

# Fixed Points of Generalized $(\alpha, \Psi)$ - Contractive Maps

Ch.Prabhakara Rao<sup>1</sup>, B.CH.K.Preethi<sup>2</sup>, S.V.V.Rama Devi<sup>3</sup>, Ch.Pragathi<sup>4</sup>

Professor Dept. of Mathematics, Dadi Institute of Engineering and Technology, Anakapalle, Visakhapatnam, A.P, India<sup>1</sup>

Assistant Professor, Dept. of Mathematics, Dadi Institute of Engineering and Technology, Anakapalle, Visakhapatnam, A.P,  
India<sup>2,3</sup>

Associate Professor, Dept. of Mathematics, GITAM Institute of Technology, GITAM University, Visakhapatnam,  
A.P, India<sup>4</sup>

**ABSTRACT:** In this paper, we introduce generalized  $(\alpha, \psi)$ - contractive maps and prove fixed point theorems in complete metric spaces. Examples are given to illustrate our results. Our results generalize the results of Samet, Vetro, and Vetro [1].

**KEYWORDS:** fixed point, complete metric space,  $\alpha$ -admissible,  $(\alpha, \psi)$ -Contractive maps.

## I. INTRODUCTION

In 2012, Samet, Vetro and Vetro [1] introduced a new concept on contractive maps namely  $(\alpha, \psi)$ -contractive mappings and established fixed point theorems in complete metric space. Here we note that  $(\alpha, \psi)$ -contractive maps generalize contraction maps so that the fixed point results of  $(\alpha, \psi)$ -contractive maps generalize Banach contraction principle.

In section 2, we introduce generalized  $(\alpha, \psi)$ -contractive maps and prove fixed point theorems in complete metric spaces. Supporting examples are given. Our results generalize the results of Samet, Vetro, Vetro, [1] and Banach contraction principle.

Throughout this paper we denote

$$\Psi = \{\psi: [0, +\infty) \rightarrow [0, +\infty) | \psi \text{ is non decreasing, } \sum_{n=1}^{\infty} \psi^n(t) < \infty \forall t > 0\}.$$

**Lemma 1.1.** Let  $\psi \in \Psi$ . Then  $\psi(t) < t$  for all  $t > 0$ .

**Remark 1.2.**  $\lim_{t \rightarrow 0} \psi(t) = 0$ .

**Definition 1.3.** [1] Let  $(X, d)$  be a metric space. Let  $T : X \rightarrow X$  be a self map

of  $X$ . If there exist two functions  $\alpha : X \times X \rightarrow [0, \infty)$  and  $\psi \in \Psi$  such that

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)) \text{ for all } x, y \in X, \quad (1)$$

then  $T$  is an  $(\alpha, \psi)$ - contractive map.

**Definition 1.4.** [1] Let  $T : X \rightarrow X$  and  $\alpha : X \times X \rightarrow [0, \infty)$ . We say that  $T$  is  $\alpha$ -admissible if  $\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1$  (2)

for all  $x, y \in X$ .

The following are the main theorems of Samet, Vetro and Vetro [1]

**Theorem 1.5.** [1] Let  $(X, d)$  be a complete metric space and  $T : X \rightarrow X$  be an  $(\alpha, \psi)$ -

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contractive map satisfying the following conditions: (i)  $T$  is  $\alpha$ -admissible, (ii) there exists  $x_0 \in X$  such that  $\alpha(x_0, Tx_0) \geq 1$ , and (iii)  $T$  is continuous. Then  $T$  has a fixed point in  $X$ .

**Theorem 1.6.** [1] Let  $(X, d)$  be a complete metric space and  $T : X \rightarrow X$  be an

$(\alpha, \psi)$ -contractive map satisfying the following conditions : (i)  $T$  is  $\alpha$ -admissible, (ii) there exists  $x_0 \in X$  such that  $\alpha(x_0, Tx_0) \geq 1$ , and (iii) if  $\{x_n\}$  is a sequence in  $X$  such that  $\alpha(x_n, x_{n+1}) \geq 1$  for all  $n$ , and  $x_n \rightarrow x$  for all  $x \in X$  as  $n \rightarrow +\infty$  then  $\alpha(x_n, x) \geq 1$  for all  $n$ . Then  $T$  has a fixed point in  $X$ .

To get the uniqueness of fixed point the authors assumed the following hypothesis: (H): for all  $x, y \in X$  there exists  $z \in X$  such that  $\alpha(x, z) \geq 1$  and

$$\alpha(y, z) \geq 1.$$

**Theorem 1.7.** [1] Adding condition (H) to the hypotheses of Theorem 1.5 (Theorem 1.6) we obtain the uniqueness of the fixed point of  $T$ .

## II. FIXED POINTS OF GENERALIZED $(\alpha, \Psi)$ - CONTRACTIVE MAPS

**Definition 2.1.** Let  $(X, d)$  be a metric space. Let  $T : X \rightarrow X$  be a self map of  $X$ . If there exist two functions  $\alpha : X \times X \rightarrow [0, \infty)$  and  $\psi \in \Psi$  such that

$$\alpha(x, y) d(Tx, Ty) \leq \psi(\max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\}) \quad (3)$$

for all  $x, y \in X$ , then we say that  $T$  is a generalized  $(\alpha, \psi)$ -contractive map.

**Remark 2.2.** Here we note that “every  $(\alpha, \psi)$ -contractive map” is a “generalized  $(\alpha, \psi)$ -contractive map”. But its converse need not be true. (Example 2.5)

**Theorem 2.3.** Let  $(X, d)$  be a complete metric space and  $T : X \rightarrow X$  be a generalized  $(\alpha, \psi)$ -contractive map satisfying the following conditions: (i)  $T$  is  $\alpha$ -admissible, (ii) there exists  $x_0 \in X$  such that  $\alpha(x_0, Tx_0) \geq 1$ , and (iii)  $T$  is continuous. Then  $T$  has a fixed point in  $X$  that is, there exists  $x^* \in X$  such that  $Tx^* = x^*$ .

**Proof.** Let  $x_0 \in X$  be such that  $\alpha(x_0, Tx_0) \geq 1$  by (ii). We define a sequence  $\{x_n\} \subseteq X$  by  $x_{n+1} = Tx_n, n = 0, 1, 2, \dots$ . If  $x_n = x_{n+1}$  for some  $n$  then,  $x_n = x_{n+1} = Tx_n$ , and  $x_n$  is a fixed point of  $T$ . Hence w.l.g. we suppose that  $x_n \neq x_{n+1}$  for all  $n$ . Since  $T$  is  $\alpha$ -admissible,  $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1$

$\Rightarrow \alpha(Tx_0, Tx_1) \geq 1$  so that  $\alpha(x_1, x_2) \geq 1$ . Now, by induction it is easy to see that  $\alpha(x_n, x_{n+1}) \geq 1$  for all  $n$ . (4)

By taking  $x = x_{n-1}$  and  $y = x_n$  in (4), we get  $d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$

$$\leq \alpha(x_{n-1}, x_n) d(Tx_{n-1}, Tx_n)$$

$$\leq \psi(\max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), \frac{d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})}{2}\})$$

$$\leq \psi(\max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_n), d(x_n, x_{n+1})\}) = \psi(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\})$$

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By remark 1.2, it follows that,  $d(x_n, x_{n+1}) \leq \psi(d(x_n, x_{n-1}))$ . It is true for each  $n = 1, 2, 3, \dots$  and hence it follows that  $d(x_n, x_{n+1}) \leq \psi^n(d(x_1, x_0))$  for each  $n = 1, 2, 3, \dots$

Let  $\epsilon > 0$  be given. Since  $\sum_{n=1}^{\infty} \psi^n(d(x_0, x_1)) < \infty$ .  
We have  $\lim_{n \rightarrow \infty} \sum_{k=1}^n \psi^k(d(x_0, x_1)) = 0$

Hence there exists  $N(\epsilon) \in \mathbb{Z}^+$  such that  
 $\sum_{k=n}^{\infty} \psi^k(d(x_0, x_1)) < \epsilon$  for all  $n \geq N(\epsilon)$ .

We now show that  $\{x_n\}$  is a Cauchy sequence in  $X$ .

Let  $m, n \in \mathbb{Z}^+$  with  $m > n \geq N$ .

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots + d(x_{m-1}, x_m) \\ &\leq \psi^n(d(x_1, x_0)) + \psi^{n+1}(d(x_1, x_0)) + \dots + \psi^{m-1}(d(x_1, x_0)) \\ &= \sum_{k=1}^{m-n} \psi^k(d(x_0, x_1)) \\ &\leq \sum_{k=n}^{\infty} \psi^k(d(x_0, x_1)) < \epsilon \end{aligned}$$

Hence  $\{x_n\}$  is a Cauchy sequence in  $X$ . Since  $X$  is complete, there exists  $x^* \in X$  such that  $\lim_{n \rightarrow \infty} x_n = x^*$ .  
By (iii) we have  $T$  is continuous.  $x^* = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} T(x_n) = T(\lim_{n \rightarrow \infty} x_n) = T x^*$ .  
Therefore  $x^* = T x^*$ . Hence the theorem follows.

**Remark 2.4.** Theorem 1.5 follows as a corollary to Theorem 2.3, since every  $(\alpha, \psi)$ -contractive map is a generalized  $(\alpha, \psi)$ -contractive map'.

**Example 2.5.** Let  $X = \mathbb{R}$  with the usual metric. We define  $T : X \rightarrow X$  by

$$T x = \begin{cases} 1, & \text{if } x < 0 \\ 1 - x, & \text{if } 0 \leq x \leq \frac{1}{3} \\ \frac{2}{3}, & \text{if } \frac{1}{3} < x \leq \frac{2}{3} \\ \frac{1}{3} + \frac{x}{2}, & \text{if } \frac{2}{3} < x \leq 1 \\ 2x - \frac{7}{6}, & \text{if } x > 1 \end{cases}$$

We define  $\alpha : X \times X \rightarrow [0, \infty)$  by  $\alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in [0, \frac{2}{3}] \\ 0, & \text{otherwise} \end{cases}$

Also, we define  $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  by  $\psi(t) = \frac{t}{2}$ ,  $t \geq 0$ . Clearly  $\psi \in \Psi$ . We first show that  $T$  satisfies condition (i) – (iii) of Theorem 2.3.

- (i)  $T$  is  $\alpha$ -admissible: for any  $x, y \in X$  with  $x, y \in [0, \frac{2}{3}]$  we have  $\alpha(x, y) \geq 1$ .  
Hence  $\alpha(Tx, Ty) = \alpha(1-x, \frac{2}{3}) = 1$ . So that  $T$  is  $\alpha$ -admissible.

- (ii) there exists  $x_0 \in X$  such that  $\alpha(x_0, Tx_0) \geq 1$ . We choose  $x_0 = \frac{1}{3}$ , then  
 $\alpha(\frac{1}{3}, T\frac{1}{3}) = \alpha(\frac{1}{3}, \frac{2}{3}) = 1$ .

- (iii)  $T$  is continuous on  $\mathbb{R}$ . We now show that  $T$  is generalized  $(\alpha, \psi)$ -contractive map. We have the following four cases.

Case (i):  $x, y \in [0, \frac{1}{3}]$ , then  $Tx = 1-x$  and  $Ty = 1-y$ . We assume w.l.g that  $x \geq y$ .  
Then  $\alpha(x, y) d(Tx, Ty) = x-y$

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$$\begin{aligned} &\leq \frac{1-2y}{2} \\ &= \psi (d(y, Ty)) \\ &\leq \psi ( \max \{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \}) \end{aligned}$$

Case (ii) :  $x \in [0, \frac{1}{3}]$ ,  $y \in [\frac{1}{3}, \frac{2}{3}]$ , then  $Tx = 1-x$  and  $Ty = \frac{2}{3}$ .

$$\begin{aligned} \alpha(x, y) d(Tx, Ty) &= \frac{1}{3} - x \\ &\leq \frac{1-2x}{2} \\ &= \psi (d(x, Tx)) \\ &\leq \psi ( \max \{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \}) \end{aligned}$$

Case (iii) :  $y \in [0, \frac{1}{3}]$ ,  $x \in [\frac{1}{3}, \frac{2}{3}]$ , since the inequality is symmetric in  $x$  and  $y$  the inequality holds by case (ii).

Case (iv) :  $x \in [0, \frac{1}{3}]$ ,  $y \in [\frac{2}{3}, 1]$ , then  $Tx = 1-x$ ,  $Ty = \frac{1}{3} + \frac{y}{2}$ . This case is trivial, since  $\alpha(x, y) = 0$ .

Therefore  $T$  is generalized  $(\alpha, \psi)$ - contractive map. Thus  $T$  satisfies all the hypotheses of Theorem 2.3 and  $T$  has two fixed points  $\frac{7}{6}$  and  $\frac{2}{3}$ . Here we obtain that when  $x = 0$  and  $y = \frac{1}{3}$  we have  $Tx = 1$  and  $Ty = \frac{2}{3}$ , and hence

$$\alpha(x, y) d(Tx, Ty) = \alpha(0, \frac{1}{3}) d(1, \frac{2}{3}) = \frac{1}{3} \not\leq \psi(d(0, \frac{1}{3})) = \psi(d(x, y)) \text{ for any } \psi \in \Psi.$$

Hence  $T$  is not an  $(\alpha, \psi)$ - contractive map, so that theorem 1.5 is not applicable. Hence Theorem 2.3 generalizes Theorem 1.5.

**Theorem 2.6.** Let  $(X, d)$  be a complete metric space and  $T : X \rightarrow X$  be a generalized  $(\alpha, \psi)$ - contractive map satisfying the following conditions: (i)  $T$  is  $\alpha$ -admissible, (ii) there exists  $x_0 \in X$  such that  $\alpha(x_0, Tx_0) \geq 1$ , and (iii) if  $\{x_n\}$  is a sequence in  $X$  such that  $\alpha(x_n, x_{n+1}) \geq 1$  for all  $n$ , and  $x_n \rightarrow x \in X$  as  $n \rightarrow +\infty$  then  $\alpha(x_n, x) \geq 1$  for all  $n$ . (iv)  $T$  is continuous, Then  $T$  has a fixed point in  $X$ .

**Proof.** On proceeding as in the proof of Theorem 2.3, we get  $\{x_n\}$  is a Cauchy sequence in  $X$ . Since  $X$  is complete, there exists  $x^* \in X$  such  $\lim_{n \rightarrow \infty} x_n = x^*$ . In the proof of Theorem 2.3, we proved that  $\alpha(x_n, x_{n+1}) \geq 1$  for all  $n$ . Hence from (iii) we have  $\alpha(x_n, x^*) \geq 1$  for all  $n$ . Suppose that  $Tx^* \neq x^*$ . We consider

$$\begin{aligned} d(Tx^*, x^*) &\leq d(Tx^*, Tx_n) + d(Tx_n, x^*) \\ &\leq \alpha(x^*, x_n) d(Tx^*, Tx_n) + d(Tx_n, x^*) \\ &\leq \psi ( \max \{ d(x^*, x_n), d(x^*, Tx^*), d(x_n, Tx_n), \frac{d(x^*, Tx_n) + d(x_n, Tx^*)}{2} \}) + d(x_{n+1}, x^*) \end{aligned}$$

$$\text{On letting } n \rightarrow \infty, \text{ we get } d(Tx^*, x^*) \leq \psi ( \max \{ 0, d(x^*, Tx^*), 0, \frac{d(x^*, Tx^*)}{2} \}) + 0$$

$$= \psi (d(x^*, Tx^*) < d(x^*, Tx^*), \text{ a contradiction.}$$

Hence  $Tx^* = x^*$ . Hence the Theorem is proved.

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**Example 2.7.** Let  $X = \mathbb{R}$  with the usual metric. We define  $T : X \rightarrow X$  by

$$T_x = \begin{cases} 1, & \text{if } x < 0 \\ 1 - x, & \text{if } 0 \leq x \leq \frac{1}{3} \\ \frac{2}{3}, & \text{if } \frac{1}{3} < x \leq \frac{2}{3} \\ 1, & \text{if } \frac{2}{3} < x \leq 1 \\ 2x - \frac{7}{6}, & \text{if } x > 1 \end{cases}$$

We define  $\alpha : X \times X \rightarrow [0, \infty)$  by  $\alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in [0, \frac{2}{3}] \\ 0, & \text{otherwise} \end{cases}$

Also, we define  $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  by  $\psi(t) = \frac{t}{2}, t \geq 0$ . Clearly  $\psi \in \Psi$ . Clearly  $T$  is not continuous on  $\mathbb{R}$ . As in example 2.5, it is easy to see that  $T$  is a generalized  $(\alpha, \psi)$ - contractive map. Clearly  $T$  is  $\alpha$ -admissible and for  $x_0 = \frac{1}{3}$ , we have

$\alpha(x_0, Tx_0) = \alpha(\frac{1}{3}, T\frac{1}{3}) = \alpha(\frac{1}{3}, \frac{2}{3}) = 1$ . Hence conditions (i) and (ii) of Theorem 2.6 hold. Let

$\{x_n\} \subseteq X$  such that  $\alpha(x_n, x_{n+1}) \geq 1$ . Then by the definition of  $\alpha$ , we have  $x_n \in [0, \frac{2}{3}]$  for all  $n$ .

Now if  $x_n \rightarrow x$  as  $n \rightarrow \infty$ , since  $[0, \frac{2}{3}]$  is closed, it follows that  $x \in [0, \frac{2}{3}]$  and hence  $\alpha(x_n, x) \geq 1$  for all  $n$ . Thus condition (iii) of Theorem 2.6 holds.

Thus  $T$  satisfies all the hypotheses of Theorem 2.6 and  $T$  has three fixed points  $1, \frac{7}{6}$  and  $\frac{2}{3}$ .

We now prove the uniqueness of fixed points of generalized  $(\alpha, \psi)$ - contractive maps, by assuming the hypotheses(H).

**Theorem2.8.** Adding the hypotheses (H) to the hypotheses of Theorem 2.3 (Theorem2.6), we obtain the uniqueness of the fixed point of  $T$ .

**Proof.** By Theorem 2.3 ( Theorem2.6 ),  $T$  has a fixed point in  $X$ . Suppose that  $T$  has two fixed points  $x^*, y^*$  (say) in  $X$ . By above condition (H), there exists  $z$  in  $X$  such that  $\alpha(x^*, z) \geq 1$  and  $\alpha(y^*, z) \geq 1$ .  
(5)

Since  $T$  is  $\alpha$ -admissible and from (5) we get  $\alpha(x^*, T^n z) \geq 1$  and  $\alpha(y^*, T^n z) \geq 1$  (6)

Now by generalized  $(\alpha, \psi)$ - contractive map and using (6), we get  $d(x^*, T^n z) = d(Tx^*, T(T^{n-1}z))$

$$\leq \alpha(x^*, T^{n-1}z) d(Tx^*, T(T^{n-1}z)) \\ \leq \psi(\max \{d(x^*, T^{n-1}z), d(x^*, Tx^*), d(T^{n-1}z, T^n z), \frac{d(x^*, T^n z) + d(T^{n-1}z, Tx^*)}{2}\})$$

$$\leq \psi(\max \{d(x^*, T^{n-1}z), 0, d(T^{n-1}z, T^n z), \frac{d(x^*, T^n z) + d(T^{n-1}z, Tx^*)}{2}\})$$

$$\leq \psi(\max \{d(x^*, T^{n-1}z), 0, d(T^{n-1}z, x^*), d(x^*, T^n z)\}) \\ \leq \psi(d(x^*, T^{n-1}z))$$

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$$Tx = \begin{cases} 1, & \text{if } x < 0 \\ 1 - x, & \text{if } 0 \leq x \leq \frac{1}{3} \\ \frac{2}{3}, & \text{if } \frac{1}{3} < x \leq \frac{2}{3} \\ \frac{1}{3} + \frac{x}{2}, & \text{if } \frac{2}{3} < x \leq 1 \\ x - \frac{1}{6}, & \text{if } x > 1 \end{cases}$$

We define  $\alpha : X \times X \rightarrow [0, \infty)$  by  $\alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in [0, \frac{2}{3}] \\ 0, & \text{otherwise} \end{cases}$

Also, we define  $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  by  $\psi(t) = \frac{t}{2}$ ,  $t \geq 0$ . Clearly  $\psi \in \Psi$ . As in example 2.5, it is easy to see that  $T$  is a generalized  $(\alpha, \psi)$  contractive map. Clearly  $T$  is  $\alpha$ -admissible and for  $x_0 = \frac{1}{3}$ , we have  $\alpha(x_0, Tx_0) = \alpha(\frac{1}{3}, T\frac{1}{3}) = \alpha(\frac{1}{3}, \frac{2}{3}) = 1$ . Clearly  $T$  is continuous on  $\mathbb{R}$  and condition (H) holds trivially. Hence  $T$  satisfies all the hypotheses of Theorem 2.8 and  $T$  has a unique fixed point  $\frac{2}{3}$ .

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