

On Two Complete Metric Spaces

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ABSTRACT:We prove common fixed point theorems for a pair of contractive mappings in two complete metric linear spaces. The presented results extend some known existence results from the literature.

KEYWORDS:Fixed point, contractive mapping, Non-expansive, complete metric spaces

I. INTRODUCTION

Fixed point theorems are broadly studied for three classes of mappings namely continuous, contractive and non-expansive mappings. And fixed point theory has generally developed in three main directions: generalization of conditions which ensure existence, and, if possible, uniqueness of fixed points. Some fixed point theorems on complete metric spaces have studied by L.Kikina et al.[5],V.Popa [9], R.K.Namdeo, et al[6]. In this paper, the generalization is given on two complete metric spaces.

II. PRELIMINARIES

Definition 2.1. Let X is any non-empty set. A metric on X is a real valued function $d: X \times X \rightarrow \mathbb{R}$ which satisfies the following conditions: for all $x, y \in X$,

- (i) $d(x, y) \geq 0$;
- (ii) $d(x, y) = 0$ iff $x = y$;
- (iii) $d(x, y) = d(y, x)$;
- (iv) $d(x, z) \leq d(x, y) + d(y, z)$.

A metric space is a non- empty set X equipped with a metric d on X and is denoted by (X, d) .

Example 2.2. The set \mathbb{R} of real numbers with the distance function $d(x, y) = |x - y|$ is a metric space.

Example 2.3. Every normed linear space is a metric space under the metric $d(x, y) = \|x - y\|$.

Definition 2.4. A sequence $\{x_n\}$ of a metric space (x, d) is said to be

- (i) Convergent to a point $x \in X$ if $\lim_{n \rightarrow \infty} d(x_n, x) = 0$.
- (ii) Cauchy sequence if $\lim_{m, n \rightarrow \infty} d(x_n, x_m) = 0$.

Definition 2.5. A metric space (X, d) is said to be complete if every Cauchy sequence in (X, d) is convergent in X . The following theorems were proved by B.Fisher [4].

Theorem 2.6. Let (X, d) and (Y, ρ) be complete metric spaces. If $T: X \rightarrow Y$ and $S: Y \rightarrow X$ satisfying the inequalities,
$$\rho(Tx, TSy) \leq c \max(d(x, Sy), \rho(y, Tx), \rho(y, TSy))$$
$$d(Sy, STx) \leq c \max(\rho(y, Tx), d(x, Sy), d(x, STx))$$

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$\forall x \in X$ and $y \in Y$ where $0 \leq c < 1$ then ST has a unique fixed point $z \in X$ and TS has a unique fixed point $w \in Y$. Further $Tz=w$ and $Sw=z$.

III. MAIN RESULTS

Theorems 3.1. Let (X, d) and (Y, e) be complete metric spaces. If T is a mapping of X into Y and S is a mapping of Y into X satisfying the inequalities

$$e^4(Tx, TSy) \leq a_1(d(x, Sy)e^3(y, Tx)) + b_1(d(x, Sy)e^3(y, TSy)) + c_1(e(y, Tx)e^3(y, TSy))$$

$$d^4(Sy, STx) \leq a_2(e(y, Tx)d^3(x, Sy)) + b_2(e(y, Tx)d^3(x, STx)) + c_2(d(x, Sy)d^3(x, STx))$$

For all x in X and y in Y , where $a_1, b_1, c_1, a_2, b_2, c_2 > 0$ and $(a_1 + b_1 + c_1) \cdot (a_2 + b_2 + c_2) < 1$, then ST has a unique fixed point z in X and TS has a unique fixed point w in Y . Further, $Tz=w$ and $Sw=z$.

Proof: Let x be an arbitrary point in X . Define sequences $\{x_n\}$ and $\{y_n\}$ in X and Y respectively by

$$(ST)^n x = x_n, T(ST)^{n-1} x = y_n \text{ for } n=1,2,\dots$$

We have

$$d^4(x_n, x_{n+1}) = d(Sy_n, STx_n)$$

$$\leq a_2(e(y_n, y_n)d^3(x_n, x_n)) + b_2(e(y_n, y_{n+1})d^3(x_n, x_{n+1})) + c_2(d(x_n, x_n)d^3(x_n, x_{n+1}))$$

$$= b_2e(y_n, y_{n+1})d^3(x_n, x_{n+1})$$

This implies

$$d(x_n, x_{n+1}) \leq b_2e(y_n, y_{n+1})$$

If $d(x_n, x_{n+1}) \neq 0$ and by using above inequality, we have

$$e^4(y_n, y_{n+1}) = e^4(Tx_{n-1}, TSy_n)$$

$$\leq a_1(d(x_{n-1}, x_n)e^3(y_n, y_n)) + b_1(d(x_{n-1}, x_n)e^3(y_n, y_{n+1})) + c_1(e^3(y_n, y_n)e(y_{n+1}, y_n))$$

$$= b_1d(x_{n-1}, x_n)e^3(y_n, y_{n+1})$$

This implies

$$e(y_n, y_{n+1}) \leq b_1d(x_{n-1}, x_n)$$

If $e(y_n, y_{n+1}) \neq 0$. it follows that

$$d(x_n, x_{n+1}) \leq b_2e(y_n, y_{n+1}) \leq b_1b_2d(x_{n-1}, x_n) \leq \dots \leq (b_1b_2)^n d(x_0, x_1)$$

And since $0 \leq b_1, b_2 < 1$, $\{x_n\}$ is a Cauchy sequence with a limit z in X and $\{y_n\}$ is a Cauchy sequence with a limit w in Y .

Now, we have

$$e^4(Tz, y_n) \leq a_1(d(z, x_{n-1})e^3(y_{n+1}, Tz)) + b_1(d(z, x_{n-1})e^3(y_{n-1}, y_n)) + c_1(e(y_{n-1}, y_n)e^3(y_{n-1}, Tz))$$

Letting n tend to infinity we have $e^4(Tz, w) \leq 0$ and so $Tz=w$. Similarly, we can prove that $Sw=z$ and so $STz=Sw=z$, $TSw=Tz=w$. Thus ST has a fixed point z and TS has a fixed point w . Now suppose that ST has a second fixed point z' . Then we have

$$d^4(z, z') = d^4(STz', STz) \leq a_2(e(Tz', Tz)d^3(z, STz')) + b_2(d^3(z, STz')d(z, STz)) + c_2(e^3(Tz', Tz)d^3(z, STz))$$

$$= a_2e(Tz', Tz)d^3(z, z')$$

Which implies $d(z', z) \leq c_2e(Tz', Tz)$. But by using inequality

$$e^4(Tz, Tz') = e^4(Tz', TSTz)$$

$$\leq a_1(d(z', STz)e^3(Tz, Tz')) + b_1(d^3(z, STz)e(Tz, TSTz)) + c_1(e^3(Tz, Tz')e(Tz, TSTz))$$

$$= a_1d(z', STz)e^3(Tz', Tz)$$

This implies

$$e(Tz', Tz) \leq a_1d(z', STz) = a_1d(z, z')$$

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And so

$$d(z', z) \leq a_2 e(Tz', Tz) = a_1 d(z, z')$$

Since, $0 \leq a_1 a_2 < 1$ the uniqueness of z follows. Similarly, w is the unique fixed point of TS . If $\exists n \in \mathbb{N}$ that $d(x_n, x_{n+1}) = 0$ or $e(y_n, y_{n+1}) = 0$ the theorem is evident.

Corollary 3.2. Let (X, d) be complete metric spaces. If T is a mapping of X into X satisfying the inequalities

$$d^4(Tx, TSy) \leq a_1 d((x, Sy)d^3(y, Tx)) + b_1(d(x, Sy)d^3(y, TSy)) + c_1(d(y, Tx)d^3(y, TSy))$$

$$d^4(Sy, STx) \leq a_2(d(y, Tx)d^3(x, Sy)) + b_2(d(y, Tx)d^3(x, STx)) + c_2(d(x, Sy)d^3(x, STx))$$

for all x, y in X , where $a_1, b_1, c_1, a_2, b_2, c_2 > 0$ and $(a_1 + b_1 + c_1) \cdot (a_2 + b_2 + c_2) < 1$, then ST has a unique fixed point z in X and TS has a unique fixed point w in Y . Further, $Tz = w$ and $Sw = z$.

Proof. The proof of corollary follows immediate.

Theorem 3.3. Let (X, d) and (Y, e) be complete metric spaces. If T is a mapping of X into Y and S is a mapping of Y into X satisfying the inequalities

$$e^4(Tx, TSy) \leq c_1 M_1 \tag{3.1}$$

$$d^4(Sy, STx) \leq c_2 M_2 \tag{3.2}$$

Where

$$M_1 = \max\{d(x, Sy)e^3(y, Tx), d(x, Sy)e^3(y, TSy), e(y, Tx)e^3(y, TSy)\}$$

$$M_2 = \max\{e(y, Tx)d^3(x, Sy), e(y, Tx)d^3(x, STx), d(x, Sy)d^3(x, STx)\}$$

For all x in X and y in Y , where $0 \leq c_1 \cdot c_2 < 1$, then ST has a unique fixed point z in X and TS has a unique fixed point w in Y . Further, $Tz = w$ and $Sw = z$.

Proof. Let x be an arbitrary point in X . Define sequence $\{x_n\}$ and $\{y_n\}$ in X and Y respectively by

$$(ST)^n x = x_n, \quad T(ST)^{n-1} x = y_n$$

for $n=1, 2, \dots$. Using inequality (3.2) we have

$$\begin{aligned} d^4(x_n, x_{n+1}) &= d(Sy_n, STx) \leq c_2 \max\{e(y_{n+1}, y_n)d^3(x_n, x_n), e(y_n, y_{n+1})d^3(x_n, x_{n+1}), d(x_n, x_n)d^3(x_n, x_{n+1})\} \\ &= c_1 e(y_n, y_{n+1})d^3(x_n, x_{n+1}) \end{aligned}$$

This implies

$$d(x_n, x_{n+1}) \leq c_2 e(y_n, y_{n+1})$$

If $d(x_n, x_{n+1}) \neq 0$ and by using inequality (3.1). We have

$$\begin{aligned} e^4(y_n, y_{n+1}) &= e^4(Tx_{n-1}, TSy_n) \leq c_1 \max\{d(x_{n-1}, x_n)e^3(y_n, y_n), d(x_{n-1}, x_n)e^3(y_n, y_{n+1}), e^3(y_n, y_n)e(y_{n+1}, y_n)\} \\ &= c_1 d(x_{n-1}, x_n)e^3(y_n, y_{n+1}) \end{aligned}$$

This implies

$$e(y_n, y_{n+1}) \leq c_1 d(x_{n-1}, x_n)$$

If $e(y_n, y_{n+1}) \neq 0$. It follows that

$$d(x_n, x_{n+1}) \leq c_2 e(y_n, y_{n+1}) \leq c_1 c_2 d(x_{n-1}, x_n) \leq \dots \leq (c_1 c_2)^n d(x_0, x_1)$$

And since $0 \leq c_1 \cdot c_2 < 1$, $\{x_n\}$ is a Cauchy sequence with a limit z in X and $\{y_n\}$ is a Cauchy sequence with a limit w in Y .

Now, we have

$$e^4(Tz, y_n) \leq c_1 \max\{d(z, x_{n-1})e^3(y_{n+1}, Tz), d(z, x_{n-1})e^3(y_{n-1}, y_n), e(y_{n-1}, y_n)e^3(y_{n-1}, Tz)\}$$

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Letting n tend to infinity we have $e^4(Tz, w) \leq 0$ and so $Tz=w$. Similarly, we can prove that $Sw=z$ and so $STz=Sw=z$, $TSw=Tz=w$. Thus ST has a fixed point z and TS has a fixed point w . Now suppose that ST has a second fixed point z' . Then we have

$$d^4(z, z') = d^4(STz', STz) \leq c_2 \max\{e(Tz', Tz)d^3(z, STz'), d^3(z, STz')d(z, STz), e^3(Tz', Tz)d^3(z, STz)\} \\ = c_2 e(Tz', Tz)d^3(z, z')$$

This implies

$$d(z', z) \leq c_2 e(Tz', Tz).$$

$$e^4(Tz, Tz') = e^4(Tz', TSTz) \leq c_1 \max\{d(z', STz)e^3(Tz, Tz'), d^3(z, STz)e(Tz, TSTz), e^3(Tz, Tz')e(Tz, TSTz)\} \\ = c_1 d(z', STz)e^3(Tz', Tz)$$

This implies

$$e(Tz', Tz) \leq c_1 d(z', STz) = c_1 d(z, z')$$

And so

$$d(z', z) \leq c_2 e(Tz', Tz) \leq c_1 c_2 d(z, z').$$

Since, $0 \leq c_1 c_2 < 1$ the uniqueness of z follows. Similarly, w is the unique fixed point of TS . If $\exists n \in \mathbb{N}$ that $d(x_n, x_{n+1}) = 0$ or $e(y_n, y_{n+1}) = 0$ the theorem is evident.

Corollary 3.4. Let (X, d) be complete metric spaces. If T is a mapping of X into X satisfying the inequalities

$$d^4(Tx, TSy) \leq c_1 M_1 \quad (3.3)$$

$$d^4(Sy, STx) \leq c_2 M_2 \quad (3.4)$$

Where

$$M_1 = \max\{d(x, Sy)d^3(y, Tx), d(x, Sy)d^3(y, TSy), d(y, Tx)d^3(y, TSy)\}$$

$$M_2 = \max\{d(y, Tx)d^3(x, Sy), d(y, Tx)d^3(x, STx), d(x, Sy)d^3(x, STx)\}$$

For all x in X and y in Y , where $0 \leq c_1, c_2 < 1$, then ST has a unique fixed point z in X and TS has a unique fixed point w in X . Further, $Tz=w$ and $Sw=z$, and if $z=w$, z is the unique fixed point of S and T .

Proof. The existence of z and w follows from above theorem. If $z=w$, then z is of course a common fixed point of S and T . Now suppose that T has a second fixed point z_0 . Then, by using inequality (3.3), we have,

$$d^4(z, z_0) = d^4(Tz_0, TSz) \leq c_1 \max\{d^3(z, Tz_0)d(z_0, Sz), d^3(z, Tz_0)d(z, TSz), d^3(z, TSz)d(z_0, Sz)\} = c_1 d^3(z_0, z)$$

Since $0 \leq c_1 < 1$, the uniqueness of z follows. Similarly we can show z is the unique fixed point of S .

IV. CONCLUSION

We have proved common fixed point theorems for a pair of contractive mappings in two complete metric spaces. The presented results generalize the results proved in single metric spaces and extend some results from the literature.

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