

The Inverse of Perturbed M-Matrices and Diagonal of M^{-1}

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ABSTRACT: We consider the inverse of perturbed M-matrix and the effect of changing single elements inside the diagonal of M^{-1} . The largest interval is given by which the diagonal entries of the inverse of tridiagonal M-matrices can be perturbed without losing the property of total nonnegativity.

KEYWORDS: Tridiagonal Matrices, Compound Matrix, Element-wise Perturbations.

I. INTRODUCTION

In many mathematical problems, Z-matrices and M-matrices play an important role. It is often useful to know the properties of their inverses, especially when the Z-matrices and the M-matrices have a special combinatorial structure, for more details we refer the reader [1]. M-matrices have important applications, for instance, in iterative methods, in numerical analysis, in the analysis of dynamical systems, in economics, and in mathematical programming. One of the most important properties of some kinds of M-matrices is the non-negativity of their inverses, which plays central role in many of mathematical problems.

For k, n we denote by $Q_{k,n}$ the set of all strictly increasing sequences of k integers chosen from $\{1, 2, \dots, n\}$. For $\alpha = \{\alpha_1, \dots, \alpha_k\}, \beta = \{\beta_1, \dots, \beta_k\} \in Q_{k,n}$, we denote by $A[\alpha | \beta]$ the $k \times k$ submatrix of A contained in the rows indexed by $\alpha_1, \dots, \alpha_k$ and columns indexed by β_1, \dots, β_k . A matrix A is called totally positive (abbreviated TP) if $\det A[\alpha | \beta] > 0$ and totally nonnegative (abbreviated TN) if $\det A[\alpha | \beta] \geq 0$ for all $\alpha, \beta \in Q_{k,n}, k=1, 2, \dots, n$. For a given index $\alpha = \{\alpha_1, \dots, \alpha_k\}$, with property $\alpha_j < \alpha_{j+1}, j=1, \dots, k-1$, the dispersion of α , denoted by $d(\alpha)$, is defined to be $\alpha_i = \alpha \setminus \{n-i+1\}, i = \{1, \dots, n\}$.

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II. NOTATIONS

We use the following notation for general tridiagonal M-matrix:

$$M = \begin{bmatrix} a_1 & -b_1 & 0 & \cdots & 0 \\ -c_1 & a_2 & -b_2 & \cdots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \cdots & -c_{n-2} & a_{n-1} & -b_{n-1} \\ 0 & \cdots & 0 & -c_{n-1} & a_n \end{bmatrix}$$

Where a_i, b_i and $c_i > 0$, and each a_i is large enough that M is strictly diagonally dominant. We let E_{ij} to be the square standard basis matrix whose only non zero entry is 1 that occurs in the (i,j) position.

III. COMPOUND MATRICES

Definition: 3.1

An $n \times n$ real matrix $M = (m_{ij})$ is called M-matrix if $m_{ii} > 0, i=1,2,\dots,n$ and $m_{ij} \leq 0, i \neq j$, over the years, M-matrices have considerable attention, in large part.

Definition: 3.2

A matrix is said to be generalized (strictly) diagonally dominant if $m_{ii} > \sum_{\substack{i=1 \\ i \neq j}}^n m_{ij}$. Of particular importance to

us is the fact that since M is an M-matrix it is non-singular and $M^{-1} > 0$ where the inequality is satisfied element-wise.

Definition: 3.3

Let $A = (a_{ij})_{i,j=1}^n$ be a square matrix of order n . Let $\alpha = \{1, \dots, n\}$ be the index set of cardinality n , defining $\alpha_i = \alpha \setminus \{n - i + 1\}, i = \{1, \dots, n\}$ are the index sets of cardinality $n-1$. Construct the following table which depends on α_i .

	α_1	α_2	\cdots	α_n
α_1	$\det A(\alpha_1, \alpha_1)$	$\det A(\alpha_1, \alpha_2)$	\cdots	$\det A(\alpha_1, \alpha_n)$
α_2	$\det A(\alpha_2, \alpha_1)$	$\det A(\alpha_2, \alpha_2)$	\cdots	$\det A(\alpha_2, \alpha_n)$
\vdots	\vdots	\vdots	\cdots	\vdots
α_n	$\det A(\alpha_n, \alpha_1)$	$\det A(\alpha_n, \alpha_2)$	\cdots	$\det A(\alpha_n, \alpha_n)$

The created matrix

$$c_{n-1}(A) = \begin{bmatrix} \det A(\alpha_1, \alpha_1) & \det A(\alpha_1, \alpha_2) & \cdots & \det A(\alpha_1, \alpha_n) \\ \det A(\alpha_2, \alpha_1) & \det A(\alpha_2, \alpha_2) & \cdots & \det A(\alpha_2, \alpha_n) \\ \vdots & \vdots & \cdots & \vdots \\ \det A(\alpha_n, \alpha_1) & \det A(\alpha_n, \alpha_2) & \cdots & \det A(\alpha_n, \alpha_n) \end{bmatrix}$$

is called $(n-1)^{th}$, compound matrix of A .

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For example, if $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ with indexed sets $\alpha_1 = \{1,2\}, \alpha_2 = \{1,3\}$ and $\alpha_3 = \{2,3\}$

$$\text{Then } C_2(A) = \begin{bmatrix} \det A(\alpha_1, \alpha_1) & \det A(\alpha_1, \alpha_2) & \det A(\alpha_1, \alpha_3) \\ \det A(\alpha_2, \alpha_1) & \det A(\alpha_2, \alpha_2) & \det A(\alpha_2, \alpha_3) \\ \det A(\alpha_3, \alpha_1) & \det A(\alpha_3, \alpha_2) & \det A(\alpha_3, \alpha_3) \end{bmatrix} = \begin{bmatrix} -3 & -6 & -3 \\ -6 & -12 & -6 \\ -3 & -6 & -3 \end{bmatrix}$$

IV. TRIDIAGONAL M-MATRICES

Theorem 4.1

Let $M = \begin{bmatrix} a_1 & -b_1 & 0 & \dots & 0 \\ -c_1 & a_2 & -b_2 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & -c_{n-2} & a_{n-1} & -b_{n-1} \\ 0 & \dots & 0 & -c_{n-1} & a_n \end{bmatrix}$ be strictly diagonally dominant M-matrix.

If $C_{n-1}(M)$ is the $(n-1)^{\text{th}}$ compound matrix of M then the matrix $S(C_{n-1}(M))^T S$ is totally nonnegative matrix. Moreover, $\det S(C_{n-1}(M))^T S = (\det M)^{n-1}$ where $S = \text{diag}(1, -1, \dots, (-1)^{n+1})$.

Proof:

Let M be strictly diagonally dominant M-matrix.

Then $M^{-1} = \frac{S(C_{n-1}(M))^T S}{\det M}$ is totally nonnegative matrix. So is $S(C_{n-1}(M))^T S$.

There is an explicit formula for the determinant of $\det S(C_{n-1}(M))^T S = (\det M)^{n-1}$ given as

$$\det S(C_{n-1}(M))^T S = \begin{vmatrix} \det M(\alpha_1, \alpha_1) & -\det M(\alpha_2, \alpha_1) & \dots & (-1)^{n+1} \det M(\alpha_n, \alpha_1) \\ -\det M(\alpha_1, \alpha_2) & \det M(\alpha_2, \alpha_2) & \dots & (-1)^{n+2} \det M(\alpha_n, \alpha_2) \\ \vdots & \vdots & \dots & \vdots \\ (-1)^{n+1} \det M(\alpha_1, \alpha_n) & (-1)^{n+2} \det M(\alpha_2, \alpha_n) & \dots & (-1)^{2n} \det M(\alpha_n, \alpha_n) \end{vmatrix}$$

$$= \det M(\alpha_1, \alpha_1) \begin{vmatrix} 1 & \frac{-\det M(\alpha_2, \alpha_1)}{\det M(\alpha_1, \alpha_1)} & \dots & \frac{(-1)^{n+1} \det M(\alpha_n, \alpha_1)}{\det M(\alpha_1, \alpha_1)} \\ -\det M(\alpha_1, \alpha_2) & \det M(\alpha_2, \alpha_2) & \dots & (-1)^{n+2} \det M(\alpha_n, \alpha_2) \\ \vdots & \vdots & \dots & \vdots \\ (-1)^{n+1} \det M(\alpha_1, \alpha_n) & (-1)^{n+2} \det M(\alpha_2, \alpha_n) & \dots & (-1)^{2n} \det M(\alpha_n, \alpha_n) \end{vmatrix}$$

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$$= \det M \begin{pmatrix} \alpha_1, \alpha_1 \\ \vdots \\ \alpha_n, \alpha_n \end{pmatrix} = \begin{vmatrix} 1 & \frac{-\det M(\alpha_2, \alpha_1)}{\det M(\alpha_1, \alpha_1)} & \dots & \frac{(-1)^{n+1} \det M(\alpha_n, \alpha_1)}{\det M(\alpha_1, \alpha_1)} \\ 0 & \frac{\det M(\alpha_2, \alpha_2) \det M(\alpha_1, \alpha_1) - \det M(\alpha_2, \alpha_1) \det M(\alpha_1, \alpha_2)}{\det M(\alpha_1, \alpha_1)} & \dots & \frac{(-1)^{n+2} \det M(\alpha_n, \alpha_2) \det M(\alpha_1, \alpha_1) + (-1)^{n+1} \det M(\alpha_2, \alpha_1) \det M(\alpha_1, \alpha_2)}{\det M(\alpha_1, \alpha_1)} \\ \vdots & \vdots & \dots & \vdots \\ 0 & \frac{(-1)^{n+2} \det M(\alpha_2, \alpha_n) \det M(\alpha_1, \alpha_1) + (-1)^{n+1} \det M(\alpha_1, \alpha_n) \det M(\alpha_2, \alpha_1)}{\det M(\alpha_1, \alpha_1)} & \dots & \frac{\det M(\alpha_n, \alpha_n) \det M(\alpha_1, \alpha_1) + (-1)^{2n+1} \det M(\alpha_1, \alpha_n) \det M(\alpha_n, \alpha_1)}{\det M(\alpha_1, \alpha_1)} \end{vmatrix}$$

$$= \frac{1}{(\det M(\alpha_1, \alpha_1))^{n-1}} \begin{vmatrix} 1 & \frac{-\det M(\alpha_2, \alpha_1)}{\det M(\alpha_1, \alpha_1)} & \dots & \frac{(-1)^{n+1} \det M(\alpha_n, \alpha_1)}{\det M(\alpha_1, \alpha_1)} \\ 0 & \det M((\alpha_1)_1, (\alpha_1)_1) \det M & \dots & (-1)^n \det M((\alpha_1)_1, (\alpha_1)_{n-1}) \det M \\ \vdots & \vdots & \dots & \vdots \\ 0 & (-1)^n \det M((\alpha_1)_1, (\alpha_1)_{n-1}) \det M & \dots & \det M((\alpha_1)_{n-1}, (\alpha_1)_{n-1}) \det M \end{vmatrix}$$

Where

$$(\alpha_i)_i = \alpha_i \setminus \{n-i\} = \frac{(\det M)^{n-1}}{(\det M(\alpha_1, \alpha_1))^{n-1}} \begin{vmatrix} 1 & \frac{-\det M(\alpha_2, \alpha_1)}{\det M(\alpha_1, \alpha_1)} & \dots & \frac{(-1)^{n+1} \det M(\alpha_n, \alpha_1)}{\det M(\alpha_1, \alpha_1)} \\ 0 & \det M((\alpha_1)_1, (\alpha_1)_1) & \dots & (-1)^n \det M((\alpha_1)_1, (\alpha_1)_{n-1}) \\ \vdots & \vdots & \dots & \vdots \\ 0 & (-1)^n \det M((\alpha_1)_1, (\alpha_1)_{n-1}) & \dots & \det M((\alpha_1)_{n-1}, (\alpha_1)_{n-1}) \end{vmatrix}$$

and now apply an induction argument to get the result.

Example: 4.1

Let $M = \begin{bmatrix} 4 & -1 & 0 \\ -2 & 5 & -1 \\ 0 & -2 & 4 \end{bmatrix}$ be strictly diagonally dominant M-matrix, then $C_2(M) = \begin{bmatrix} 18 & -4 & 1 \\ -8 & 16 & -4 \\ 4 & -8 & 18 \end{bmatrix}$ and

$S(C_2(M))^T S = \begin{bmatrix} 18 & 8 & 4 \\ 4 & 16 & 8 \\ 1 & 4 & 18 \end{bmatrix}$ is totally nonnegative. $\det(M) = 64$ and $\det S(C_2(M))^T S = 64^2 = 4096$.

Example: 4.2

Let $M = \begin{bmatrix} 3 & -1 & 0 & 0 \\ -2 & 3 & -1 & 0 \\ 0 & -2 & 5 & -1 \\ 0 & 0 & -2 & 3 \end{bmatrix}$ be strictly diagonally dominant M-matrix, then

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$$C_3(M) = \begin{bmatrix} 29 & -7 & 3 & -1 \\ -14 & 21 & -9 & 3 \\ 12 & -18 & 39 & 3 \\ -8 & 12 & -26 & 33 \end{bmatrix} \text{ and } S(C_3(M))^T S = \begin{bmatrix} 29 & 14 & 2 & 8 \\ 7 & 21 & 18 & 12 \\ 3 & 9 & 39 & 26 \\ 1 & 3 & 13 & 33 \end{bmatrix} \text{ is totally nonnegative.}$$

$\det(M)=73$ and $\det S(C_2(M))^T S=73^3 = 389017$.

Models:

$$\det M^{-1}(1,2,\dots,n-1/2,3,\dots,n) = \det M^{-1}(2,3,\dots,n/1,2,\dots,n-1) = 0 \text{ for } n \geq 3$$

Moreover, $\det M^{-1} = \frac{1}{\det M}$

Suppose M is nonsingular then $M^{-1} = \frac{S(C_{n-1}(M))^T S}{\det M}$, so $\det M^{-1} = \det \left(\frac{S(C_{n-1}(M))^T S}{\det M} \right) =$

$$\frac{\det(S(C_{n-1}(M))^T S)}{(\det M)^n} = \frac{(\det M)^{n-1}}{(\det M)^n} = \frac{1}{\det M}$$

Example: 4.3

When $n=4$, the M -matrix of our form

Let $M = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -2 & 3 & -1 & 0 \\ 0 & -2 & 4 & -1 \\ 0 & 0 & -2 & 5 \end{bmatrix}$ be a tridiagonal M -matrix and has an inverse given as

$$M^{-1} = \begin{bmatrix} m_{11}^{-1} & m_{12}^{-1} & m_{13}^{-1} & m_{14}^{-1} \\ m_{21}^{-1} & m_{22}^{-1} & m_{23}^{-1} & m_{24}^{-1} \\ m_{31}^{-1} & m_{32}^{-1} & m_{33}^{-1} & m_{34}^{-1} \\ m_{41}^{-1} & m_{42}^{-1} & m_{43}^{-1} & m_{44}^{-1} \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} \frac{\det M(2,3,4/2,3,4)}{\det M} & -\frac{\det M(1,3,4/2,3,4)}{\det M} & \frac{\det M(1,2,4/2,3,4)}{\det M} & -\frac{\det M(1,2,3/2,3,4)}{\det M} \\ \frac{\det M(2,3,4/1,3,4)}{\det M} & \frac{\det M(1,3,4/1,3,4)}{\det M} & \frac{\det M(1,2,4/1,3,4)}{\det M} & \frac{\det M(1,2,3/1,3,4)}{\det M} \\ \frac{\det M(2,3,4/1,2,4)}{\det M} & -\frac{\det M(1,3,4/1,2,4)}{\det M} & \frac{\det M(1,2,4/1,2,4)}{\det M} & -\frac{\det M(1,2,3/1,2,4)}{\det M} \\ \frac{\det M(2,3,4/1,2,3)}{\det M} & \frac{\det M(1,3,4/1,2,3)}{\det M} & -\frac{\det M(1,2,4/1,2,3)}{\det M} & \frac{\det M(1,2,3/1,2,3)}{\det M} \end{bmatrix}$$

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$$M^{-1} = \begin{bmatrix} 0.8462 & 0.3462 & 0.0962 & 0.0192 \\ 0.6923 & 0.6923 & 0.1923 & 0.0385 \\ 0.3846 & 0.3846 & 0.3846 & 0.1538 \\ 0.1538 & 0.1538 & 0.1538 & 0.2308 \end{bmatrix}$$

$$\det M^{-1}(1,2,3/2,3,4) = \det M^{-1}(2,3,4/1,2,3) \cong 0$$

Observe that the error came from the rounded to the nearest part of 10,000.

Theorem: 4.2

Let M be the M-matrix defined above then $\det(M^{-1} - xE_{ii}) = \det M^{-1} - \frac{x \times a_i}{\det M}$

Example: 4.4

When n=5, the M-matrix of our form

Let

$$M = \begin{bmatrix} 3 & -2 & 0 & 0 & 0 \\ -1 & 5 & -2 & 0 & 0 \\ 0 & -3 & 7 & -2 & 0 \\ 0 & 0 & -1 & 9 & -2 \\ 0 & 0 & 0 & -3 & 11 \end{bmatrix}$$

be a tridiagonal M-matrix and has an inverse given as

$$M^{-1} = \begin{bmatrix} 0.5694 & 0.1934 & 0.0572 & 0.0135 & 0.0025 \\ 0.0967 & 0.3003 & 0.0858 & 0.0203 & 0.0037 \\ 0.0429 & 0.1287 & 0.1859 & 0.0440 & 0.0080 \\ 0.0051 & 0.0152 & 0.0220 & 0.1235 & 0.0225 \\ 0.0014 & 0.0042 & 1.2818 & 0.1235 & 0.1469 \end{bmatrix}$$

$$\text{Let } x=0.2 \text{ then } M^{-1} - 0.2E_{11} = \begin{bmatrix} 0.3694 & 0.1934 & 0.0572 & 0.0135 & 0.0025 \\ 0.0967 & 0.3003 & 0.0858 & 0.0203 & 0.0037 \\ 0.0429 & 0.1287 & 0.1859 & 0.0440 & 0.0080 \\ 0.0051 & 0.0152 & 0.0220 & 0.1235 & 0.0225 \\ 0.0014 & 0.0042 & 1.2818 & 0.1235 & 0.1469 \end{bmatrix}$$

$$\det(M^{-1} - 0.2E_{11}) = 0.0002 \text{ and } \det M^{-1} - \frac{x \times a_1}{\det M} = 0.0003 - \frac{0.2 \times 3}{6503} = 0.0002$$

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V. CONCLUSION

We conclude that, the above definition and examples are using in applications of the M- matrix, because M-matrices have considerable attention, in larger part because they arise in many applications. Recently, a noticeable amount of attention has turned to the inverse of tridiagonal M-matrices, those matrices which happen to be inverse of special form of M-matrices with property $a_{ij}=0$ whenever $|i-j|>1$ and M is generalized strictly diagonally dominant.

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