

# Common Fixed Point theorems for OWC Mappings satisfying a General Contractive Condition of Integral Type

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**ABSTRACT:** In this paper , we obtain a common fixed point theorem for two pairs of occasionally weakly compatible single and set-valued maps defined on a metric space satisfying a contractive condition of integral type. Our result unifies, extends and complements many results existing in the literature.

**KEYWORDS:** Occasionally weakly compatible maps, weakly compatible maps, compatible maps, common fixed point, metric space.

**2000 Mathematics subject classification.** 47H10, 54H25.

## I. INTRODUCTION

The concepts of weak commutativity, compatibility, noncompatibility and weak compatibility were frequently used to prove existence theorems in fixed and common fixed points for single and set-valued maps satisfying certain conditions in different spaces. Generalizing the concept of commuting mappings, Sessa [10] was introduced the concept of weakly commuting mappings. He has defined that  $f$  and  $g$  to be weakly commuting, if  $d(fgx, gfx) \leq d(gx, fx)$  for all  $x \in X$ , where  $f$  and  $g$  are two self –maps of  $(X,d)$ .

In 1986 Jungck [4] made more generalized commuting and weakly commuting maps called compatible maps.  $f$  and  $g$  are said to be compatible if  $\lim_{n \rightarrow \infty} d(fgx_n, gfx_n) = 0$  whenever  $(x_n)_{n \in \mathbb{N}}$  is a sequence in  $X$  such that  $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = t$  for some  $t \in X$ . This concept has been useful as a tool for obtaining more comprehensive fixed point theorems. Clearly, commuting maps are weakly commuting and weakly commuting maps are compatible, but neither implication is reversible (see [4]). Afterwards, Jungck [5] generalized the compatibility by introducing the concept of weak compatibility. He defines  $f$  and  $g$  to be weakly compatible if  $f t = g t$ ,  $t \in X$  implies  $f g t = g f t$ . Clearly, compatibility imply weak compatibility but there exist weakly compatible maps which are not compatible .

The study of common fixed points on occasionally weakly compatible maps is new and also interesting. In the recent paper of Jungck and Rhoades [7], the concept of occasionally weakly commuting maps (owc) was introduced. In that paper, it was shown that essentially every theorem involving four maps becomes a special case of one of the results on owc maps. Branciari [3] obtained a fixed point theorem for a single valued mapping satisfying an analogue of Banach's contraction principle for an integral-type inequality. Rhoades [9] proved two fixed point theorems involving more general contractive conditions .

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## II. PRELIMINARIES

Throughout this paper,  $(X,d)$  denotes a metric space and  $CB(X)$  the class of all nonempty bounded closed subsets of  $X$ . We take these usual notations: for  $x \in X$  and  $A \subseteq X$ ,

$$d(x,A) = \inf \{d(x,y) : y \in A\}$$

and let  $H$  be the associated Hausdorff metric on  $CB(X)$ : for every  $A,B$  in  $CB(X)$ ,

$$H(A,B) = \max \{ \sup_{x \in A} d(x,B), \sup_{y \in B} d(A,y) \}.$$

In the following, we use small letters:  $f, g, \dots$  to denote maps from  $X$  into  $X$  and capital letters:  $F,G,S,T, \dots$  for set-valued maps, that is, maps from  $X$  into  $CB(X)$  and we write  $fx$  for  $f(x)$  and  $Fx$  for  $F(x)$ .

The same author with Rhoades [7] weakened the concept of weakly compatible maps by giving the new concept of occasionally weakly compatible maps. Two self-maps  $f$  and  $g$  of  $X$  are said to be occasionally weakly compatible maps (shortly owc) if there is a point  $x$  in  $X$  such that  $fx = gx$  at which  $f$  and  $g$  commute.

In their paper [8], Kaneko and Sessa extended the compatibility to the setting of single and set-valued maps as follows:  $f : X \rightarrow X$  and  $F : X \rightarrow CB(X)$  are said to be compatible if  $f Fx \in CB(X)$  for all  $x \in X$  and  $\lim_{n \rightarrow \infty} H(Ffx_n, f Fx_n) = 0$  whenever  $(x_n)_{n \in \mathbb{N}}$  is a sequence in  $X$  such that  $fx_n \rightarrow t, Fx_n \rightarrow A \in CB(X)$  and  $t \in A$ .

After, in [6] Jungck and Rhoades made an extension of the concept of compatible single and set-valued maps by giving the concept of weak compatibility. Maps  $f$  and  $F$  above are weakly compatible if they commute at their coincidence points, i.e., if  $f Fx = F f x$  whenever  $fx \in Fx$ .

More recently, Abbas and Rhoades [1] extended the owc maps to the setting of single and set-valued maps and they proved some common fixed point theorems satisfying generalized contractive condition of integral type. Previous maps  $f$  and  $F$  are said to be owc if and only if there exists some points  $x$  in  $X$  such that  $fx \in Fx$  and  $f Fx \subseteq Ffx$ . Clearly, weakly compatible maps are occasionally weakly compatible, However, the converse is not true in general. The example below illustrate this fact.

**Example 1.1.** Let  $X = [1, \infty)$  with the usual metric. Define  $f : X \rightarrow X$  and  $F : X \rightarrow CB(X)$  by, for all  $x \in X$ ,

$$fx = 2x + 1, \quad Fx = [1, 2x + 1].$$

$$fx = 2x + 1 \in Fx \quad \text{and} \quad f Fx = [3, 4x + 3] \subset F f x = [1, 4x + 3].$$

Hence,  $f$  and  $F$  are occasionally weakly compatible but not weakly compatible.

The following property is to be needed. It is immediately proved from definition of the Hausdorff metric  $H$ .

**Property 1.2.** Let  $A$  and  $B \in CB(X)$ , then for any  $a \in A$ , we have

$$d(a, B) \leq H(A, B).$$

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**Definition 1.3.** Let  $F : X \rightarrow 2^X$  be a set-valued map on  $X$ . A point  $x \in X$  is a fixed point of  $F$  if  $x \in Fx$  and is an absolutely fixed point of  $F$  if  $Fx = \{x\}$ .

### III. MAIN RESULT

**Theorem 2.1.** Let  $f, g : X \rightarrow X$  be mappings and  $F, G : X \rightarrow CB(X)$  be set-valued mappings such that the pairs  $\{f, F\}$  and  $\{g, G\}$  are owc. Let  $\psi : \mathbb{R}^5 \rightarrow \mathbb{R}^+$  be a real map satisfying the following conditions :

- (1)  $\psi$  is nondecreasing in the 1<sup>st</sup>, 4<sup>th</sup> and 5<sup>th</sup> variables ,
- (2) if  $u \in \mathbb{R}^+$  is such that  $\int_0^u \varphi(t) dt \leq \int_0^{\psi(u,0,0,u,u)} \varphi(t) dt$  then  $u = 0$ .

If for all  $x, y \in X$ , we have

$$(*) \quad \int_0^{H(Fx,Gy)} \varphi(t) dt < \int_0^{\psi(d(fx,gy),d(fx,Fx),d(gy,Gy),d(fx,Gy),d(gy,Fx))} \varphi(t) dt$$

where  $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}$  is a Lebesgue integrable mapping which is summable, nonnegative and such that, for each  $\varepsilon > 0$ ,  $\int_0^\varepsilon \varphi(t) dt > 0$  and for which the right-hand side of (\*) is not zero. Then  $f, g, F$  and  $G$  have a unique common fixed point which is an absolutely fixed point for  $F$  and  $G$ .

**Proof.** (i) We begin to show existence of a common fixed point. Since the pairs  $\{f, F\}$  and  $\{g, G\}$  are owc, then there exist  $u, v$  in  $X$  such that  $fu \in Fu, gv \in Gv, fFu \subseteq Ffu$  and  $gGv \subseteq Ggv$ . Also, using the triangle inequality and property 1.2, we obtain

$$d(fu,gv) \leq H(Fu,Gv) \tag{3}$$

and  $d(f^2u,gv) \leq H(Ffu,Gv) \tag{4}$

First, we show that  $gv = fu$ . Suppose not, then condition (\*) implies that

$$\int_0^{H(Fu,Gv)} \varphi(t) dt < \int_0^{\psi(d(fu,gv),d(fu,Fu),d(gv,Gv),d(fu,Gv),d(gv,Fu))} \varphi(t) dt .$$

By (1) we have

$$\int_0^{H(Fu,Gv)} \varphi(t) dt \leq \int_0^{\psi(H(Fu,Gv),0,0,H(Fu,Gv),H(Fu,Gv))} \varphi(t) dt ,$$

which from (2) gives  $H(Fu, Gv) = 0$ , which from (3) implies that  $d(fu,gv) = 0$ , so  $fu = gv$ . Hence the claim follows. Again we have  $d(f^2u,fu) \leq H(Ffu,Gv)$ . Next, we claim that  $f^2u = fu$ . If not then condition (\*) implies that

$$\begin{aligned} \int_0^{H(Ffu,Gv)} \varphi(t) dt &< \int_0^{\psi(d(f2u,gv),d(f2u,Ffu),d(gv,Gv),d(f2u,Gv),d(gv,Ffu))} \varphi(t) dt \\ &= \int_0^{\psi(d(f2u,fu),0,0,d(f2u,Gv),d(fu,Ffu))} \varphi(t) dt . \end{aligned}$$

By (1), we have

$$\int_0^{H(Ffu,Gv)} \varphi(t) dt \leq \int_0^{\psi(H(Ffu,Gv),0,0,H(Ffu,Gv),H(Ffu,Gv))} \varphi(t) dt ,$$

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which from (2) gives  $H(Ffu, Gv) = 0$ , by (4) we obtain  $d(f^2u, fu) = 0$ , so  $f^2u = fu$ . Hence the claim follows. Since  $(f, F)$  and  $(g, G)$  have the same role, we have :  $Fu = Ggv$  and  $gv = g^2v$ . Therefore,  $ffu = fu = gv = gg v = gfu$ ,  $fu = f^2u \in fFu \subset Ffu$ , so  $fu \in Ffu$  and  $fu = gfu \in Gfu$ . Then  $fu$  is a common fixed point of  $f, g, F$  and  $G$ .

(ii) Now we show uniqueness of the common fixed point. Put  $fu = w$  and let  $w'$  be another common fixed point of the four maps such that  $w \neq w'$ , then we have

$d(w, w') = d(fw, gw') \leq H(Fw, Gw')$ ; by (\*), we get

$$\begin{aligned} \int_0^{H(Fw, Gw')} \varphi(t) dt &< \int_0^{\psi(d(fw, gw'), d(fw, Fw), d(gw', Gw'), d(fw, Gw'), d(gw', Fw))} \varphi(t) dt \\ &= \int_0^{\psi(d(fw, gw'), 0, 0, d(fw, Gw'), d(gw', Fw))} \varphi(t) dt. \end{aligned}$$

By (1), we get

$$\int_0^{H(Fw, Gw')} \varphi(t) dt \leq \int_0^{\psi(H(Fw, Gw'), 0, 0, H(Fw, Gw'), H(Fw, Gw'))} \varphi(t) dt,$$

which from (2) gives  $H(Fw, Gw') = 0$  and thus  $d(fw, gw') = d(w, w') = 0$ .

Therefore  $w = fu$  is a unique common fixed point of  $f, g, F$  and  $G$ .

(iii) Let  $w \in Ffu$ . Using the triangle inequality and property 1.2, we have

$$d(fu, w) \leq d(fu, Ffu) + H(Ffu, Gv) + d(w, Gv).$$

Since  $fu \in Ffu$  and  $H(Ffu, Gv) = 0$ , we have  $d(w, fu) \leq d(w, Gv) \leq H(Ffu, Gv) = 0$ . Hence  $w = fu$  and  $Ffu = \{fu\} = \{gv\} = Ggv$ , this last equality is given thanks to the same role of  $F$  and  $G$ . Hence  $fu$  is an absolutely fixed point for  $F$  and  $G$ . This completes the proof of our theorem.

If we let  $f = g$  and  $F = G$  in theorem 2.1, we get the following corollary

**Corollary 2.2.** Let  $f : X \rightarrow X$  and  $F : X \rightarrow CB(X)$  such that the pair  $\{f, F\}$  is owc. Let  $\psi : \mathbb{R}^5 \rightarrow \mathbb{R}^+$  be a real map satisfying the following conditions :

- (1)  $\psi$  is nondecreasing in the 1<sup>st</sup>, 4<sup>th</sup> and 5<sup>th</sup> variables ,
- (2) if  $u \in \mathbb{R}^+$  is such that  $\int_0^u \varphi(t) dt \leq \int_0^{\psi(u, 0, 0, u, u)} \varphi(t) dt$  then  $u = 0$ .

If for all  $x, y \in X$ , we have

$$(*) \quad \int_0^{H(Fx, Fy)} \varphi(t) dt < \int_0^{\psi(d(fx, fy), d(fx, Fx), d(fy, Fy), d(fx, Fy), d(fy, Fx))} \varphi(t) dt$$

where  $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}$  is a Lebesgue integrable mapping which is summable, nonnegative and such that, for each  $\varepsilon > 0$ ,  $\int_0^\varepsilon \varphi(t) dt > 0$  and for which the right-hand side of (\*) is not zero. Then  $f$  and  $F$  have a unique common fixed point .

Now , if we let  $f = g$  in theorem 2.1 , we get the next result

**Corollary 2.3.** Let  $f$  be a self-mapping of a metric space  $(X, d)$  and let  $F, G : X \rightarrow CB(X)$  be set-valued mappings such that the pairs  $\{f, F\}$  and  $\{f, G\}$  are owc. Let  $\psi : \mathbb{R}^5 \rightarrow \mathbb{R}^+$  be a real map satisfying the following conditions :

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- (1)  $\psi$  is nondecreasing in the 1<sup>st</sup>, 4<sup>th</sup> and 5<sup>th</sup> variables ,
- (2) if  $u \in \mathbb{R}^+$  is such that  $\int_0^u \varphi(t)dt \leq \int_0^{\psi(u,0,0,u,u)} \varphi(t)dt$  then  $u = 0$ .

If for all  $x, y \in X$ , we have

$$(*) \int_0^{H(Fx,Gy)} \varphi(t)dt < \int_0^{\psi(d(fx,fy),d(fx,Fx),d(fy,Gy),d(fx,Gy),d(fy,Fx))} \varphi(t)dt$$

where  $\varphi: \mathbb{R}^+ \rightarrow \mathbb{R}$  is a Lebesgue integrable mapping which is summable, nonnegative and such that, for each  $\varepsilon > 0$ ,  $\int_0^\varepsilon \varphi(t)dt > 0$  and for which the right-hand side of (\*) is not zero. Then  $f$ ,  $F$  and  $G$  have a unique common fixed point .

**Corollary 2.4.** Let  $f, g: X \rightarrow X$  be mappings and  $F, G: X \rightarrow CB(X)$  be set-valued mappings such that the pairs  $\{f, F\}$  and  $\{g, G\}$  are owc. Let  $\psi: \mathbb{R}^5 \rightarrow \mathbb{R}^+$  be a real map satisfying the following conditions :

- (1)  $\psi$  is nondecreasing in the 1<sup>st</sup>, 4<sup>th</sup> and 5<sup>th</sup> variables ,
- (2) if  $u \in \mathbb{R}^+$  is such that  $u \leq \psi(u,0,0,u,u)$  then  $u = 0$ .

If for all  $x, y \in X$ , we have

$$(*) H(Fx, Gy) < \psi(d(fx, gy), d(fx, Fx), d(gy, Gy), d(fx, Gy), d(gy, Fx))$$

for which the right-hand side of (\*) is not zero. Then  $f, g, F$  and  $G$  have a unique common fixed point which is an absolutely fixed point for  $F$  and  $G$ .

## IV. CONCLUSION

Our main theorem 2.1 is a generalization of Abbas and Rhoades [[1], theorem 2.6]. Put  $\varphi(t) = 1$ , in theorem 2.1, we obtain the corollary 2.4, which improves Bouhadjera and Godet – Thobie [[2], theorem 2.2] .

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