

# Some Properties of Sub Semigroups of an M-Semigroup

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**ABSTRACT:** In this paper, we discuss the various properties of sub semi groups of an M-semigroup. It is clear that every M-semigroup contains at least one sub semigroup and at least one sub-M-semigroup .By discussing the properties of an M-semigroup through its sub semigroup or its sub-M-semigroup we get some interesting results.

**KEYWORDS:** Semi group, M- semi group, sub M-semi group, semi deflation, Fine Band.

## I. INTRODUCTION

Tamura et al proved that an M – Groupoid S is the direct product of a right singular semigroup and a groupoid with a two sides identity. By an M-semigroup we mean a groupoid M in which

P<sub>1</sub>: There is atleast one  $e \in M$  such that  $ex=x$  for all  $x \in M$ .

P<sub>2</sub>:  $x(yz)=(xy)z$  for all  $x,y,z \in M$ (Assosiative law)

P<sub>3</sub>: For each  $x \in M$ , there is unique left identity  $e$  such that  $xe=x$ .

In this paper we discuss some properties of M-semigroup through its sub M- semi group.

## II. PRELIMINARIES

**2.1** A binary operation on a non-empty set s is a mapping of  $S \times S$  into S, where  $S \times S$  is defined as  $\{(x, y)/x, y \in S\}$ ; The image of (x, y) under this mapping is denoted by  $x.y$  (or by  $xy$ ) and is called the product of x and y;  $x,y \in S$ .

**2.2** A semigroup S is a system consisting of s nonempty set S, together with an associative binary operation on S, so that

$$x(yz) = (xy)z \text{ for all } x,y,z \text{ in } S.$$

If the operation is denoted by ‘.’ Then the semigroup is denoted by (S,.) or sometimes by just S, when the operation is understood.

**2.3** By a total semigroup S is meant one in which

$$S = S^2 = \{xy/x,y \in S\}.$$

**2.4** By an idempotent element of a semigroup S, is meant an element x in S, such that  $x = x^2$ ; a semigroup S is said to be an idempotent semigroup if every element of S is idempotent.

**2.5** A band is an idempotent semigroup.

**2.6** By an M - groupoid S we mean a groupoid which satisfies the following conditions:

(i) There is atleast one  $e \in S$ .

(ii) If y or z is a left identity of S then  $x(yz) = (xy)z$  for all  $x \in S$ .

(iii) For any  $x \in S$ , there is a unique left identity  $e$  such that  $xe = x$ .

**2.7** By an M – semigroup M we mean a semigroup that satisfies the following two conditions:

P<sub>1</sub> : There is at least one  $e \in M$  such that  $ex =x$  for all  $x \in M$ .

P<sub>2</sub> : For every  $x \in M$  there is a unique left identity  $e$  such that  $xe = x$ .

**2.8** An M – semigroup M is isomorphic to the direct product of a right zero semigroup and a semigroup with a two sided identity; and conversely.

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That is  $M \cong R \times S$  where  $R$  is a right zero semi – group and  $S$  is a semigroup with a two sided identity. The right zero semigroup  $R$  is the set of all left identities of  $M$  and  $S \cong Me$ , for a left identity  $e$  of  $M$ .

**2.9** By a sub –  $M$  – semigroup  $M'$  of a semigroup  $S$  we mean a subsemigroup  $M'$  of  $S$  which is also an  $M$  – semigroup.

**2.10** By an induced sub –  $M$  – semigroup  $M'$  of an  $M$  – semigroup  $M$ , we mean a sub –  $M$  – semigroup  $M'$  of  $M$  whose set of left identities is same as that of  $M$ .

**2.11** By a normal sub –  $M$  – semigroup  $K$  of an  $M$  – semigroup we mean that  $K$  is a sub –  $M$  – semigroup of  $M$  satisfies:

- (i)  $ka \subseteq ak$ ; for all  $a$  belong to  $M$ , and
- (ii) Every left identity of  $K$  is a left identity of  $M$ .

**2.12** A band is an idempotent semigroup.

**2.13** Let  $S = \bigcup_{i \in A} S_i$  be a decomposition of a semigroup  $S_i$  into subsemigroups over an index  $A$ . If for each  $(i,j)$  in  $A \times A$ , there exists an element  $k$  in  $A$  such that  $S_i S_j \subseteq S_k^2$  then  $A$  becomes thereby a band. Then  $S$  is called as the union of the fine band  $A$  of semigroups  $S_i (i \in A)$  or  $S$  is a fine band  $A$  of semigroups  $S_i$ .

### III. PROPERTIES OF AN M-SEMIGROUP

#### M-Semigroup and Sub-Semigroup Of a Semigroup

##### 3.1 Lemma:

Every semi-deflation of an  $M$ -semigroup to its sub semigroup is an  $M$ -semigroup.

##### Proof :

Let  $M \equiv R \times S$  be an  $M$ -semigroup such that  $M$  is semi-deflatable to  $M'$ , a sub semigroup of  $M$ . If  $\varphi$  is the semi-deflation map then there is a homomorphism,  $\varphi : M \rightarrow M'$  such that  $x\varphi = x$  for all  $x \in M'$ .

Let  $e \in R$ . Obviously any element  $x$  of  $M'$  is also an element of  $M$ . Therefore  $e.x = x$  for all  $x \in M'$ . That is,  $e\varphi$  is a left identities of  $M'$ . Hence  $M'$  contains left identities.

For any  $x \in M'$ ,  $x$  being an element of  $M$ , there exists a unique  $e$  such that  $x.e = x$

Now,  $(X.e)\varphi = x\varphi .e\varphi = x.e\varphi$  and  $(X.e)\varphi = x.\varphi = x$

Therefore,  $x = x.(e\varphi)$  and hence for each  $x \in M'$  there exists a unique  $e\varphi$  such that

$x.e\varphi = x$ . This proves that  $M'$  is an  $M$ -semigroup and hence the lemma.

##### 3.2 Lemma:

In a  $M$ -semigroup  $M \equiv R \times S$ ,  $R$  is a maxima rectangular band.

##### Proof:

If  $R'$  is a rectangular band such that  $R' \supset R$ , let  $x \in R' \setminus R$ . Then  $X$  being an element of  $M$ , there exists an  $e \in R'$  such that  $xe = x$ .

Now,  $e.xe.e = e$  since  $e, xe = x \in R'$  and  $R'$  is a rectangular band.

Also  $e.xe.e = xe = x$ . Therefore  $x = e \in R$ .

This is,  $R' \subset R$  that is  $R' = R$ . Therefore  $R$  is a maximal rectangular band.

##### 3.3 Lemma :

In an  $M$ -semigroup  $M \equiv R \times S$ , we can always define a congruence relation.

##### Proof:

Define a relation  $R^*$  on  $M$  such that for any two  $a, b$  of  $M$ ,  $aR^* b$  if and only if  $ae = be$  for all  $e \in R$ . Clearly  $R^*$  is an equivalence relation. Let  $aR^* b$ , and let  $c = cf$  be any element of  $M$ . Since

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$aR^*b, ae = be$  for all  $e \in R$ .

i.e.,  $ae.cf = be.cf$ .

i.e.,  $acf = bef$ .

i.e.,  $acfe = befe$ ; for any  $e \in R$

i.e.,  $ace = bce$ ; for all  $e \in R$ .

i.e.,  $acR^*bc$ .

i.e.,  $R^*$  is right compatible.

Let  $aR^*b$  and let  $c = cf$  be any element of  $M$ .

$aR^*b \Rightarrow ae = be$ ; for all  $e \in R$

$\Rightarrow cae = cbe$ ; for all  $e \in R$

$\Rightarrow (ca)e = (cb)e$ ; for all  $e \in R$

$\Rightarrow caR^*cb$

i.e.,  $R^*$  is an left compatible.

Therefore  $R^*$  is a congruence relation on  $M$ . Hence the lemma.

Sub-M-semigroup of any semigroup, in particular that of an M-semigroup, plays an important role in the study of the semigroup. The following lemmas are the characterizations for a semigroup S to have a sub-M-semigroup.

### 3.4 Lemma:

The necessary and sufficient condition for a semigroup S to have a sub-M-semigroup is that it contains a right zero semigroup.

#### Proof:

Let S contain a sub-M-semigroup M. The set of left identities of M forms a right zero semigroup and hence S contains a right zero semigroup.

Conversely, if S contains a right zero semigroup R, then R itself is a sub-M-semigroup of S.

### 3.5 Lemma:

The necessary and sufficient condition for a semigroup S to have a sub-M-semigroup is that S contains an idempotent element.

#### Proof:

If S contains an idempotent element 'a' then {a} is an M-semigroup and hence S contains a sub-M-semigroup M then every left identity of M is idempotent element of S. This proves the lemma.

### 3.6 Lemma:

A semigroup S with R as its set of left identities will have a sub-M-semigroup M which consists of the elements of S that commute with any one of  $e \in R$ , if and only if S satisfies the condition that any  $x \in S$  can commute with atmost one  $e \in R$ .

#### Proof:

Let S contains a sub-M-semigroup M, consists of the elements of S which commute with any one of  $e \in R$ . Since e commutes with itself for all  $e \in R$ ,  $R \subseteq M$ , and R is the set of left identities of M. M being an M-semi-group  $Me \cap cMF = \phi$  for all  $e, f \in R$ . Let any

$x \in S$ , commute with two left identities e and f of R.

Therefore,  $xe = ex = x$  and  $xf = fx = x$ . That is,  $x \in Me \cap Mf$ , which is a contradiction.

Therefore, x can commute with atmost one  $e \in R$ .

Conversely, let S satisfied the condition that any  $x \in S$  can commute with atmost one  $e \in R$ . Denote the set of element of S which commute with an e belong to R as  $A_e$ . since  $e \in A_e$ ,  $A_e$  is not empty for all  $e \in R$ . For any x,y belong to  $A_e$ ,

$xye = xey = exy$ .

Therefore xy is an element of  $A_e$  and hence  $A_e$  is a subsemigroup.

Consider  $\bigcup_{e \in R} A_e = M$

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For any  $x \in Ae$  and  $y \in Af$ ,  $e f \in R$ ,

$(xy)f = x(fy) = (xf)y = fxy = xy$ .

Therefore  $xy \in M$ , which proves that  $M$  is a subsemigroup of  $S$ .

Since  $e.xf = xf$ ,  $e$  is a left identity of  $M$ .

Similarly, each element of  $R$  is a left identity of  $M$ . Obviously, by the condition on  $S$ , for each  $x$  belong to  $M$ , there exist a unique  $e$  belong to  $R$  such that

$x.e = e.x = x$ .

Therefore  $M$  is a sub-M-semigroup of  $S$  consists of the element of  $S$  which commute with any one of  $e$  belongs to  $R$ . This proves the lemma.

The following theorem gives a necessary condition for a semigroup  $S$  to have a sub-M-semigroup.

### 3.7 Theorem:

If a semigroup  $S$  satisfied the condition that for any  $x \in S$ ,  $xe \neq xf$ ;  $e f \in R$ , where  $R$  is the set of left identities of  $S$ , then  $S$  contains an  $M$ -semi-group,  $M = \bigcup_{e \in R} Se$ , With  $R$  as its set of left identities.

#### Proof:

Consider  $M = \bigcup_{e \in R} Se$

$Se \cap Sf = \emptyset$  because  $xe \neq xf$ ; for all  $x \in S$  and  $e, f \in R$

Clearly  $R \subset M$

For any  $xe, yf \in M$ ,  $xe.yf = xyf \in Sf \subset M$ , proves that  $M$  is a subsemigroup.

If  $M$  contains a left identity  $xf \in M \setminus R$ , then  $xf$  being an element of  $M = \bigcup_{e \in R} Se$

$Xf.f = x(ff) = xf$  and  $xf.f = f$

i.e.,  $f = xf \in R$ , Therefore,  $R$  is the set of left identity of  $M$

For any  $xe \in M$ ,  $xe.e = xe = e.xe$

And if  $xe.f = f(xe)$  then  $xf = xe$ , which is not so. Therefore for any  $xe \in M$ , there exists a unique  $e \in R$  such that  $Xe.e = e.xe = xe$ .

This proves that  $M$  is an  $M$ - semigroup, with  $R$  as its set of left identities.

Hence the theorem.

The following lemma, gives a condition on a semigroup  $S$  to be a semi-inflation of its sub-M-semigroup.

### 3.8 Lemma:

$Me$  is a sub semigroup of  $M$ .

### 3.9 Theorem:

An  $M$ -semigroup  $M$  is a disjoint union of its sub-M-semigroups which are left ideals of  $M$  containing exactly one identity and each such sub-M-semigroup is a semi deflation of  $M$ .

### 3.10 Lemma:

If a semigroup  $S$  with  $R$  as its set of left identities satisfies the conditions,

(i) For any  $x \in S$ ,  $Xe \neq xf$ ;  $e f \in R$ ,  $e \neq f$  and

(ii).  $xy.f = f.xy$  implies  $yf = fy$  for all  $f \in R$  then  $S$  is a semi-inflation of its sub-M-semigroup  $M = \bigcup_{e \in R} Se$

#### Proof:

From 3.7  $M = \bigcup_{e \in R} Se$  is a sub-M-semigroup of  $S$ .

Define  $\varphi: S \rightarrow M$  such that  $x\varphi = x$ ; for all  $x \in M$  and  $a\varphi = ae$ , for a fixed  $e \in R$ ; for all  $a \in S \setminus M$ . For any two  $xe, yf \in M$ ,  $(xe.yf)\varphi = (xyf)\varphi = xyf$ .

$xe\varphi.yf\varphi = xe.yf = xyf$ .

For any  $xf \in M$  and  $a \in S \setminus M$ ;  $xf.a \notin M$  because if  $xf.a \in M$ ,  $M$  being an  $M$ -semigroup there exists a unique  $g \in R$  such that  $xf.a = xfa.g$

i.e.,  $g[(xf)(a)] = [(xf)(a)]g$

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i.e.,  $ag = gaby(ii)$

i.e.,  $a = ag \in m$  a contradiction .

therefore,  $xf \in S \setminus M$ .

$$(xf)a\varphi = xfae = xae$$

and  $(xf)\varphi a\varphi = xf.ae = xae$ .

For any  $a \in S \setminus M$  and  $xf \in M$ ,  $axf \in M$ , because  $axf$  commute with  $f$ .

Therefore

$$(axf)\varphi = axf \text{ and } a\varphi.(xf)\varphi = ae.xf = axf.$$

For any two  $a, b \in S \setminus M$ ,  $ab \notin M$  because if  $ab \in M$ , then there exists a  $g \in R$  such that

$$abg = ab = g.ab$$

i.e.,  $gb = bg$  by(ii).

i.e.,  $b = bg \in M$ , a contradiction.

Therefore,  $ab \in S \setminus M$ ,

$$(ab)\varphi = abe \text{ and } a\varphi.b\varphi = ae.be = abe.$$

Therefore  $\varphi$  is a homomorphism fixing the elements of  $M$ . Hence  $S$  is a semi-inflation of  $M$ .

From 3.8 and 3.9,  $M$  is a fine band  $R$  of sub- $M$ -semigroups. Combining this with 3.10, we have the following theorem.

### 3.11 Theorem:

A semigroup  $S$  which satisfies the conditions (i) and (ii) of 3.10 is a semi-inflation of its sub- $m$ -semigroup which is a fine band  $R$  of sub- $M$ -semigroups.

The following theorem gives a structure of a semigroup  $S$  to be an inflation of its set of factorizable elements which is a sub- $M$ -semigroup of  $S$ .

### 3.12 Theorem:

Any semigroup  $S$  which satisfies the following four conditions:

- i.  $S$  is a disjoint union of its subsets  $U_i$  ( $i \in \Lambda$ ) where (a) and  $U_i$  is a subsemigroup satisfies

$$U_i^2 = \{\text{singleton}\}$$

$$\text{And (b) } S^2 = \bigcup_{i \in \Lambda} U_i^2$$

- ii. There exists at least one  $U_i$  ( $i \in \Lambda$ ) such that each  $x \in U_i$  implies  $xy \in U_j$ ; where  $y \in U_j$ .
- iii. For each  $x \in U_j$ ; there exists a unique  $U_i$  satisfies (ii) such that  $x(U_i^2) = x$  and
- iv. If  $xy \in U_k$  for any  $x \in U_i, y \in U_j$  then  $xy \in U_k$ ; for all  $x \in U_i, y \in U_j$   
is an inflation of its set of factorizable elements which is an  $M$ -semigroup.

**Proof:** Consider  $S^2$ . Denote the factorizable elements of the subsets of  $S$  satisfying (ii) as  $e, f, \dots$  since there is at least one such set, there exists at least one  $e$  such that  $e.y = y$ , for all  $y \in S^2$ . That is  $e$  is a left identity of  $S^2$ . By (iii) for any  $x \in S^2$ , there exists a unique  $e \in S^2$  such that  $xe = x$ . therefore  $S^2$  is an  $M$ -semigroup.

Define  $\varphi: S \rightarrow S^2$  such that  $x\varphi = \{U_i\}^2$  where  $x \in U_i$

$\varphi$  is well defined.

If  $x \in U_i, y \in U_j$  and  $xy \in U_k$  then

$$(xy)\varphi = (U_k)^2$$

$$X\varphi.y\varphi = (U_i)^2.(U_j)^2 = (U_k)^2 \text{ by(iv)}$$

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i.e.,  $\varphi$  is a homomorphism. Since  $S^2 = \bigcup_{i \in \Lambda} U_i^2$ ;  $\varphi$  fixes the elements of  $S^2$ . Hence  $\varphi$  is an inflation of its set of factorizable elements which is an M-semigroup.

## IV. CONCLUSION

Thus we have structured of an M- Semi group through the semi deflation of its sub M- Semi group.

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