

On Generalized of Laplace-Weierstrass Transform

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ABSTRACT: The present paper generalizes Laplace-Weierstrass transform to the space of generalized function by defining testing function space $LW_{a,b}$. Laplace-Weierstrass transform is defined by taking $e^{-st - \frac{(x-y)^2}{4}}$ as a kernel, as it happens to the most natural pair of transformations. We have proved boundedness theorem called characterization theorem for Laplace-Weierstrass transform. Also representation theorem for Laplace-Weierstrass transform is given which states that every abstract structure with certain properties is isomorphic to a concrete structure

KEYWORDS: Laplace transform, Weierstrass transform, Laplace-Weierstrass transform, Integral transform, Testing function space

I. INTRODUCTION

Integral transform is that it transforms difficult mathematical problems to relatively easy problems. Extensions of some transformations to generalized functions have been done from time to time and their properties have been studied by various Mathematicians. Zemanian [8, 9] extended Laplace and Weierstrass transformations to generalized functions. Mathurkar *et.al* [2,3] discussed the analyticity of Laplace Weierstrass transform with elementary properties. Pathak [4] developed representation theorem for a class of stieltjes transformable generalized function. Pollard [6] studied representation as a Gaussian integral. The representation theorem states that every abstract structure with certain properties is isomorphic to a concrete structure. Zayed [7] also explained various transform to generalized function. There is much scope in extending double transformation to a certain class of generalized functions. Gudadhe and Gulhane [1] created distributional Laplace-Stieltjes transform.

In the present paper we established the boundedness theorem as well as representation theorem. In section [II], we have defined testing function space. We have given the lemma in section [III]. Section [IV] is devoted the boundedness theorem. Representation theorem is given in section [V]. Lastly conclusions are given in section [VI]. Notation and terminology as per Zemanian.

In this work, define the Laplace-Weierstrass transform $F(s, x)$ of a generalized function f directly as the application

of $f(t, y)$ to $e^{-st - \frac{(x-y)^2}{4}}$

$$\text{i.e. } F(s, x) = \frac{1}{\sqrt{4\pi}} \left\langle f(t, y), e^{-st - \frac{(x-y)^2}{4}} \right\rangle$$

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For this purpose we construct a testing function space $LW_{a,b}$ which contains the kernel $e^{-st - \frac{(x-y)^2}{4}}$ for all $f(t, y)$ in some restricted domain.

II. TESTING FUNCTION SPACE

The Testing Function Space $LW_{a,b}$

$LW_{a,b}$ as the linear space of all complex valued smooth functions $\phi(t, y)$ on $0 < t < \infty$, $0 < y < \infty$ such that for each $p, q = 0, 1, 2, \dots$

$$\gamma_{a,b,p,q} \phi(t, y) = \sup_{\substack{0 < t < \infty \\ 0 < y < \infty}} \left| e^{at - \frac{by + y^2}{4}} D_t^p D_y^q \phi(t, y) \right| < \infty, \quad (2.1)$$

for some fixed numbers a, b in \mathbb{R}

The space $LW_{a,b}$ is complete and a Frechet space. This topology is generated by the total families of countably multinorms space given by (2.1).

III. LEMMA

For sufficient condition of boundedness theorem require following lemma

If, on the half plane $\{s : a \leq \text{Re } s\}$ and $\{x : b \leq \text{Re } x\}$, $G(s, x)$ is analytic and satisfies $|G(s, x)| \leq K_1 K_2 |(s, x)|^{-2}$,

where K_1, K_2 are constants and if $g(t, y) = \frac{-1}{4\pi\sqrt{\pi}} \int_{\sigma-i\infty}^{\sigma+i\infty} \int_{\sigma'-i\infty}^{\sigma'+i\infty} G(s, x) e^{st + \frac{(x-y)^2}{4}} dx ds$, $a < \sigma$ & $b < \sigma'$

$$(3.1)$$

then $g(t, y)$ is a continuous function that does not depend on the choice of σ & σ' and generates a regular generalized function in $LW_{a,b}^*$. Moreover $LW\{g(t, y)\} = G(s, x)$ for $a < \text{Re } s$ and $b < \text{Re } x$.

Proof: The fact that $g(t, y)$ does not depend on σ & σ' follows by Cauchy's theorem and the bound on $G(s, x)$. Now from equation (3.1)

$$e^{-\sigma t - \frac{(\sigma' - y)^2}{4}} g(t, y) = \frac{-1}{4\pi\sqrt{\pi}} \int_{-\infty - i\infty}^{\infty - i\infty} \int_{-\infty - i\infty}^{\infty - i\infty} G(\sigma + i\omega, \sigma' + i\omega') e^{i\omega t + \frac{i(\omega' - y)^2}{4}} d\omega d\omega', \quad a < \sigma \text{ \& } b < \sigma'$$

$$(3.2)$$

In view of the bound on $G(s, x)$ it follows that the last integral converges uniformly for all $t, y \geq 0$, which implies the continuity of $g(t, y)$. In view of equation (3.2) and the bound on $G(s, x)$ it can easily be seen that $g(t, y)$ is a regular generalized function in $LW_{a,b}^*$. Finally from inversion theorem it also follows that $G(s, x) = LW\{g(t, y)\}$ for $a < \text{Re } s$ and $b < \text{Re } x$.

This completes the proof of the lemma.

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IV. CHARACTERISTIC THEOREM FOR LAPLACE-WEIERSTRASS TRANSFORM

Theorem:

If $F(s, x) = LW\{f(t, y)\}$ for, $(s, x) \in \Omega_f = \{(s, x) / \sigma_1 < \text{Re } s < \sigma_2, \sigma'_1 < \text{Re } x < \sigma'_2\}$, $F(s, x)$ is bounded on any subset $\Omega' = \{(s, x) / a < \text{Re } s \text{ \& } b < \text{Re } x\}$ and $\sigma < a < b < \sigma'$ of Ω according to $|F(s, x)| \leq K P(|s, x|)$ for some polynomial P depending on a & b and K is constant.

Proof: Necessary condition

We know that $F(s, x)$ is analytic on Ω_f by analyticity theorem [2, 3]. By the boundedness property of generalized function, there exist a constant K and non negative integers r and r' .

$$\begin{aligned}
 |F(s, x)| &= \left| \frac{1}{\sqrt{4\pi}} \left\langle f(t, y), e^{-st - \frac{(x-y)^2}{4}} \right\rangle \right| \\
 &\leq \frac{K'}{\sqrt{4\pi}} \max_{\substack{0 \leq p \leq r, \\ 0 \leq q \leq r'}} \gamma_{a,b,p,q} \left[e^{-st - \frac{(x-y)^2}{4}} \right] \\
 &\leq K \max_{\substack{0 \leq p \leq r, \\ 0 \leq q \leq r'}} \text{Sup}_{\substack{0 < t < \infty \\ 0 < y < \infty}} \left[e^{at - \frac{by + y^2}{2} - \frac{y^2}{4}} D_t^p D_y^q e^{-st - \frac{(x-y)^2}{4}} \right] \\
 &\leq K \max_{\substack{0 \leq p \leq r, \\ 0 \leq q \leq r'}} |s|^p |p_q(x-y)| \text{Sup}_{\substack{0 < t < \infty \\ 0 < y < \infty}} \left[e^{-(s-a)t} e^{-\frac{by + y^2}{2} - \frac{(x-y)^2}{4}} \right]
 \end{aligned}$$

$$|F(s, x)| \leq K P(|s, x|)$$

where $P(|s, x|)$ depends on a and b . This proves necessary part.

Sufficient condition

Now to complete the proof of the theorem, we assume that $\{s : a \leq \text{Re } s\} \subset \Omega_f$ and $\{x : b \leq \text{Re } x\} \subset \Omega_f$. Next assume that $Q(s, x)$ is a polynomial which has no zeros in the half plane $a \leq \text{Re } s$ and $b \leq \text{Re } x$ and which satisfies

$$\left| \frac{F(s, x)e^{x^2}}{Q(s, x)} \right| \leq \frac{K_1 K_2}{|(s, x)|^2}, \quad a < \text{Re } s \text{ and } b < \text{Re } x$$

where K_1, K_2 are constants.

Set $G(s, x) = \frac{F(s, x)e^{x^2}}{Q(s, x)}$. Then, $g(t, y) = \frac{-1}{4\pi\sqrt{\pi}} \int_{\sigma-i\infty}^{\sigma+i\infty} \int_{\sigma'-i\infty}^{\sigma'+i\infty} G(s, x) e^{st + \frac{(x-y)^2}{4}} dx ds$,

$$a < \sigma \text{ and } b < \sigma'$$

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Satisfies conditions of above lemma and is a member of $LW_{a,b}^*$. Now let $f(t, y) = Q(D)g(t, y)$, where D represent the generalized differentiation in $LW_{a,b}^*$. Then $f(t, y)$ is also a member of $LW_{a,b}^*$ and $LW\{f(t, y)\} = Q(s, x)G(s, x) = F(s, x)$ for $a < \text{Re } s$ and $b < \text{Re } x$.

V. REPRESENTATION THEOREM FOR LAPLACE-WEIERSTRASS TRANSFORM

Theorem:

Let $f(t, y)$ be an arbitrary element of $LW_{a,b}^*$ and $\phi(t, y)$ be an element of $D(\Omega)$, the space of infinitely differentiable functions with compact support on Ω . Then there exist bounded measurable functions $h_{m,n}(t, y)$ defined over Ω such that,

$$\langle f, \phi \rangle = \left\langle \sum_{m=0}^{r+1} \sum_{n=0}^{v+1} (-1)^{m+n} e^{at - \frac{by}{2} + \frac{y^2}{4}} D_t^m D_y^n h_{m,n}(t, y), \phi(t, y) \right\rangle,$$

where r and v are appropriate non-negative integers satisfying $m \leq r + 1$ and $n \leq v + 1$

Proof: Let $\{\gamma_{a,b,p,q}\}_{p,q=0}^{\infty}$ be the sequence of seminorms.

Let $f(t, y)$ be an arbitrary element of $LW_{a,b}^*$ and $\phi(t, y)$ be an element of $D(\Omega)$. Then by the boundedness property of generalized functions we have for an appropriate constant K and a non negative integers r and v satisfying $|p| \leq r$ and $|q| \leq v$

$$\begin{aligned} |\langle f, \phi \rangle| &\leq K \max_{\substack{|p| \leq r \\ |q| \leq v}} \gamma_{a,b,p,q} \phi(t, y) \\ &\leq K \max_{\substack{|p| \leq r \\ |q| \leq v}} \text{Sup}_{\substack{0 < t < \infty \\ 0 < y < \infty}} e^{at - \frac{by}{2} + \frac{y^2}{4}} |D_t^p D_y^q \phi(t, y)| \\ &\leq K \max_{\substack{|p| \leq r \\ |q| \leq v}} \text{Sup}_{\substack{0 < t < \infty \\ 0 < y < \infty}} e^{at - \frac{by}{2} + \frac{y^2}{4}} \max_{\substack{m \leq p \\ n \leq q}} D_t^m D_y^n \phi(t, y) \end{aligned}$$

where K is a constant which depends only on m, n and hence p, q

$$|\langle f, \phi \rangle| \leq K \max_{\substack{m \leq r \\ n \leq v}} \text{Sup}_{\substack{0 < t < \infty \\ 0 < y < \infty}} e^{at - \frac{by}{2} + \frac{y^2}{4}} D_t^m D_y^n \phi(t, y) \tag{5.1}$$

Now let us set,

$$\phi_{r,v}(t, y) = e^{at - \frac{by}{2} + \frac{y^2}{4}} \phi(t, y), \quad m \leq r, n \leq v \tag{5.2}$$

Then clearly,

$$\phi_{r,v}(t, y) \in D(\Omega)$$

Also,

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$$\phi(t, y) = e^{-at + \frac{by}{2} - \frac{y^2}{4}} \phi_{r,v}(t, y)$$

On differentiating above equation with respect to t and y successively we get,

$$D_t D_y \phi(t, y) = \frac{\partial}{\partial y} \left[\frac{\partial}{\partial t} e^{-at + \frac{by}{2} - \frac{y^2}{4}} \phi_{r,v}(t, y) \right]$$

$$= e^{-at + \frac{by}{2} - \frac{y^2}{4}} \left\{ \begin{aligned} & \left[\frac{\partial^2 \phi_{r,v}(t, y)}{\partial y \partial t} + \left(\frac{b-y}{2} \right) \frac{\partial \phi_{r,v}(t, y)}{\partial t} \right] \\ & \left[-a \frac{\partial \phi_{r,v}(t, y)}{\partial y} - a \phi_{r,v}(t, y) \left(\frac{b-y}{2} \right) \right] \end{aligned} \right\}$$

Let us suppose that in Ω ,

$$Sup \phi = Sup \phi_{r,v} = \Omega_2 = \{(t, y) / C' \leq t \leq D', A' \leq y \leq B'\}$$

Then since,

$$e^{-at + \frac{by}{2} - \frac{y^2}{4}} > 0$$

$$|D_t D_y \phi| \leq e^{-at + \frac{by}{2} - \frac{y^2}{4}} \left\{ \begin{aligned} & \left| -a \phi_{r,v}(t, y) \right| \left| \frac{b-y}{2} \right| + \left| \frac{b-y}{2} \right| \left| \frac{\partial \phi_{r,v}(t, y)}{\partial t} \right| + \left| -a \right| \left| \frac{\partial \phi_{r,v}(t, y)}{\partial y} \right| + \left| \frac{\partial^2 \phi_{r,v}(t, y)}{\partial y \partial t} \right| \end{aligned} \right\}$$

$$\leq e^{-at + \frac{by}{2} - \frac{y^2}{4}} \left\{ \begin{aligned} & \left| a \right| \left| \phi_{r,v}(t, y) \right| \left| \frac{b-A'}{2} \right| + \left| \frac{b-A'}{2} \right| \left| \frac{\partial \phi_{r,v}(t, y)}{\partial t} \right| + \left| a \right| \left| \frac{\partial \phi_{r,v}(t, y)}{\partial y} \right| + \left| \frac{\partial^2 \phi_{r,v}(t, y)}{\partial y \partial t} \right| \end{aligned} \right\}$$

$$\leq K''' e^{-at + \frac{by}{2} - \frac{y^2}{4}} \left\{ \begin{aligned} & \left| \phi_{r,v}(t, y) \right| + \left| \frac{\partial \phi_{r,v}(t, y)}{\partial t} \right| + \left| \frac{\partial \phi_{r,v}(t, y)}{\partial y} \right| + \left| \frac{\partial^2 \phi_{r,v}(t, y)}{\partial y \partial t} \right| \end{aligned} \right\}$$

where $K''' = \max \left\{ \left| a \right| \left| \frac{b-A'}{2} \right|, \left| \frac{b-A'}{2} \right|, \left| a \right|, 1 \right\}$

Hence by induction we can prove that in Ω for obvious constant K^{iv} , which depends on a and b then,

$$|D_t^m D_y^n \phi(t, y)| \leq K^{iv} e^{-at + \frac{by}{2} - \frac{y^2}{4}} \sum_{\substack{c \leq m \\ d \leq n}} |D_t^c D_y^d \phi_{r,v}(t, y)| \tag{5.3}$$

Using equation (5.3) in equation (5.1), we get

$$\langle f, \phi \rangle \leq K'' \max_{\substack{m \leq r \\ n \leq v}} Sup_{\substack{0 < t < \infty \\ 0 < y < \infty}} e^{at - \frac{by}{2} + \frac{y^2}{4}} K^{iv} e^{-at + \frac{by}{2} - \frac{y^2}{4}} \sum_{\substack{c \leq m \\ d \leq n}} |D_t^c D_y^d \phi_{r,v}(t, y)|$$

$$\leq K^v \max_{\substack{m \leq r \\ n \leq v}} Sup_{\substack{0 < t < \infty \\ 0 < y < \infty}} |D_t^c D_y^d \phi_{r,v}(t, y)| \tag{5.4}$$

where $c \leq m$ and $d \leq n$ are suitable constants.

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Now we can write,

$$\begin{aligned} \sup_{\substack{0 < t < \infty \\ 0 < y < \infty}} |\phi(t, y)| &\leq \sup_{\substack{0 < t < \infty \\ 0 < y < \infty}} \left| \int_t^\infty \int_y^\infty D_t D_y \phi(t, y) dt dy \right| \\ &\leq \|D_t D_y \phi(t, y)\|_{L \times L} \end{aligned} \tag{5.5}$$

Hence from equation (5.1), equation (5.4) becomes,

$$\langle f, \phi \rangle \leq K^{vi} \max_{\substack{m \leq r+1 \\ n \leq v+1}} \sup_{\substack{0 < t < \infty \\ 0 < y < \infty}} \|D_t^m D_y^n \phi_{r,v}(t, y)\|_{L \times L} \tag{5.6}$$

Let the product space $L \times L$ be denoted by $(L')^2$. We consider the linear one to one mapping $\mathfrak{S} : \phi \rightarrow \{D_t^m D_y^n \phi(t, y)\}_{\substack{m \leq r+1 \\ n \leq v+1}}$ of $D(\Omega)$ into $(L')^2$. In view of equation (5.6) we see that the linear function

$\mathfrak{S} : \phi_{r,v} \rightarrow \langle f, \phi \rangle$ is continuous on $\mathfrak{S}D(\Omega)$ for the topology induced by $(L')^2$. Hence by Hahn Banach theorem, it can be a continuous linear functional in the whole of $(L')^2$. But the dual of $(L')^2$ is isomorphic with L^∞ , therefore there exists L^∞ -function $h_{m,n} (m \leq r+1, n \leq v+1)$ such that,

$$\langle f, \phi \rangle \leq \sum_{\substack{m \leq r+1 \\ n \leq v+1}} \langle h_{m,n}(t, y), D_t^m D_y^n \phi_{r,v}(t, y) \rangle$$

From equation (5.2) we have,

$$\langle f, \phi \rangle \leq \sum_{\substack{m \leq r+1 \\ n \leq v+1}} \left\langle h_{m,n}(t, y), D_t^m D_y^n e^{at - \frac{by}{2} + \frac{y^2}{4}} \phi(t, y) \right\rangle$$

Using property of differentiation of a distribution by an infinitely smooth function, we get

$$\langle f, \phi \rangle \leq \sum_{\substack{m \leq r+1 \\ n \leq v+1}} \left\langle (-1)^{m+n} D_t^m D_y^n h_{m,n}(t, y) e^{at - \frac{by}{2} + \frac{y^2}{4}}, \phi(t, y) \right\rangle$$

Now by using property of multiplication of a distribution by an infinitely smooth function, we get

$$\langle f, \phi \rangle \leq \sum_{\substack{m \leq r+1 \\ n \leq v+1}} \left\langle (-1)^{m+n} e^{at - \frac{by}{2} + \frac{y^2}{4}} D_t^m D_y^n h_{m,n}(t, y), \phi(t, y) \right\rangle$$

V. CONCLUSION

In the present paper the boundedness theorem and representation theorem for the Laplace-Weierstrass are proved

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REFERENCES

- [1] Gulhane P. A., Gudadhe A. S., "Representation theorem for the distributional Laplace-Stieltjes transform", Science journal of GVISH, Vol. II, pp 29-32, 2005.
- [2] Mathurkar S. S., Gulhane P. A., "Elementary properties of Laplace-Weierstrass transform with analytic behaviour", Proceeding of National Conference on Recent Application on Mathematical Tool in Science and Technology (RAMT-2014), May8-9, 2014.
- [3] Mathurkar S. S., Dagwal V. J., Gulhane P. A., "Analytic behavior of Laplace Weierstrass transform", International Journal of Mathematical Archive-5(10), pp 243-246, ISSN 2229- 5046, Oct. 2014.
- [4] Pathak R .S., "A representation theorem for a class of Stieltjes transformable generalized Function", 1974.
- [5] Pathak R. S., "Integral transformation of generalized functions and their applications", Hordon and Breach Science Publishers, Netherland.
- [6] Pollard H., "Representation as a Gaussian integral", Duke Math. J. Vol. 10, pp. 59-65, 1943.
- [7] Zayed A. I., "Handbook of function and generalized function transformations", Mathematical Sciences Reference Series, CRC Press, Boca Raton, FL, 1996.
- [8] Zemanian A. H., "A generalized Weierstrass transformation", SIAM J. Appl. Math.15, 1088- 1105, 1967.
- [9] Zemanian A. H., "Generalized integral transformations", Pure and Applied mathematics, Vol.XVIII, Interscience Publishers, New York London- Sydney, 1968.