

Observations on Ternary Quadratic Diophantine Equation

$$6(x^2 + y^2) - 11xy + 3x + 3y + 9 = 72z^2$$

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ABSTRACT: The Ternary Quadratic Diophantine Equation representing non-homogeneous cone given by $6(x^2 + y^2) - 11xy + 3x + 3y + 9 = 72z^2$ is analyzed for its non-zero distinct integer points on it. Six different patterns of integral solutions satisfying the cone under consideration are obtained. A few interesting relations between the solutions and some special number patterns are presented.

KEYWORDS: Ternary non-homogeneous Quadratic, Integral solutions.

I. INTRODUCTION

Ternary quadratic equations are rich in variety. For an extensive review of various problems one may refer [1-7]. In [8], the ternary quadratic Diophantine equation of the form $kxy + m(x + y) = z^2$ has been studied for non-trivial integral solutions. In [9,10], a special Pythagorean triangle problem have been discussed for its integral solutions. In [11,12], ternary quadratic diophantine equations have been discussed for its integral solutions. This communication concerns with yet another interesting ternary quadratic equation $6(x^2 + y^2) - 11xy + 3x + 3y + 9 = 72z^2$ representing a cone for determining its infinitely many non-zero integral points. Also, a few interesting relations among the solutions are presented.

II. NOTATIONS

Pr_n = Pronic number of rank 'n'.

$T_{m,n}$ = Polygonal number of rank 'n' with sides 'm'.

$4DF_n$ = Four Dimensional Figurate number of rank 'n'.

CS_n = Centered Square number of rank 'n'.

SO_n = Stella octangula number of rank 'n'.

O_n = Octahedral number of rank 'n'.

P_n^5 = Pyramidal number of rank 'n' with sides 'm'.

Gno_n = Gnomonic number of rank 'n'.

$Star_n$ = Star number of rank 'n'.

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III. METHOD OF ANALYSIS

The ternary quadratic diophantine equation to be solved for its non-zero integral solutions is

$$6(x^2 + y^2) - 11xy + 3x + 3y + 9 = 72z^2 \quad (1)$$

The substitution of linear transformations $x = u + v$ and $y = u - v$ (2)

in (1) leads to, $(u + 3)^2 + 23v^2 = 72z^2$ (3)

We illustrate below six different patterns of non-zero distinct integer solutions to (1).

PATTERN : 1

Assume $z = z(a, b) = a^2 + 23b^2$ (4)

where a and b are non-zero integers.

and write $72 = (7 + i\sqrt{23})(7 - i\sqrt{23})$ (5)

Using (4) and (5) in (3), and using factorization method,

$$\left((u + 3) + i\sqrt{23}v\right)\left((u + 3) - i\sqrt{23}v\right) = (7 + i\sqrt{23})(7 - i\sqrt{23})\left[(a + i\sqrt{23}b)^2(a - i\sqrt{23}b)^2\right] \quad (6)$$

Equating the like terms and comparing real and imaginary parts, we get

$$u = u(a, b) = 7a^2 - 161b^2 - 46ab - 3$$

$$v = v(a, b) = a^2 - 23b^2 + 14ab$$

Substituting the above values of u & v in equation (2), the corresponding integer solutions of (1) are given by

$$x = x(a, b) = 8a^2 - 184b^2 - 32ab - 3$$

$$y = y(a, b) = 6a^2 - 138b^2 - 60ab - 3$$

$$z = z(a, b) = a^2 + 23b^2$$

OBSERVATIONS:

1. $x(a, 1) + y(a, 1) + z(a, 1) + Pr_a - T_{8,a} - T_{28,a} - 305Gno_a \equiv 0 \pmod{687}$
2. $6(y(a, a) - x(a, a)) + 2z(a, a)$ is a perfect square.
3. $6x(a, 1) \cdot z(a, 1) - x(a, 1) \cdot y(a, 1) - 240SO_a - 52Star_a + 9798Gno_a \equiv 0 \pmod{62023}$
4. $x(a, 1) + y(a, 1) - 2z(a, 1) - 2Star_a + 40Gno_a \equiv 0 \pmod{416}$
5. $3x(a, 1) - 4y(a, 1) + z(a, 1) \cdot y(a, 1) - 72(4DF_a) + 3O_a + 116P_a^5 - 122T_{3,a} - 3240Gno_a \equiv 0 \pmod{7776}$
6. Each of the following expressions represents a Nasty number.
 - i. $3x(a, a) - 5y(a, a) - 14z(a, a)$
 - ii. $6[6(y(a, a) - x(a, a)) + 2z(a, a)]$

PATTERN : 2

Instead of (5), we write 72 as

$$72 = \frac{(5 + i7\sqrt{23})(5 - i7\sqrt{23})}{16} \quad (7)$$

Following the procedure presented above, the corresponding non-zero distinct integer solutions to (1) are obtained as

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$$x = x(A, B) = 12A^2 - 276B^2 - 312AB - 3$$

$$y = y(A, B) = -2A^2 + 46B^2 - 332AB - 3$$

$$z = z(A, B) = 4A^2 + 92B^2$$

OBSERVATIONS:

1. $3z(A, A) + y(A, A) - x(A, A)$ is a perfect square.
2. $2y(A, 1) + z(A, 1) + 332Gno_A \equiv 0 \pmod{154}$
3. $y(A, 1) + z(A, 1) - x(A, 1) + 2T_{12,A} + 14Gno_A \equiv 0 \pmod{20^2}$
4. $x(A, A) - y(A, A) + 4z(A, A) - 16Star_A - 48Gno_A \equiv 0 \pmod{32}$
5. Each of the following expressions represents a Nasty number
 - i. $y(A, A) - x(A, A) - 2z(A, A)$
 - ii. $6[3z(A, A) + y(A, A) - x(A, A)]$

PATTERN : 3

Equation (3) can be written as

$$(u + 3)^2 = 72z^2 - 23v^2 \quad (8)$$

Introducing the linear transformations,

$$z = X + 23T \quad \text{and} \quad v = X + 72T \quad (9)$$

in (8), we get

$$X^2 = 1656T^2 + \left(\frac{u + 3}{7}\right)^2 \quad (10)$$

which is satisfied by

$$T(r, s) = 2rs$$

$$u(r, s) = 11592r^2 - 7s^2 - 3$$

$$X(r, s) = 1656r^2 + s^2$$

Employing the above values in (9) and using (2), the corresponding integer solutions of (1) are given by

$$x = x(r, s) = 13248r^2 - 6s^2 + 144rs - 3$$

$$y = y(r, s) = 9936r^2 - 8s^2 - 144rs - 3$$

$$z = z(r, s) = 1656r^2 + s^2 + 46rs$$

OBSERVATIONS:

1. $x(r, r) - y(r, r) - 2z(r, r)$ is a perfect square.
2. $6[x(r, r) - y(r, r) - 2z(r, r)]$ represents a Nasty number.
3. $6z(r, 1) - y(r, 1) - 210Gno_r \equiv 0 \pmod{227}$
4. $x(r, 1) - y(r, 1) - 1656CS_r - 1800Gno_r \equiv 0 \pmod{146}$
5. $x(r, 1) - y(r, 1) - 2z(r, 1) \equiv 0 \pmod{14^2}$
6. $3y(r, 1) - 4x(r, 1) + 14z(r, 1) + 182Gno_A \equiv 0 \pmod{165}$

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PATTERN : 4

Consider (3) as

$$(u+3)^2 - 49v^2 = 72(z^2 - v^2) \quad (11)$$

Write (12) in the form of ratio as

$$\frac{(u+3)+7v}{z+v} = \frac{72(z-v)}{(u+3)-7v} = \frac{A}{B}, B \neq 0$$

which is equivalent to the following two equations

$$B(u+3) + (7B-A)v - Az = 0$$

$$A(u+3) + (72B-7A)v - 72Bz = 0$$

On employing the method of cross multiplication, we get

$$u = u(A, B) = 7A^2 - 144AB + 504B^2 - 3$$

$$v = v(A, B) = A^2 - 72B^2$$

$$z = z(A, B) = A^2 - 14AB + 72B^2$$

Applying the above values of u & v in equation (2), the corresponding integer solutions of (1) are given by

$$x = x(A, B) = 8A^2 - 144AB + 432B^2 - 3$$

$$y = y(A, B) = 6A^2 - 144AB + 576B^2 - 3$$

$$z = z(A, B) = A^2 - 14AB + 72B^2$$

OBSERVATIONS:

1. $y(A, A) - x(A, A) - 2z(A, A)$ represents a Nasty number.
2. $y(A, 1) - 6z(A, 1) + 30Gno_A \equiv 0 \pmod{111}$
3. $x(A, 1) + y(A, 1) - 7CS_A + 137Gno_A \equiv 0 \pmod{858}$
4. $z(A, 1) - x(A, 1) + Pr_A + Star_A - 356Gno_A \equiv 0 \pmod{587}$
5. $x(A, 1) - y(A, 1) + z(A, 1) - T_{8,A} + 6Gno_A \equiv 0 \pmod{78}$

Remarkable Note:

Equation (12) can also be expressed in the form of ratio in two different ways as follows.

$$(i). \frac{(u+3)+7v}{8(z+v)} = \frac{9(z-v)}{(u+3)-7v} = \frac{A}{B}, B \neq 0$$

$$(ii). \frac{(u+3)+7v}{9(z+v)} = \frac{8(z-v)}{(u+3)-7v} = \frac{A}{B}, B \neq 0$$

Repeating the analysis as above the corresponding integer solutions along with properties are presented below.

Solutions of (i):

$$x = x(A, B) = 64A^2 - 144AB + 54B^2 - 3$$

$$y = y(A, B) = 48A^2 - 144AB + 72B^2 - 3$$

$$z = z(A, B) = 8A^2 - 14AB + 9B^2$$

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OBSERVATIONS :

1. $9[y(A, A) - x(A, A)] + 2z(A, A)$ represents a Nasty number.
2. $8[y(A, A) - x(A, A)] + 3z(A, A)$ is a perfect square.
3. $y(A, 1) - x(A, 1) - z(A, 1) + 12CS_A + 5Gno_A \equiv 0 \pmod{4^2}$.

Solutions of (ii):

$$x = x(A, B) = 72A^2 - 144AB + 48B^2 - 3$$

$$y = y(A, B) = 54A^2 - 144AB + 64B^2 - 3$$

$$z = z(A, B) = 9A^2 - 14AB + 8B^2$$

OBSERVATIONS :

1. $y(A, 1) - x(A, 1) + 2T_{20,A} + 8Gno_A \equiv 0 \pmod{8}$.
2. $y(A, 1) - 6z(A, 1) + 30Gno_A \equiv 0 \pmod{17}$.
3. $y(A, 1) + 2z(A, 1) - 12Star_A + 50Gno_A \equiv 0 \pmod{15}$.

IV. CONCLUSION

In this paper, we have presented six different patterns of non-zero distinct integer solutions of the non-homogeneous cone given by $6(x^2 + y^2) - 11xy + 3x + 3y + 9 = 72z^2$.

To conclude, one may search for other patterns of non-zero integer distinct solutions and their corresponding properties for other choices of ternary quadratic diophantine equations.

REFERENCES

- [1] Batta.B and Singh.A.N, History of Hindu Mathematics, Asia Publishing House 1938.
- [2] Carmichael, R.D., "The Theory of Numbers and Diophantine Analysis", Dover Publications, New York, 1959.
- [3] Dickson, L.E., "History of the theory of numbers", Chelsia Publishing Co., Vol.II, New York, 1952.
- [4] Mollin.R.A., "All solutions of the Diophantine equation $x^2 - Dy^2 = n$ ", For East J.Math. Sci., Social Volume, 1998, Part III, pages 257-293.
- [5] Mordell.L.J., "Diophantine Equations", Academic Press, London 1969.
- [6] Telang.S.G., "Number Theory", Tata McGraw-Hill Publishing Company, New Delhi 1996.
- [7] Nigel.P.Smart, "The Algorithmic Resolutions of Diophantine Equations", Cambridge University Press, London 1999.
- [8] Gopalan.M.A., Manju Somnath, and Vanitha.N., "Integral Solutions of $kxy + m(x+y) = z^2$ ", Acta Ciencia Indica, Volume 33;number 4, pages 1287-1290, 2007.
- [9] Gopalan.M.A., and Anbuselvi.R., "A special Pythagorean triangle", Acta Ciencia Indica, Volume 31, number 1, pages 53-54, 2005.
- [10] Gopalan.M.A., and Devibala.S., "A special Pythagorean triangle", Acta Ciencia Indica, Volume 31, number 1, pages 39-40, 2005.
- [11] Gopalan.M.A., Vidhyalakshmi.S and Umarani.J., "On ternary Quadratic Diophantine equation $6(x^2 + y^2) - 8xy = 21z^2$ ", Sch.J. Eng. Tech. 2(2A); 108- 112,2014
- [12] Gopalan.M.A., Vidhyalakshmi.S and Shanthi.J "On ternary quadratic Diophantine equation $3(x^2 + y^2) - 5xy + x + y + 1 = 15z^2$ ", International Journal of Innovative Science, Engineering &Technology, Vol 1, Issue 6, 2014.

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