g(x_1) - g(x_2)| < \epsilon$.

DEFINITION: 4

Let X be a Banach space. A subset $C \subseteq X$, is said to be **Convex** if for each pair $a, b \in C$, the line segment joining them is defined by $(1 - \lambda)a + \lambda b, 0 \leq \lambda \leq 1$ is in C .

SCHAUDER FIXED POINT THEOREM: 5

Let C be a bounded, closed and convex set in a Banachspace X . If $f : C \rightarrow C$ is a continuous map, then f has a fixed point in C .

LEMMA:

Let $\alpha \in (n - 1, n), \sigma \in [0, \alpha - 1), (1 - t)^\sigma f(t, u)$ be continuous on $[0, 1] \times [0, \infty)$ and $f(t, u) \in C^1((0, 1) \times (0, \infty))$ is non-negative. Assume that (1.3) and (3.2) hold. If $u \in C^1(0, 1) \cap C[0, 1]$ is non-negative and satisfies $u = Tu$, where

$$Tu(t) = \int_0^1 G(t, s) f(s, u(s)) ds,$$

then $u(t)$ is n -times differentiable on $(0, 1)$ and is a solution of (1.2).

MAIN RESULTS

Consider the problem,

$$\begin{aligned} {}^c D_{0+}^\alpha u(t) + \tilde{f}(t) &= 0, \\ u'(0) = \dots = u^{(n-1)}(0) &= u(1) = 0 \end{aligned} \quad \text{---(3.1)}$$

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By the lemma 2.26, we know that, under some conditions on \tilde{f} , the function

$$\frac{-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \tilde{f}(s) ds$$

satisfies the equation in (3.1).

In order that the boundary condition at $t = 1$ holds, we take

$$\frac{-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \tilde{f}(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \tilde{f}(s) ds.$$

Therefore, the Green function of (3.1) is (The validity of the boundary conditions at $t = 0$ in the lemma 3.1):

$$G_0(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (1-s)^{\alpha-1}, & 0 \leq t \leq s \leq 1. \\ (1-s)^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1. \end{cases}$$

The Green function of equation (1.2) is then given as

$$G(t,s) = \frac{1}{1-k} \int_0^1 G_0(\tau,s) d\mu(\tau) + G_0(t,s),$$

where

$$k = \int_0^1 d\mu(\tau)$$

(We refer the lemma 3.1 for the validity of the integral boundary condition at $t = 1$).

We assume that,

$$\int_0^1 G_0(\tau,s) d\mu(\tau) \geq 0 \text{ for } s \in [0,1]. \quad \text{_____ (3.2)}$$

The equation (1.3) and (3.2), we have

$$G(t,s) \geq 0.$$

For non-negative numbers a and b with $a < b$, we define the upper-control function

$$H_a(t,u) = \sup_{a \leq \eta \leq u} f(t,\eta), \quad \text{and} \quad \text{lower-control function}$$

$$h_b(t,u) = \inf_{u \leq \eta \leq b} f(t,\eta), \quad \text{where the sup and inf are taken for each fixed } t \in [0,1].$$

$H_a(t,u)$, $h_b(t,u)$ are monotone non-decreasing on u and

$$h_b(t,u) \leq f(t,u) \leq H_a(t,u) \text{ for } (t,u) \in [0,1] \times [a,b].$$

THEOREM:

Suppose that the conditions in lemma 3.1 hold. Assume that $\hat{u}(t)$ and $\tilde{u}(t)$ are a pair of order upper and lower solutions for (1.2) with respect to the non-negative constants a and b , then (1.2) has at least one solution $u(t)$ which is n -times differentiable. In addition, the following holds:

$$a \leq \hat{u}(t) \leq u(t) \leq \tilde{u}(t) \leq b \text{ for } t \in [0,1].$$

PROOF:

Let $S = \{v(t) \mid v(t) \in P, \hat{u}(t) \leq v(t) \leq \tilde{u}(t), t \in [0,1]\}$.

S is convex, bounded and closed. We claim that $T : S \rightarrow S$.

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For $v(t) \in \mathcal{S}$, we have

$$\begin{aligned} Tv(t) &= \int_0^1 G(t,s)f(s,v(s))ds \\ &\leq \int_0^1 G(t,s)H_a(s,v(s))ds \\ &\leq \int_0^1 G(t,s)H_a(s,\hat{u}(s))ds \\ &\leq \hat{u}(t) \end{aligned}$$

Similarly,

$$\begin{aligned} Tv(t) &= \int_0^1 G(t,s)f(s,v(s))ds \\ &\geq \int_0^1 G(t,s)h_b(s,v(s))ds \\ &\geq \int_0^1 G(t,s)h_b(s,\hat{u}(t))ds \\ &\geq \hat{u}(t) \end{aligned}$$

By applying Schauder fixed point theorem, we conclude that T has a fixed point $u(t)$ in \mathcal{S} , which, by lemma 3.1, is a solution of (1.2).

Hence the proof..

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