

On the Construction of Two Dimensional Cantor like Sets and Determination of Hausdorff Dimension

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ABSTRACT: In this paper we construct two dimensional Cantor like set S from any sequence $\{\epsilon_n\}$ with $0 < \epsilon_n < 1$, $\sum \epsilon_n = \infty$ with the help of sequence of sets $\{E_n\}$ of subsets of unit square $[0, 1] \times [0, 1]$ such that $E_n \supset E_{n+1}$, $S = \bigcap E_n$ and $m(S) = 0$. Two dimensional Cantor ternary set comes out to be a particular case of this construction of two dimensional Cantor like sets by choosing $\epsilon_n = \frac{1}{3}$ for all n . Similarly we can construct two dimensional Cantor like set by choosing $\epsilon_n = \frac{1}{5}$ for all n , which we have called two dimensional Cantor - $\frac{2}{5}$ set, we can construct two dimensional Cantor like set by choosing $\epsilon_n = \frac{1}{7}$ for all n , which we have called two dimensional Cantor - $\frac{3}{7}$ set. Also we can construct two dimensional Cantor like set by choosing $\epsilon_n = \frac{1}{2}$ for all n , which we have called two dimensional Cantor - $\frac{1}{4}$ set.

Further we determine the estimates for the Hausdorff dimension of such two dimensional Cantor like sets. In particular it is proved that the Hausdorff dimension of the set S defined above is $\leq \frac{-2 \log 2}{\gamma}$, where $\gamma = \lim_{k \rightarrow \infty} \log \left[\prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{1/k}$. In particular $\epsilon_n = \epsilon$, a constant, $0 < \epsilon < 1$, it is proved that the Hausdorff dimension of the set S defined above is $\geq \frac{-\log 4}{\log(\frac{1-\epsilon}{2})}$.

KEYWORDS: Cantor set, Cantor like sets, Hausdorff dimension.

I. INTRODUCTION

Definition 1.1 Hausdorff measure:

If U is any non-empty subset of n -dimensional Euclidean space, \mathbb{R}^n , then diameter of U is defined as $|U| = \sup\{|x - y| : x, y \in U\}$. If $\{U_i\}$ is a countable (or finite) collection of sets of diameter at most δ that cover F , i.e. $F \subset \bigcup_{i=1}^{\infty} U_i$ with $0 < |U_i| \leq \delta$ for each i , we say that $\{U_i\}$ is a δ -cover of F .

Suppose that F is a subset of \mathbb{R}^n and s is a non-negative number. For any $\delta > 0$ we define

$$H^s_\delta(F) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } F \right\}$$

We define

$$H^s(F) = \lim_{\delta \rightarrow 0} H^s_\delta(F)$$

This limit exists for any subset F of \mathbb{R}^n , though the limiting value can be 0 or ∞ .

We call $H^s(F)$ the s -dimensional Hausdorff measure of F .

OR $H^s(F) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a countable cover of } F \right\}$

Definition 1.2 Hausdorff dimension:

The Hausdorff dimension of F is defined as

$$\dim_H(F) = \inf \{ s \geq 0 : H^s(F) = 0 \}.$$

It is known that $\inf \{ s \geq 0 : H^s(F) = 0 \} = \sup \{ s \geq 0 : H^s(F) = \infty \}$.

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Above definitions and properties are taken from Kenneth Falconer [2].

Theorem 1.3

Let $\{E_i\}$ be a sequence of measurable sets. We have

i) If $E_1 \subseteq E_2 \subseteq \dots$, then $\mu(\bigcup_{n=1}^{\infty} E_n) = \lim \mu(E_n)$.

ii) If $E_1 \supseteq E_2 \supseteq \dots$, then $\mu(E_i) < \infty$, $\mu(\bigcap_{n=1}^{\infty} E_n) = \lim \mu(E_n)$.

This is Theorem 10 in Section 5.5 on page 103 of Chapter 5, from G. de Barra [1].

II. MAIN RESULTS

A. CONSTRUCTION OF TWO DIMENSIONAL GENERALIZED CANTOR LIKE SET

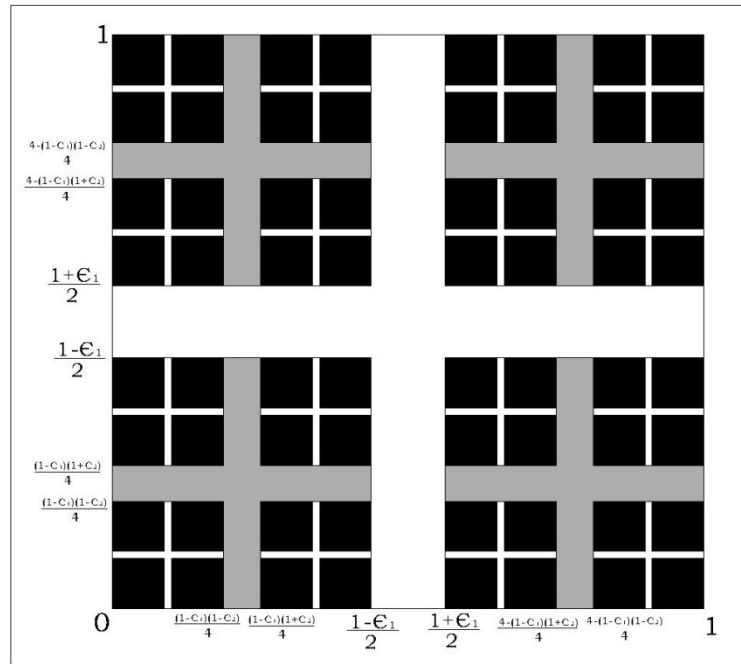


Fig. 2.1: Construction of two dimensional generalized Cantor like set

At first stage we divide unit square E_0 into three columns and three rows consisting of nine rectangles out of which 4 are corner squares each of side $\frac{1-\epsilon_1}{2}$, 4 are rectangles each of size $\epsilon_1 \times (\frac{1-\epsilon_1}{2})$ and 1 centre square is of side ϵ_1 .

Let E_1 be a set obtained by removing middle rectangle from each column and each row. Thus E_1 is the union of 4 corner squares each of side $\frac{1-\epsilon_1}{2}$.

At second stage we divide each remaining square again into three columns and three rows consisting of nine rectangles out of which 4 are corner squares each of side $\frac{(1-\epsilon_1)(1-\epsilon_2)}{4}$, 4 are rectangles each of length $\frac{(1-\epsilon_1)\epsilon_2}{2}$ and breadth $\frac{(1-\epsilon_1)(1-\epsilon_2)}{4}$ and 1 centre square is of side $\frac{(1-\epsilon_1)\epsilon_2}{2}$.

Let E_2 be a set obtained by removing middle rectangle from each column and each row from each remaining square. Thus E_2 is union of 4^2 squares each of side $\frac{(1-\epsilon_1)(1-\epsilon_2)}{4}$.

At third stage we divide each remaining squares again into three columns and three rows consisting of nine rectangles out of which 4 are corner squares each of side $\frac{(1-\epsilon_1)(1-\epsilon_2)(1-\epsilon_3)}{8}$, 4 are rectangles each of length $\frac{(1-\epsilon_1)(1-\epsilon_2)\epsilon_3}{4}$ and breadth $\frac{(1-\epsilon_1)(1-\epsilon_2)(1-\epsilon_3)}{8}$ and 1 centre square is of side $\frac{(1-\epsilon_1)(1-\epsilon_2)\epsilon_3}{4}$.

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Let E_3 be a set obtained by removing middle rectangle from each column and each row from each remaining square. Thus E_3 is union of 4^3 squares each of side $\frac{(1-\epsilon_1)(1-\epsilon_2)(1-\epsilon_3)}{8}$.

Continuing in this way at k^{th} stage E_k consists of union of 4^k squares each of side $\prod_{i=1}^k \frac{(1-\epsilon_i)}{2}$.

We define

$$S = \bigcap_{k=0}^{\infty} E_k$$

Where $E_k = \bigcup_{j,k} J_{j,k}$, $j = 1, 2, \dots, 4^k$, each $J_{j,k}$ being a square of side $\prod_{i=1}^k \frac{(1-\epsilon_i)}{2}$.

Remark 2.2

Two dimensional Cantor- $\frac{2}{5}$ set is the particular case of two dimensional generalized Cantor like set constructed in 2.1 when $\epsilon_n = \frac{1}{5}$.

Remark 2.3

Two dimensional Cantor- $\frac{3}{7}$ set is the particular case of two dimensional generalized Cantor like set constructed in 2.1 when $\epsilon_n = \frac{1}{7}$.

Remark 2.4

Two dimensional Cantor- $\frac{1}{4}$ set is the particular case of two dimensional generalized Cantor like set constructed in 2.1 when $\epsilon_n = \frac{1}{2}$.

Lemma 2.5

Given any sequence $\{\epsilon_n\}$ with $0 < \epsilon_n < 1$, $\sum \epsilon_n = \infty$ if and only if $\lim_{n \rightarrow \infty} \prod_{j=1}^n (1 - \epsilon_j) = 0$.

This is from [4].

Now we determine the measure of two dimensional generalized Cantor like set constructed above in following theorem. Here m is used to indicate the Lebesgue measure of two dimensional set.

Theorem 2.6

Measure of two dimensional generalized Cantor like set S from a given sequence $\{\epsilon_n\}$ of positive numbers less than 1 is $[\lim_{k \rightarrow \infty} \prod_{i=1}^k (1 - \epsilon_i)]^2$ i.e.

$$m(S) = [\lim_{k \rightarrow \infty} \prod_{i=1}^k (1 - \epsilon_i)]^2. \text{ In particular if } \sum \epsilon_n < \infty, m(S) \neq 0 \text{ and if } \sum \epsilon_n = \infty, m(S) = 0.$$

Proof:-

Since at first stage E_1 is union of 4 corner closed squares each having area $[\frac{1-\epsilon_1}{2}]^2$.

$$\therefore m(E_1) = 4[\frac{1-\epsilon_1}{2}]^2.$$

Since at second stage E_2 is union of 4^2 corner closed squares each having area $[\frac{(1-\epsilon_1)(1-\epsilon_2)}{4}]^2$.

$$\therefore m(E_2) = 4^2[\frac{(1-\epsilon_1)(1-\epsilon_2)}{4}]^2.$$

Since at third stage E_3 is union of 4^3 corner closed squares each having area $[\frac{(1-\epsilon_1)(1-\epsilon_2)(1-\epsilon_3)}{8}]^2$.

$$\therefore m(E_3) = 4^3[\frac{(1-\epsilon_1)(1-\epsilon_2)(1-\epsilon_3)}{8}]^2.$$

Continuing in this way at k^{th} stage E_k is union of we get 4^k corner closed squares each of area $[\prod_{i=1}^k \frac{(1-\epsilon_i)}{2}]^2$.

$$\therefore m(E_k) = 4^k [\prod_{i=1}^k \frac{(1-\epsilon_i)}{2}]^2.$$

Since $S = \bigcap E_k$, $E_k \supset E_{k+1}$, $m(E_k) < \infty$ for all $k \geq 1$, by Theorem 1.3, we have

$$m(S) = m(\bigcap E_k) = \lim_{k \rightarrow \infty} m(E_k) \dots \dots \dots (1)$$

$$\text{But } m(E_k) = \sum_{j=1}^{4^k} m(J_{j,k}) = \sum_{j=1}^{4^k} [\prod_{i=1}^k \frac{(1-\epsilon_i)}{2}]^2 = 4^k [\prod_{i=1}^k \frac{(1-\epsilon_i)}{2}]^2$$

$$\therefore m(E_k) = 4^k [\prod_{i=1}^k \frac{(1-\epsilon_i)}{2}]^2$$

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Putting in equation (1) we get,

$$\begin{aligned} m(S) &= \lim_{k \rightarrow \infty} 4^k \left[\prod_{i=1}^k \frac{(1-\epsilon_i)}{2} \right]^2 \\ &= \lim_{k \rightarrow \infty} 4^k \left[\prod_{i=1}^k \left(\frac{1-\epsilon_i}{2} \right)^2 \right] \\ &= \lim_{k \rightarrow \infty} 4^k \left[\prod_{i=1}^k \frac{(1-\epsilon_i)^2}{4} \right] \\ &= \lim_{k \rightarrow \infty} 4^k \frac{[\prod_{i=1}^k (1-\epsilon_i)^2]}{4^k} \\ &= \lim_{k \rightarrow \infty} [\prod_{i=1}^k (1-\epsilon_i)^2] \\ m(S) &= \lim_{k \rightarrow \infty} [\prod_{i=1}^k (1-\epsilon_i)]^2 \\ \therefore m(S) &= \left[\lim_{k \rightarrow \infty} \prod_{i=1}^k (1-\epsilon_i) \right]^2 \text{ when } \sum \epsilon_n < \infty. \end{aligned}$$

In particular when $\sum \epsilon_n < \infty$, by Lemma 2.5 we get,

$$\begin{aligned} \lim_{k \rightarrow \infty} \prod_{i=1}^k (1-\epsilon_i) \neq 0 &\Rightarrow \left[\prod_{i=1}^{\infty} (1-\epsilon_i) \right]^2 \neq 0. \\ \therefore m(S) &\neq 0. \end{aligned}$$

In particular when $\sum \epsilon_n = \infty$, by Lemma 2.5, we get,

$$\begin{aligned} \prod_{i=1}^{\infty} (1-\epsilon_i) = 0 &\Rightarrow \left[\prod_{i=1}^{\infty} (1-\epsilon_i) \right]^2 = 0. \\ \therefore m(S) &= 0. \end{aligned}$$

Particular Cases

Case I : For $\epsilon_n = \frac{1}{3}$ we get two dimensional Cantor ternary set. Also measure of this two dimensional Cantor ternary set is obtained by choosing $\epsilon_n = \frac{1}{3}$ which is zero.

Case II : For $\epsilon_n = \frac{1}{5}$ we get two dimensional Cantor - $\frac{2}{5}$ set. Also measure of this two dimensional Cantor - $\frac{2}{5}$ set is obtained by choosing $\epsilon_n = \frac{1}{5}$ which is zero.

Case III : For $\epsilon_n = \frac{1}{7}$ we get two dimensional Cantor - $\frac{3}{7}$ set. Also measure of this two dimensional Cantor - $\frac{3}{7}$ set is obtained by choosing $\epsilon_n = \frac{1}{7}$ which is zero.

Case IV : For $\epsilon_n = \frac{1}{2}$ we get two dimensional Cantor - $\frac{1}{4}$ set. Also measure of this two dimensional Cantor - $\frac{1}{4}$ set is obtained by choosing $\epsilon_n = \frac{1}{2}$ which is zero.

Now we determine the upper bound for Hausdorff dimension of two dimensional generalized Cantor like sets in following theorem.

Theorem 2.7

Hausdorff dimension of two dimensional generalized Cantor like set S constructed in 2.1 for $0 < \epsilon_n < 1$, with $\sum \epsilon_n = \infty$ satisfies $dim_H(S) \leq \frac{-2 \log 2}{\lim_{k \rightarrow \infty} \log \left[\prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{1/k}}$ i.e. $\frac{-2 \log 2}{\gamma}$ if $\lim_{k \rightarrow \infty} \log \left[\prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{1/k} = \gamma$ exists.

Proof

Let $\alpha = \frac{-2 \log 2}{\gamma}$, where $\gamma = \lim_{k \rightarrow \infty} \log \left[\prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{1/k}$.

We first show that if $\beta > \alpha$ then $H^\beta(S) = 0$.

Take $k \geq 1$ and let $J_{1,k}, J_{2,k}, J_{3,k}, \dots, J_{4^k,k}$ be the family of 4^k closed squares remaining at k^{th} stage in the construction of two dimensional generalized Cantor like set S whose union is E_k .

$$|J_{j,k}| = \text{Area of each square remaining at } k^{th} \text{ stage } J_{j,k} = \left[\prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^2$$

Since $E_k = \cup_{j=1}^{4^k} J_{j,k}$ and $S = \cap_{k=1}^{\infty} E_k$

$\therefore S \subset E_k$ for every k

$$\text{Now } |J_{j,k}| = \left[\prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^2, j = 1, 2, \dots, 4^k$$

$$\therefore |J_{j,k}|^\beta = \left\{ \left[\prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^2 \right\}^\beta, j = 1, 2, \dots, 4^k$$

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$$\begin{aligned} \therefore |J_{j,k}|^\beta &= \left[\prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{2\beta}, j = 1, 2, \dots, 4^k \\ \therefore \sum_{j=1}^{4^k} |J_{j,k}|^\beta &= \sum_{j=1}^{4^k} \left[\prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{2\beta} = 4^k \left[\prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{2\beta} \end{aligned} \quad \dots\dots\dots (1)$$

Since $\gamma = \lim_{k \rightarrow \infty} \log \left[\prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{1/k}$

$$\begin{aligned} \gamma &\leq \lim_{k \rightarrow \infty} \log \left[\left(\frac{1}{2} \right)^k \right]^{1/k} \quad (\because (1 - \epsilon_n) < 1 \forall n = 1, 2, 3, \dots) \\ \gamma &\leq \lim_{k \rightarrow \infty} \log \left(\frac{1}{2} \right) = \log \left(\frac{1}{2} \right) = -\log 2 \\ \therefore \gamma &\leq -\log 2. \text{ Thus } \gamma < 0 \\ &\implies 2\gamma < 0 \end{aligned} \quad \dots\dots\dots (2)$$

Now, since $\beta > \alpha = \frac{-2\log 2}{\gamma}$

$$\begin{aligned} \therefore \beta\gamma &< -2\log 2 \quad (\because \gamma < 0) \\ \implies 2\log 2 + \beta\gamma &< 0 \end{aligned}$$

$$\implies 2\log 2 + \beta \lim_{k \rightarrow \infty} \log \left[\prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{1/k} < 0 \quad \dots\dots\dots (3)$$

Now,

$$\begin{aligned} 4^k \left[\prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{2\beta} &= e^{\log \{ 4^k \left[\prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{2\beta} \}} \\ &= e^{\{ \log 4^k + \log \left[\prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{2\beta} \}} \\ &= e^{\{ k \log 4 + \beta \log \left[\prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^2 \}} \\ &= e^{\{ \log 4 + \frac{\beta}{k} \log \left[\prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^2 \}} \\ &= e^{\{ \log 4 + \beta \log \left[\prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{2/k} \}} \end{aligned}$$

Putting in equation (1) we get,

$$\sum_{j=1}^{4^k} |J_{j,k}|^\beta = e^{\{ \log 4 + \beta \log \left[\prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{2/k} \}} \quad \dots\dots\dots (4)$$

Since $\beta > \alpha = \frac{-2\log 2}{\lim_{k \rightarrow \infty} \log \left[\prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{1/k}} \geq \frac{-2\log 2}{\lim_{k \rightarrow \infty} \log \left[\prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{2/k}}$

$$\begin{aligned} \therefore \beta &> \frac{-2\log 2}{\lim_{k \rightarrow \infty} \log \left[\prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{2/k}} \\ \therefore \beta \lim_{k \rightarrow \infty} \log \left[\prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{2/k} &< -\log 4 \\ \therefore \log 4 + \beta \lim_{k \rightarrow \infty} \log \left[\prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{2/k} &< 0 \end{aligned} \quad \dots\dots\dots (5)$$

We show that $\lim_{k \rightarrow \infty} e^{\{ \log 4 + \beta \log \left[\prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{2/k} \}} = 0$.

$$\begin{aligned} \text{Let } \gamma_k &= 2\log \left[\prod_{n=1}^k \frac{(1-\epsilon_n)}{2} \right]^{1/k} \\ \therefore \log 4 + 2\beta\gamma &= \log 4 + \beta \lim_{k \rightarrow \infty} \gamma_k < 0 \quad (\text{By equation (5)}) \\ \text{Let } \log 4 + 2\beta\gamma &= -\delta_1 \text{ (Say), where } \delta_1 > 0 \\ \therefore \exists K_1 &\text{ such that } k \geq K_1 \\ \implies |(\log 4 + \beta\gamma_k) + \delta_1| &< \frac{\delta_1}{2} \\ \implies |(\log 4 + \beta\gamma_k) - (-\delta_1)| &< \frac{\delta_1}{2} \\ \implies -\delta_1 - \frac{\delta_1}{2} &< (\log 4 + \beta\gamma_k) < -\delta_1 + \frac{\delta_1}{2} \\ &= -\frac{3\delta_1}{2} < (\log 4 + \beta\gamma_k) < -\frac{\delta_1}{2} \\ \therefore \log 4 + \beta\gamma_k &< -\frac{\delta_1}{2} \\ \implies k(\log 4 + \beta\gamma_k) &< -\frac{k\delta_1}{2} \end{aligned}$$

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$$\therefore e^{k(\log 4 + \beta \gamma_k)} < e^{-\frac{k\delta_1}{2}}, k \geq 1.$$

$$\text{Since } \lim_{k \rightarrow \infty} e^{-\frac{k\delta_1}{2}} = 0, \lim_{k \rightarrow \infty} e^{k(\log 2 + \beta \gamma_k)} = 0.$$

This proves that $\lim_{k \rightarrow \infty} e^{k(\log 2 + \beta \log[\prod_{n=1}^k \frac{(1-\epsilon_n)}{2}])^{2/k}} = 0$ (6)

We now prove that for every $\delta > 0$, $H^\beta_\delta(S) = 0$.

Suppose $\delta > 0$ is given. Since $\prod_{n=1}^\infty \frac{(1-\epsilon_n)}{2} = 0, \exists k^1 \geq 1, \prod_{n=1}^{k^1} \frac{(1-\epsilon_n)}{2} < \frac{\delta}{\sqrt{2}}$.

$$\therefore |J_{j,k^1}| = \sqrt{2} \prod_{n=1}^{k^1} \frac{(1-\epsilon_n)}{2} < \delta \text{ for each } j = 1, 2, \dots, 4^{k^1}.$$

Thus $\{J_{j,k^1}, j = 1, 2, \dots, 4^{k^1}\}$ is a δ -cover of E_k and hence covers S .

By definition 1.1 we get,

$$H^\beta_\delta(S) = \inf \{ \sum_{i=1}^\infty |U_i|^\beta : \{U_i\} \text{ is a } \delta\text{-cover of } S \} \leq \sum_{j=1}^{4^{k^1}} |J_{j,k^1}|^\beta$$

$$\therefore H^\beta_\delta(S) = e^{k^1(\log 4 + \beta \log[\prod_{n=1}^{k^1} \frac{(1-\epsilon_n)}{2}])^{2/k^1}}.$$

As $\delta \rightarrow 0, k^1 \rightarrow \infty$ and R.H.S. $\rightarrow 0$.

$$\therefore H^\beta_\delta(S) = 0. \quad (\text{By equation (6)})$$

Since this is true for every $\delta > 0, H^\beta(S) = 0$.

$$\therefore H^\beta(S) = \lim_{\delta \rightarrow 0} H^\beta_\delta(S) = 0$$

$$\therefore H^\beta(S) = 0$$

Thus $(\alpha, \infty) \subset \{ \beta > 0 : H^\beta(S) = 0 \}$

$$\therefore \inf \{ \beta > 0 : H^\beta(S) = 0 \} \leq \alpha$$

$$\therefore \dim_H(S) \leq \alpha.$$

Now we determine mass distribution on two dimensional generalized Cantor like set by using following mass distribution.

Definition 2.8 Mass distribution [3].

Let A be bounded subset of \mathbb{R}^n . Let μ be a measure on \mathbb{R}^n . The support of μ is the smallest closed set A such that $\mu(\mathbb{R}^n - A) = 0$. Then we call a measure μ on \mathbb{R}^n a mass distribution on A if $0 < \mu(\mathbb{R}^n) < \infty$ and μ has bounded support A .

2.9 Determination of mass distribution on two dimensional generalized Cantor like set S

Let S be two dimensional generalized Cantor like set constructed in 2.1.

Let μ be defined on E_k , where $E_k, k = 0, 1, 2, \dots$ are remaining squares at each stage in the construction of two dimensional generalized Cantor like set S

i.e. $E_k = \bigcup_{j=1}^{4^k} J_{j,k}, k \geq 1$ and $\mu(J_{j,k}) = \frac{1}{4^k}, j = 1, 2, \dots, 4^k$. Thus $\mu(E_k) = 1$ for all $k \geq 1$.

For each $k \geq 1$, let $\mathcal{E}_k = \{ J_{j,k} : j = 1, 2, 3, \dots, 4^k \}$.

Let $\mathcal{E} = \{ A \subseteq \mathbb{R}^2 : A = \text{Finite union of members of } \bigcup_{k=0}^\infty \mathcal{E}_k \text{ and their complements} \}$. Then \mathcal{E} is an algebra containing all \mathcal{E}_k . Extend μ on \mathcal{E} by using additivity i.e. $A = \bigcup_{i=1}^n A_i$, where $A_i \in \bigcup_{k=0}^\infty \mathcal{E}_k, A_i \cap A_j = \emptyset, i \neq j$ then $\mu(A) = \sum_{i=1}^n \mu(A_i)$ and $\mu([0,1] \times [0,1] - A) = 1 - \mu(A)$. This defines a measure μ on \mathcal{E} .

Define μ^* on $\mathcal{H}(\mathcal{E})$ by

$$\mu^*(E) = \inf \{ \sum \mu(E_n) : E_n \in \mathcal{E}, n = 1, 2, 3, \dots, E \subset \bigcup_{n=1}^\infty E_n \}, E \in \mathcal{H}(\mathcal{E}).$$

Here $\mathcal{H}(\mathcal{E})$ is hereditary σ -ring containing \mathcal{E} .

We show that $\mu^*(E) = \mu(E)$ if $E \in \mathcal{E}$

$$\text{We know that } \mu^*(E) \leq \mu(E) \quad \dots \dots \dots (1)$$

We show that $\mu^*(E) \geq \mu(E)$

Let $E \in \mathcal{E}$ and $E \subset \bigcup_{n=1}^\infty E_n$, where $E_n \in \mathcal{E}$. Then

$$E = \bigcup_{n=1}^\infty (E_n \cap E) \quad \dots \dots \dots (2)$$

Choose $F_n \in \mathcal{E}$ such that $F_n \subset E_n \cap E$, each F_n are pair wise disjoint and

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$$\begin{aligned} \bigcup_{n=1}^k F_n &= \bigcup_{n=1}^k (E_n \cap E) \text{. Then } \mu(F_n) \leq \mu(E_n \cap E) \\ \therefore \mu(E) &= \sum_{n=1}^{\infty} \mu(F_n) \leq \sum_{n=1}^{\infty} \mu(E_n) \quad \text{(From equation (2))} \\ \therefore \mu(E) &\leq \inf \{ \sum_{n=1}^{\infty} \mu(E_n) : E_n \in \mathcal{E}, n = 1, 2, 3, \dots, E \subset \bigcup_{n=1}^{\infty} E_n \} \\ \therefore \mu(E) &\leq \mu^*(E) \quad \dots\dots\dots (3) \end{aligned}$$

From equations (1) and (3), we get

$$\mu^*(E) = \mu(E) \text{ if } E \in \mathcal{E}$$

Now, we call $E \in \mathcal{E}^*$ iff $\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c) \forall A \subseteq \mathbb{R}^2$
 $\therefore \mathcal{E}^*$ is a σ -algebra containing \mathcal{E} and μ^* is restricted to \mathcal{E}^* , is a measure on \mathcal{E}^* .

Since every open interval $\in \mathcal{E}^*$
 \therefore Every Borel set $\in \mathcal{E}^*$

We know that

$$\text{Support of } \mu = \cap \{ F \text{ is closed} : \mu([0,1] \times [0,1] - F) = 0 \}$$

We show that Support of $\mu \subset \bigcap_{k=1}^{\infty} E_k$

We know that $E_k, k = 0, 1, 2, \dots$ are remaining squares at each stage in the construction of two dimensional generalized Cantor like set S i.e. $E_k = \bigcup_{j=1}^{4^k} J_{j,k}, k \geq 1$ and $\mu(J_{j,k}) = \frac{1}{4^k}, j = 1, 2, 3, \dots, 4^k$.

By definition $\mu([0, 1] \times [0, 1] - E_k) = 1 - \mu(E_k)$

$$\begin{aligned} &= 1 - \mu(\bigcup_{j=1}^{4^k} J_{j,k}) \\ &= 1 - \sum_{j=1}^{4^k} \frac{1}{4^k} = 1 - \frac{1}{4^k} \times 4^k = 1 - 1 = 0 \\ \therefore \mu([0, 1] \times [0, 1] - E_k) &= 0 \end{aligned}$$

\therefore Support of $\mu \subset E_k$ for every $k \geq 1$, since each E_k is closed.
 Thus Support of $\mu \subset S$.

Now, we show that $0 < \mu(S) < \infty$.

We know that $E_1 \supset E_2 \supset \dots, \mu(E_k) = 1 < \infty$.

$$\therefore \mu(S) = \mu(\bigcap_{k=1}^{\infty} E_k) = \lim_{k \rightarrow \infty} \mu(E_k) = 1 > 0.$$

$$\therefore 0 \leq \mu(S) < \infty.$$

$\therefore \mu$ is a mass distribution on S.

Now we determine the lower bound for Hausdorff dimension of two dimensional generalized Cantor like set when $\epsilon_n = \epsilon$, a constant, $0 < \epsilon < 1$ by using following mass distribution principle.

Mass distribution principle 2.10

Let μ be a mass distribution on F and suppose that for some s there are numbers $c > 0$ and $\delta > 0$ such that $\mu(U) \leq c |U|^s$ for all sets U with $|U| \leq \delta$. Then $H^s(F) \geq \frac{\mu(F)}{c}$ and $s \leq \dim_H(F) \leq \overline{\dim}_B(F) \leq \overline{\dim}_B(F)$.

Above principle is on page 55 of Chapter 4 from Kenneth Falconer [2].

Theorem 2.11

If S is two dimensional generalized Cantor like set constructed in 2.1, then for special case $\epsilon_n = \epsilon$, a constant, $0 < \epsilon < 1, \dim_H(S) \geq \frac{-\log 4}{\log(\frac{1-\epsilon}{2})}$.

Proof

Let S be two dimensional generalized Cantor like set constructed in 2.1.

Let $\epsilon_n = \epsilon$, a constant, where $0 < \epsilon < 1$.

$$\text{Put } s = \frac{-\log 4}{\log(\frac{1-\epsilon}{2})}$$

$$\log(\frac{1-\epsilon}{2}) = -\log 4$$

$$[(\frac{1-\epsilon}{2})]^s = 4^{-1}$$

$$\therefore (\frac{1-\epsilon}{2})^{(k+1)s} = 4^{-(k+1)} \quad \dots\dots\dots(1)$$

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At k^{th} stage the rectangle between two consecutive squares from E_k parallel to X axis or Y axis have size $(\frac{1-\epsilon}{2})^{k-1} \epsilon \times (\frac{1-\epsilon}{2})^k$. Thus the diameter of such rectangle is

$$(\frac{1-\epsilon}{2})^k \sqrt{(\frac{1-\epsilon}{2})^{-2} \epsilon^2 + 1}.$$

Now if U is a rectangle of diameter $|U|$ such that

$$(\frac{1-\epsilon}{2})^{k+1} \sqrt{(\frac{1-\epsilon}{2})^{-2} \epsilon^2 + 1} \leq |U| < (\frac{1-\epsilon}{2})^k \sqrt{(\frac{1-\epsilon}{2})^{-2} \epsilon^2 + 1} \dots\dots\dots (2)$$

Then U intersects at most one of the basic squares in E_k because if U intersects more than one of the basic squares in E_k then $|U| \geq (\frac{1-\epsilon}{2})^k \sqrt{(\frac{1-\epsilon}{2})^{-2} \epsilon^2 + 1}$ which is false.

$$\therefore \mu(U) \leq 4^{-k} \qquad \qquad \qquad \therefore \mu(U) \leq \mu(\text{One square in } E_k) = 4^{-k} \qquad \qquad \qquad (3)$$

From equation (2) we get,

$$|U| \geq (\frac{1-\epsilon}{2})^{k+1} \sqrt{(\frac{1-\epsilon}{2})^{-2} \epsilon^2 + 1} \geq (\frac{1-\epsilon}{2})^{k+1}$$

$$\therefore |U| \geq (\frac{1-\epsilon}{2})^{k+1} = \{4^{-(k+1)}\}^{1/s} \quad (\text{From equation (1)})$$

$$\therefore |U| \geq \{4^{-(k+1)}\}^{1/s}$$

$$\therefore |U|^s \geq 4^{-(k+1)}$$

$$\therefore 4|U|^s \geq 4^{-k} \geq \mu(U) \quad (\text{From equation (3)})$$

$$\therefore \mu(U) \leq 4|U|^s$$

\therefore Mass distribution principle 2.10 can be applied to the set S where $c = 4$, $s = \frac{-\log 4}{\log(\frac{1-\epsilon}{2})}$. We conclude that $H^s(S) \geq \frac{\mu(S)}{c} = \frac{1}{c}$.

$$\therefore H^s(S) \geq \frac{1}{4}.$$

$$\therefore s \in \{\gamma \geq 0 : H^\gamma(S) > 0\}.$$

$$\sup\{\gamma \geq 0 : H^\gamma(S) > 0\} \geq s.$$

$$\therefore \dim_H(S) \geq s.$$

$$\therefore \dim_H(S) \geq \frac{-\log 4}{\log(\frac{1-\epsilon}{2})}.$$

If $\epsilon_n = \epsilon$, a constant, $0 < \epsilon < 1$, Theorem 2.7, Theorem 2.11 give Hausdorff dimension of two dimensional generalized Cantor like set S is $\frac{-\log 4}{\log(\frac{1-\epsilon}{2})}$.

Now we take particular cases corresponding to different values of $\epsilon_n = \epsilon$, a constant, $0 < \epsilon < 1$, Theorem 2.7, Theorem 2.11 give Hausdorff dimension of corresponding two dimensional Cantor like set.

Particular Case 2.12

If $\epsilon_n = \frac{1}{5}$ in Theorem 2.7 then Hausdorff dimension of two dimensional

Cantor - $\frac{2}{5}$ set is $\leq \frac{-2 \log 2}{\lim_{k \rightarrow \infty} \log[\prod_{n=1}^k (\frac{1-\frac{1}{5}}{2}]^{1/k}} = \frac{-2 \log 2}{\lim_{k \rightarrow \infty} \log[\prod_{n=1}^k (\frac{4}{5})]^{1/k}} = \frac{\log 4}{\log(\frac{5}{2})}$ and in Theorem 2.11 Hausdorff dimension

of two dimensional Cantor - $\frac{2}{5}$ set is $\geq \frac{-\log 4}{\log(\frac{1-\frac{1}{5}}{2})} = \frac{-\log 4}{\log(\frac{4}{5})} = \frac{\log 4}{\log(\frac{5}{2})}$.

\therefore Hausdorff dimension of two dimensional Cantor - $\frac{2}{5}$ set is $\frac{\log 4}{\log(\frac{5}{2})}$ i.e. $\dim_H(S) = \frac{\log 4}{\log(\frac{5}{2})}$.

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Particular Case 2.13

If $\epsilon_n = \frac{1}{7}$ in Theorem 2.7 then Hausdorff dimension of two dimensional Cantor - $\frac{3}{7}$ set is $\leq \frac{-2 \log 2}{\lim_{k \rightarrow \infty} \log [\prod_{n=1}^k (\frac{1-\frac{1}{7}}{2}]^{1/k}} = \frac{-2 \log 2}{\lim_{k \rightarrow \infty} \log [\prod_{n=1}^k (\frac{6}{7})]^{1/k}} = \frac{\log 4}{\log(\frac{7}{3})}$ and in Theorem 2.11 Hausdorff dimension of two dimensional Cantor - $\frac{3}{7}$ set is $\geq \frac{-\log 4}{\log(\frac{1-\frac{1}{7}}{2})} = \frac{-\log 4}{\log(\frac{6}{7})} = \frac{\log 4}{\log(\frac{7}{3})}$.

\therefore Hausdorff dimension of two dimensional Cantor - $\frac{3}{7}$ set is $\frac{\log 4}{\log(\frac{7}{3})}$ i.e. $dim_H (S) = \frac{\log 4}{\log(\frac{7}{3})}$.

Particular Case 2.15

If $\epsilon_n = \frac{1}{2}$ in Theorem 2.7 then Hausdorff dimension of two dimensional Cantor - $\frac{1}{4}$ set is $\leq \frac{-2 \log 2}{\lim_{k \rightarrow \infty} \log [\prod_{n=1}^k (\frac{1-\frac{1}{2}}{2}]^{1/k}} = \frac{-2 \log 2}{\lim_{k \rightarrow \infty} \log [\prod_{n=1}^k (\frac{1}{2})]^{1/k}} = 1$ and in Theorem 2.11 Hausdorff dimension of two dimensional Cantor - $\frac{1}{4}$ set is $\geq \frac{-\log 4}{\log(\frac{1-\frac{1}{2}}{2})} = \frac{-\log 4}{\log(\frac{1}{2})} = 1$.

\therefore Hausdorff dimension of two dimensional Cantor - $\frac{1}{4}$ set is 1 i.e. $dim_H (S) = 1$.

III. CONCLUSION

Using two dimensional generalized Cantor like sets S we constructed two dimensional different Cantor like sets by varying or fixing ϵ_n . Further we determined the estimate for the Hausdorff dimension of such two dimensional Cantor like sets. In particular it is proved that the Hausdorff dimension of the set S defined above is $\leq \frac{-2 \log 2}{\gamma}$, where $\gamma = \lim_{k \rightarrow \infty} \log [\prod_{n=1}^k (\frac{1-\epsilon_n}{2})]^{1/k}$ and for the particular case $\epsilon_n = \epsilon$, a constant, $0 < \epsilon < 1$, the equality holds namely $dim_H (S) = \frac{-\log 4}{\log(\frac{1-\epsilon}{2})}$.

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