

Fixed Point of Cyclic Contraction Functions in Banach Spaces

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ABSTRACT:In this paper, we proved some fixed point theorems for cyclic mapping with contractive and non-expansive condition on such space analogous to the results presented in the literature.

KEYWORDS:Fixed Point, Contractive Mapping, Cyclic Contraction, Banach spaces.

I. INTRODUCTION

In 2003, Kirk et al. introduced the notation of a cyclic representation and characterized the Banach contraction principle in context of a cyclic mapping. In 2011 Karapinar and Erhan introduced Kannan type cyclic contraction and Chatterjea-type cyclic contraction. Moreover, they derive some fixed point theorems for such cyclic contractions in complete metric spaces.

Theorem 1.1. (Fixed point theorem for Kannan –type cyclic contraction). Let A and B be nonempty subsets of metric spaces (X,d) and a cyclic mapping $T: A \cup B \rightarrow A \cup B$ Satisfies

$$d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)], \forall x \in A, y \in B$$

where $0 \leq k < 1/2$. Then, T has a unique fixed point in $A \cap B$.

Theorem 1.2. (fixed point theorem for Chatterjea –type cyclic contraction) Let A and B be nonempty subsets of metric spaces (X,d) and a cyclic mapping $T: A \cup B \rightarrow A \cup B$ satisfies

$$d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)], \forall x \in A, y \in B$$

where $0 \leq k < 1/2$. Then, T has a unique fixed point in $A \cap B$.

In this paper we studied some fixed point theorems for cyclic mapping with contractive and non-expansive condition on Banach space.

II. PRELIMINARIES

Definition: 2.1. Let K be subset of a Banach space X . An operator T defined on K is said to belong to class $D(a, b)$ if

$$\|Tx - Ty\| \leq a\|x - y\| + b[\|x - Tx\| + \|y - Ty\|] \quad (2.1)$$

for all x and y in K , where. If an operator T is in class $D(k,0)$ with $0 < k < 1$, then T is a contraction with $0 < k < 1$.

Definition: 2.2. Let K_1 and K_2 be closed subsets of a Banach space X . An operator T defined on K is said to belong to class $D(a,b,c)$ if

$$\|Tx - Ty\| \leq a\|x - y\| + b[\|x - Tx\| + \|y - Ty\|] + c[\|x - Ty\| + \|Tx - y\|] \quad (2.2)$$

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For all x and y in K , where $0 \leq a, b, c \leq 1, a + 2b + 2c \leq 1$ and $b > 0$.

Definition:2.3. Let K_1 and K_2 be closed subsets of a Banach space X . an operator

$T : K_1 \cup K_2 \rightarrow K_1 \cup K_2$ with $T(K_1) \subset K_2$ and $T(K_2) \subset K_1$ is said to belong to class $D(a, b)$ if it satisfies

$$\|Tx - Ty\| \leq a\|x - y\| + b[\|x - Tx\| + \|y - Ty\|] \quad (2.3)$$

for all $x \in K_1$ and $y \in K_2$, where $0 \leq a, b \leq 1$. It is clear that if T belongs to the class $D(k, 0)$ with $0 < k < 1$, then T is a cyclic contraction.

Example:2.4. Let $K_1 = [0, \frac{1}{2}]$ and $K_2 = [\frac{1}{3}, 1]$. Define the operator T as follows:

$$T(x) = \begin{cases} \frac{2}{5}, & \text{if } 0 \leq x \leq \frac{1}{2}; \\ \frac{2}{3}(1-x), & \text{if } \frac{1}{2} < x \leq 1; \end{cases}$$

We will show that T is in the class of $D(\frac{1}{4}, \frac{1}{4})$. Take $x \in [0, \frac{1}{2}]$ and $y \in [\frac{1}{3}, \frac{1}{2}]$.

Then $\|Tx - Ty\| = |\frac{2}{5} - \frac{2}{5}| = 0$.

Now $x \in [0, \frac{1}{2}]$ and $y \in [\frac{1}{2}, 1]$. Then

$$\begin{aligned} \|Tx - Ty\| &= \left| \frac{2}{5} - \frac{2}{3} + \frac{2}{3}y \right| = \left| \frac{2}{3}y - \frac{4}{15} \right| \\ &= \left| \frac{1}{4}x - \frac{1}{4}y + \frac{1}{4}x - \frac{1}{10} + \frac{5}{12}y - \frac{1}{6} \right| \\ &\leq \frac{1}{4}|x - y| + \frac{1}{4} \left(\left| x - \frac{2}{5} \right| + \left| \frac{5}{3}y - \frac{2}{3} \right| \right) \\ &= \frac{1}{4}|x - y| + \frac{1}{4}(|x - Tx| + |y - Ty|) \end{aligned}$$

Observe that T has a unique fixed point $p = \frac{2}{5}$.

Definition 2.5. Let K_1 and K_2 be closed subsets of a Banach space X . an operator

$T : K_1 \cup K_2 \rightarrow K_1 \cup K_2$ with $T(K_1) \subset K_2$ and $T(K_2) \subset K_1$ is said to belong to class $D(a, b, c)$ if it satisfies

$$\|Tx - Ty\| \leq a\|x - y\| + b[\|x - Tx\| + \|y - Ty\|] + c[\|x - Ty\| + \|Tx - y\|] \quad (2.4)$$

for all $x \in K_1$ and $y \in K_2$, where $0 \leq a, b \leq 1$. It is clear that if T belongs to the class $D(k, 0, 0)$ with $0 < k < 1$, then T is a cyclic contraction.

III. MAIN RESULTS

Proposition 3.1. Let K_1 and K_2 be closed subsets of a Banach space X . an operator $T : K_1 \cup K_2 \rightarrow K_1 \cup K_2$ with $T(K_1) \subset K_2$ and $T(K_2) \subset K_1$ and satisfies (2.4) with $0 \leq a, b, c < 1$ and $a + 2b + 2c < 1$. If $F(T) = \{x \in K_1 \cup K_2 : Tx = x\} \neq \emptyset$, then $F(T)$ consists of a single point.

Proof :

Assume the contrary, that is, let $z, w \in K_1 \cup K_2$ be two distinct fixed points of T . Then

$$\begin{aligned} \|z - w\| &= \|Tz - Tw\| \leq a\|z - w\| + b[\|z - Tz\| + \|w - Tw\|] + c[\|z - Tw\| + \|Tz - w\|] \\ &= (a + 2c)\|z - w\| \end{aligned}$$

which implies $z = w$, since $a + 2c < 1$.

Theorem 3.2. Let K_1 and K_2 be closed subsets of a Banach space X . suppose that the operator

$T : K_1 \cup K_2 \rightarrow K_1 \cup K_2$ with $T(K_1) \subset K_2$ and $T(K_2) \subset K_1$ satisfies (2.4) with $0 \leq a, b, c < 1$, and $a + 2b + 2c < 1$. Then the sequence $\{x_n\}$ in $K_1 \cup K_2$ satisfies $\lim_{n \rightarrow \infty} (x_n - Tx_n) = 0$ and the sequence $\{x_n\}$ converges to the unique fixed point of T .

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Proof. Let $x_0 \in K_1$. Define $x_n = Tx_{n-1} = T^n x_0$. Then

$$\|Tx_n - Tx_m\| \leq a\|x_n - x_m\| + b\{\|x_n - Tx_n\| + \|x_m - Tx_m\|\} + c\{\|x_n - Tx_m\| + \|Tx_n - x_m\|\}.$$

By the triangle inequality we have

$$\begin{aligned} \|Tx_n - Tx_m\| &\leq a\{\|x_n - Tx_n\| + \|Tx_n - Tx_m\| + \|x_m - Tx_m\|\} + b\{\|x_n - Tx_n\| + \|x_m - Tx_m\|\} \\ &\quad + c\{\|x_n - Tx_n\| + \|x_m - Tx_m\| + 2\|Tx_n - Tx_m\|\} \end{aligned}$$

This implies

$$\|Tx_n - Tx_m\| \leq \left(\frac{a+b+c}{1-a-2c}\right)\{\|x_n - Tx_n\| + \|x_m - Tx_m\|\}.$$

Observe that from the hypothesis, the right hand side of the inequality tends to 0, as $n \rightarrow \infty$, hence $\{Tx_n\}$ is a Cauchy sequence. Since $K_1 \cup K_2$ is complete, then it converges to a limit, say $z \in K_1 \cup K_2$, that is

$$\lim_{n \rightarrow \infty} Tx_n = z$$

Note that the subsequence $\{x_{2n}\} \in K_1$ and the sub sequence $\{x_{2n+1}\} \in K_2$ thus $z \in K_1 \cap K_2 \neq \emptyset$. Then we employ

$$\begin{aligned} &\text{the triangle inequality and the fact that } b < 1 \text{ to get } \|z - Tx_n\| \leq \|Tx_n - Tx_n\| \leq a\|z - x_n\| + b\{\|x_n - Tx_n\| + \|z - Tx_n\|\} \\ &\quad + c\{\|x_n - Tx_n\| + \|Tx_n - z\|\} \\ &\leq a\|z - x_n\| + b\{\|x_n - Tx_n\| + \|z - Tx_n\|\} + c\{\|z - x_n\| + \|z - Tx_n\| + \|x_n - Tx_n\| + \|z - x_n\|\} \\ &\leq \left(\frac{a+2c}{1-b-c}\right)\|z - x_n\| + \left(\frac{b+c}{1-b-c}\right)\|x_n - Tx_n\| \end{aligned}$$

It follows from $\lim_{n \rightarrow \infty} x_n = z$ and $\lim_{n \rightarrow \infty} (Tx_n - x_n) = 0$ that z is the fixed point of T which is unique by the proposition 3.1.

Corollary 3.3. Let K_1 and K_2 be closed subsets of a Banach space X . suppose that the operator

$T : K_1 \cup K_2 \rightarrow K_1 \cup K_2$ with $T(K_1) \subset K_2$ and $T(K_2) \subset K_1$ satisfies (2.3) with $0 \leq a, b < 1$, and $a + 2b < 1$. Then the sequence $\{x_n\}$ in $K_1 \cup K_2$ satisfies $\lim_{n \rightarrow \infty} (x_n - Tx_n) = 0$ and the sequence $\{x_n\}$ converges to the unique fixed point of T .

Proof: The proof of corollary follows immediate, by taking $c = 0$ in the above theorem.

Theorem 3.4. Let K_1 and K_2 be closed subsets of a Banach space X . suppose that the operator

$T : K_1 \cup K_2 \rightarrow K_1 \cup K_2$ with $T(K_1) \subset K_2$ and $T(K_2) \subset K_1$ satisfies (2.4) with $0 \leq a, b, c < 1$ and $a + 2b + 2c < 1$.

Then

- (i) T has a unique fixed point p in $K_1 \cap K_2$.
- (ii) $\|Tx - p\| < \|x - p\|$ for all $x \in K_1 \cup K_2$ where p is the fixed point of T .

Proof:

- (i) Take a point $x_0 \in K_1$. Define $x_{n+1} = Tx_n$ for $n = 0, 1, 2, \dots$ then we have

$$\begin{aligned} \|x_{n+1} - Tx_{n+1}\| &= \|Tx_n - Tx_{n+1}\| \leq a\|x_n - x_{n+1}\| + b\{\|x_n - Tx_n\| + \|x_{n+1} - Tx_{n+1}\|\} \\ &\quad + c\{\|x_n - Tx_{n+1}\| + \|Tx_n - x_{n+1}\|\} \end{aligned}$$

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This inequality implies

$$\|x_{n+1} - Tx_{n+1}\| \leq \left(\frac{a+b+c}{1-b-c}\right) \|x_n - Tx_n\| \leq \dots \leq \left(\frac{a+b+c}{1-b-c}\right)^n \|x_0 - Tx_0\|.$$

Hence, we obtain

$\|x_n - Tx_n\| \rightarrow 0$ as $n \rightarrow \infty$, implying that $\lim_{n \rightarrow \infty} x_n = p$, where p is the fixed point of T .

Since the subsequence $\{x_{2n}\} \in K_1$, and the subsequence $\{x_{2n+1}\} \in K_2$, then $p \in K_1 \cap K_2$. The uniqueness follows from proposition 3.1.

(ii) Let p be the fixed point of T and $x \in K_1 \cup K_2$. Then using (2.4) and the triangle inequality, we have

$$\begin{aligned} \|Tx - p\| &\leq \|Tx - Tp\| + \|Tp - p\| \\ &\leq a\|x - p\| + b[\|x - Tx\| + \|p - Tp\|] + c[\|x - Tp\| + \|Tx - p\|] \\ &\leq a\|x - p\| + b[\|x - p\| + \|p - Tx\|] + c[\|x - p\| + \|Tx - p\|] \end{aligned}$$

This inequality implies

$$\|Tx - p\| \leq \left(\frac{a+b+c}{1-b-c}\right) \|x - p\| < \|x - p\|$$

as $\left(\frac{a+b+c}{1-b-c}\right) < 1$, which completes the proof.

Corollary 3.5. Let K_1 and K_2 be closed subsets of a Banach space X . suppose that the operator

$T : K_1 \cup K_2 \rightarrow K_1 \cup K_2$ with $T(K_1) \subset K_2$ and $T(K_2) \subset K_1$ satisfies (2.3) with $0 \leq a, b < 1$ and $a + 2b < 1$. Then

- (i) T has a unique fixed point in $K_1 \cap K_2$.
- (ii) $\|Tx - p\| < \|x - p\|$ for all $x \in K_1 \cup K_2$ where p is the fixed point of T .

Proof: The proof of corollary follows immediate, by taking $c = 0$ in the above theorem.

IV. CONCLUSION

We have proved some fixed point theorems for cyclic mapping with contractive and expansive conditions in Banach spaces. The presented results generalize the results proved in various spaces and extend some results from the literature.

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