

# On Locally Multiplication and Locally Co-Multiplication Modules

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**ABSTRACT:** In this paper, our main aim is to study the locally multiplication modules and to introduce the concept of locally co-multiplication module. In addition, we obtain some properties of locally co-multiplication modules.

**MATHEMATICAL SUBJECT CLASSIFICATION:** 13C05, 13C13, 13A15, 13C99, 13E05, 16D10, 16D70.

**KEYWORDS:** Multiplication modules, Co-multiplication modules, Locally multiplication modules, Locally co-multiplication modules..

## I. INTRODUCTION

Locally multiplication module was introduced by Jabbar, A. K. in 2013 [6]. After that in 2016 Jabbar, A.K. [7] has extended some properties of multiplication modules to locally multiplication modules and some conditions obtained under which locally multiplication modules satisfied ascending and descending chain conditions on certain types of submodules. Here we introduced the concept of locally co-multiplication module and obtained some properties of locally co-multiplication modules.

Throughout, this paper  $R$  is a commutative ring with identity and  $M$  is an  $R$ - module unless otherwise stated. A non-empty subset  $S$  of  $R$  is called a multiplication closed set in  $R$  if  $0 \notin S$  and  $a, b \in S$  implies  $ab \in S$  [1]. If  $S$  is multiplicative closed set in  $R$  then one can easily make  $M_S$  as an  $R_S$ - module under the module operations  $\frac{x}{s} + \frac{y}{t} = \frac{tx+sy}{st}$  and  $\frac{r}{u} \cdot \frac{x}{s} = \frac{rx}{us}$  for  $\frac{r}{u} \in R_S$  and  $\frac{r}{s}, \frac{y}{t} \in M_S$  [2], so that when we say that  $M_S$  is a module we mean  $M_S$  is an  $R_S$ - module. If  $I$  is an ideal of  $R$  and  $N, L$  are submodules of  $M$ , then  $(N : I) = \{m \in M : Im \subseteq N\}$  and  $(N : M) = \{r \in R : rM \subseteq N\}$ , which is an ideal of  $R$  [13], in case that  $N = 0$ , then  $(0 : M)$  is called the annihilator of  $M$  and denoted by  $Ann(M)$ , that is  $Ann(M) = (0 : M) = \{r \in R : rM = 0\}$ .  $M$  is called a faithful module if  $Ann(M) = (0 : M) = 0$  [14]. A proper submodule  $K$  of  $M$  is called a prime submodule if  $r \in R$  and  $x \in M$  such that  $rx \in K$ , then  $x \in K$  or  $r \in (K : M)$  [3] (equivalently  $rM \subseteq K$ ) and  $K$  is called a primal submodule of  $M$  if  $S_M(K)$  forms an ideal of  $R$  [15]. The prime and primal spectrum of  $M$  are defined as a  $Spec(M) = \{L : L \text{ is a prime submodule of } M\}$  [16] and  $pSpec(M) = \{L : L \text{ is a primal submodule of } M\}$  [17]. Adil Kadir Jabbar [7] has shown that locally multiplication modules satisfy the chain conditions on certain types of sub-modules. This paper continues this line of research for locally co-multiplication modules on certain types of sub-modules.

## II. RELATED WORK

In this section, we give some basic definitions for future understandings.

**Definition 2.1.**  $M$  is called a **multiplication module** if for each sub-module  $N$  of  $M$ , there exists an Ideal  $I$  of  $R$  such that  $N = IM$  [3] or equivalently, if for any submodule  $N$  of  $M$ , we have  $N = (N : M)M$  [4].

**Definition 2.2.** If  $L, N$  are submodules of  $M$ , then  $(L :_M N)$  is define as  $(L :_M N) = \{m \in M \mid (Rm)(Rn) \subseteq N \text{ for all } n \in N\}$  [5].

**Definition 2.3 :** An  $R$ -module  $M$  is defined to be a **co-multiplication module** if for each submodule  $N$  of  $M$ ,  $N = (0 :_M I)$  for same ideal  $I$  of  $R$ . In this case this can be taken  $I = Ann(N)$  [18]. Some results on co-multiplication modules can be seen in [12]

# International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 5, Issue 8, August 2016

**Definition 2.4.**  $M$  is called a **locally multiplication module** if  $M_P$  is a multiplication  $R_P$  –module for each maximal ideal  $P$  of  $R$  [6]

### III. LOCALLY CO-MULTIPLICATION MODULES THAT SATISFY THE CHAIN CONDITIONS

Now we introduce the following definition:

**Definition 3.1.**  $M$  is called a locally co-multiplication module if  $M_P$  is a co-multiplication  $R_P$  –module for each maximal ideal  $P$  of  $R$ .

**Definition 3.2:** An  $R$  –module  $M$  is locally Noetherian if  $M_P$  is a Noetherian  $R_P$  –module for each prime ideal  $P$  of  $R$  [7].

This is known that if  $M$  is Noetherian  $R$ -module then  $M_S$  is a Noetherian  $R_S$  module for every multiplicative closed set  $S$  in  $R$  [1], so that if  $P$  is any prime ideal of  $R$ , then  $S = R/P$  is a multiplicatively closed set in  $R$ , so that  $M_P$  is a Noetherian  $R_P$ -module and this proves that every Noetherian module is locally Noetherian module. However the converse is not true in general. An example of locally Noetherian ring which is not Noetherian is given in [8, example 3.4] and if we consider this ring  $R$  as an  $R$ -module, then clearly it is the requested example of a locally Noetherian module which is not a Noetherian module.

Adil Kadir Jabbar [7] has given the condition that makes a locally Noetherian module as a Noetherian module

**Theorem 3.3.** Let  $M$  be locally Noetherian  $R$ - Module. If  $R$  is a semi local ring then  $M$  is a Noetherian module.

Proof. See [7]

The same argument is true for Artinian module that makes every Artinian module is a locally Artinian. [9, Proposition 2.13], but the converse is not true in general and same condition is given, to make a locally artinian module as an artinian one.

**Theorem 3.4:** let  $M$  be a locally Artinian  $R$ -module. If  $R$  is semi local ring, then  $M$  is artinian.

Proof. See [7]

Now we take the following definition:

**Definition 3.5:** let  $M$  be an  $R$  Module and  $P$  be a maximal ideal in  $R$ , then

$$H_p = \{N : N \in \text{pspec}(M) \text{ and } S_m(N) \subseteq P\} [7]$$

Now we give the following theorem:

**Theorem 3.6:** Let  $M$  be locally co-multiplication  $R$ -module and  $P$  be a maximal ideal of  $R$ . If  $R$  is a Noetherian ring, then  $M$  satisfies acc (ascending chain condition) of  $H_p$ .

**Proof:** Let  $M$  is a locally co-multiplication  $R$ -module, so  $M_P$  is a co-multiplication  $R_P$ -module, for each maximal ideal  $P$  of  $R$  (By our definition) and as  $R$  is a Noetherian ring, we have  $R_P$  is also Noetherian. Hence by [10, Lemma 2], we get  $M_P$  is a Noetherian  $R_P$  module. Now let  $N_1 \subseteq N_2 \subseteq \dots \subseteq N_n \subseteq \dots$  be any ascending chain of submodules of  $M$  in  $H_p$ . By [11, Proposition 2.1], we have

$$(N_1)_P \subseteq (N_2)_P \subseteq (N_3)_P \dots (N_n)_P \subseteq \dots$$

and since  $M_P$  is Noetherian, there exists  $K \in \mathbb{Z}^+$  such that

$$(N_K)_P = (N_{K+1})_P = \dots$$

Now, let  $x \in N_K$ , then  $\frac{x}{1} \in (N_{K+1})_P$  and as  $N_{K+1} \in H_p$ , we have  $S_M(N_{K+1}) \subseteq P$  so by [9, Lemma 2.1], we get  $x \in N_{K+1}$ , so that  $N_K \subseteq N_{K+1}$ . Similarly we can show that  $N_{K+1} \subseteq N_K$ . Hence  $N_K = N_{K+1}$ . Since  $N_i \in H_p$  for all  $i$ .so by applying the same argument as in the above, we get that

$$N_K = N_{K+1} = N_{K+2} = \dots$$

Hence  $M$  satisfies acc (ascending chain condition) on  $H_p$ . This proves the theorem.

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(An ISO 3297: 2007 Certified Organization)

Vol. 5, Issue 8, August 2016

Now we give the corollary as follows:

**Corollary 3.7:** Let  $M$  be a locally co-multiplication  $R$ -module. If  $R$  is a semi local Noetherian ring, then  $M$  is Noetherian  $R$ -module.

**Proof:** Let  $P$  be any maximal ideal of  $R$ . Since  $M$  is locally co-multiplicative  $R$ -module so  $M_P$  is co-multiplicative  $R_P$ -module for each maximal ideal  $P$  of  $R$  and as  $R$  is Noetherian, it is locally Noetherian, so that  $R_P$  is Noetherian. Hence by [10, Lemma 2], we get that  $M_P$  Noetherian, that means  $M$  is locally Noetherian Hence by Theorem 3.3,  $M$  is a Noetherian  $R$ -module. This proves the corollary.

## IV. CONCLUSION

In this paper, we have studied locally multiplication modules and we have introduced the concept of locally co-multiplication module. In addition, we obtained some properties of locally co-multiplication modules.

## ACKNOWLEDGEMENT

This research was carried out by grant received from the funding agency **Chhattisgarh Council of Science and Technology (CCOST) Raipur** 492001 INDIA. The author would like to thank to **CCOST Raipur** for the financial support.

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