

Paired Domination in Subdivision Graphs

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ABSTRACT: A dominating set S of G is called a paired dominating set if $\langle S \rangle$ has a perfect matching. If S is a paired dominating set with perfect matching M , then two vertices u and v are said to be paired in S if the edge uv is in M . Every graph without isolated vertices has a paired dominating set. Let G be a graph without isolated vertices. The minimum cardinality of a paired dominating set of G is called the paired domination number of G and is denoted by $\gamma_{pr}(G)$.

An edge $e = uv$ of a graph G is said to be subdivided if e is replaced by the edges uw and wv for some vertex w not in $V(G)$. The graph obtained from G by subdividing each edge of G exactly once is called the subdivision graph of G and is denoted by $S(G)$. In this paper, we initiate a study of the paired domination number of subdivision graphs. We determine the paired domination number of the subdivision of some standard graphs. We also obtain bounds for the paired domination number of $S(G)$.

KEYWORDS: paired domination, paired domination number, subdivision graph, Paired domination number of a subdivision graph.

I. INTRODUCTION

Haynes and Slater [7] introduced the concept of paired domination in graphs. Arumugam and Paulraj Joseph [1] have studied the domination parameters of the subdivision graph $S(G)$.

The graphs considered in this paper are finite, undirected, without loops, multiple edges and isolated vertices. we need the following theorems.

Theorem 1.1. For any graph G , $\gamma(S(G)) \leq n - 1$

Theorem 1.2. For any graph G , $\gamma(S(G)) \leq m(G)$

Theorem 1.3. If a graph G has no isolated vertices, then $\gamma_{pr}(G) \leq 2\gamma(G)$.

Theorem 1.4. $\gamma_{pr}(P_n) = \gamma_{pr}(C_n) = 2\lceil \frac{n}{4} \rceil$

Theorem 1.5. If T is a tree of order $n \geq 3$ with s support vertices, then $\gamma_{pr}(G) \leq \frac{n+2s-1}{2}$

Since $S(C_n) = C_{2n}$ and $S(P_n) = P_{2n-1}$ it follows from Theorem 1.4 that

$$\gamma_{pr}(S(C_n)) = 2\lceil \frac{n}{2} \rceil \text{ and } \gamma_{pr}(S(P_n)) = 2\lceil \frac{2n-1}{2} \rceil$$

II. MAIN RESULTS

Theorem 2.1. For any graph G of order n , $\gamma_{pr}(S(G)) \leq 2(n - 1)$, and this bound is sharp.

Proof: Let G be any graph of order n . It follows from Theorem 1.1 and Theorem 1.3 that $\gamma_{pr}(S(G)) \leq 2(n - 1)$. For the graph $G = K_{1,n-1}$, we have $\gamma_{pr}(S(G)) = 2(n - 1)$ and hence the inequality is sharp.

Problem 2.2. Characterize graphs G for which $\gamma_{pr}(S(G)) = 2(n - 1)$.

Theorem 2.3. For any connected graph G , $\gamma_{pr}(S(G)) \leq 2m(G)$ where $m(G)$ is the number of edges of G . Further equality holds if and only if G is the star $K_{1,n}$.

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Proof. It follows from Theorem 1.2 and Theorem 1.3 that $\gamma_{pr}(S(G)) \leq 2m(G)$.

Now, suppose $\gamma_{pr}(S(G)) = 2m$. Since $\gamma_{pr}(S(G)) \leq 2(n-1)$, we have $m \leq n-1$. Since G is connected, $m \geq n-1$, so that $m = n-1$ and G is a tree. Now suppose $G \neq K_{1,n}$. Then G contains a path $P = (u_0, u_1, u_2, u_3)$.

Let v_1, v_2 and v_3 be the vertices of $S(G)$ subdividing the edges u_0u_1, u_1u_2 and u_2u_3 respectively. Let $e_i = x_iy_i, 1 \leq i \leq n-4$ be the remaining edges of G and let z_i be the vertex of $S(G)$ subdividing the edge x_iy_i . Now $D = \left(\bigcup_{i=1}^{n-4} \{x_i, y_i\} \right)$

$\bigcup \{v_1, u_1, v_3, u_3\}$ is a paired dominating set of $S(G)$ and $|D| = 2(n-2)$. Hence $\gamma_{pr}(S(G)) \leq 2(n-2)$, which is a contradiction. Hence $G = K_{1,n}$.

Conversely, if $G = K_{1,n}$, then obviously $\gamma_{pr}(S(G)) = 2m(G)$.

Theorem 2.4. Let G be a connected unicyclic graph of order n . Then $\gamma_{pr}(S(G)) = 2(n-1)$ if and only if $G = C_3$.

Proof. Let G be a connected unicyclic graph of order n with $\gamma_{pr}(S(G)) = 2(n-1)$.

Let $C = (u_1, u_2, \dots, u_k, u_1)$ be the cycle in G . We claim that $\deg u_i = 2$ for all $i = 1, 2, \dots, k$. Suppose there exists a vertex, say u_1 , with $\deg u_1 \geq 3$. Then $G = \{u_1u_2, u_1u_k\}$ has two nontrivial components, say G_1 and G_2 .

Now $\gamma_{pr}(S(G_1)) \leq 2(|V(G_1)| - 1)$ and $\gamma_{pr}(S(G_2)) \leq 2(|V(G_2)| - 1)$ so that $\gamma_{pr}(S(G)) \leq 2(n-2)$, which is a contradiction.

Thus $\deg u_i = 2$ for all $i = 1, 2, \dots, k$ and $G = C$. If $G = C_n, n \geq 4$, then $\gamma_{pr}(S(C_n)) = 2 \lceil \frac{n}{2} \rceil \leq 2(n-2)$.

Thus $n = 3$ and $G = C_3$. The converse is obvious.

Theorem 2.5. Let G be a graph of order n . Let $G^{(k)}$ be the graph obtained by attaching k pendant edges to each vertex of G . Then $\gamma_{pr}(S(G^{(k)})) = 2nk$.

Proof. Let $V(G) = \{u_1, u_2, u_3, \dots, u_n\}$. Let $V_i = \{v_{i1}, v_{i2}, \dots, v_{ik}\}$ be the set of pendant vertices which are adjacent to u_i . Let w_{ij} be the vertex in $S(G^{(k)})$ which subdivides the edge $u_i v_{ij}$.

Now $S = \{v_{ij}w_{ij} : 1 \leq i \leq k-1, 1 \leq j \leq k-1\} \cup \{u_i w_{ik} : 1 \leq i \leq n\}$ is a paired dominating set of $S(G^{(k)})$. Hence $\gamma_{pr}(S(G^{(k)})) = 2nk$. Thus $\gamma_{pr}(S(G^{(k)})) = 2nk$.

Theorem 2.6. Let T be a tree of order n . Then $s + l \leq \gamma_{pr}(S(T)) \leq n + l - 1$ where l is the number of pendant vertices of T and s is the number of support vertices of T .

Proof. Let $V_1 = \{u_1, u_2, \dots, u_l\}$ be the set of all pendant vertices of T .

Let V_2 be the set of all $\gamma_{pr}(S(G^{(k)})) \leq |S| = 2(n(k-1) + n) = 2nk$. Further, since G has nk pendant vertices, we have support vertices of T and let $V_3 = \{v_1, v_2, \dots, v_l\}$ where v_i is the vertex subdividing the edge incident with u_i . Then any

paired dominating set D of $S(T)$ contains $V_2 \cup V_3$ or $V_1 \cup V_3$ and hence $\gamma_{pr}(S(T)) \geq |D| \geq s + l$.

If T is a tree with l pendant vertices, then $S(T)$ is a tree with $(2n-1)$ vertices and l supports. Now, it follows from

Theorem 1.5 that $\gamma_{pr}(S(T)) \leq \frac{(2n-1) + 2l-1}{2} = n + l - 1$.

Thus $s + l \leq \gamma_{pr}(S(T)) \leq n + l - 1$.

Theorem 2.7. For the wheel $W_n = C_{n-1} + K_1$, we have $\gamma_{pr}(S(W_n)) = 2 + 2 \lceil \frac{n-1}{2} \rceil$

Proof. Let $V(W_n) = \{u_0, u_1, u_2, \dots, u_{n-1}\}$, and

$E(W_n) = \{u_i u_{i+1} : 1 \leq i \leq n-2\} \cup \{u_i u_{n-1}\} \cup \{u_0 u_i : 1 \leq i \leq n-1\}$

Let $v_i, (1 \leq i \leq n-1)$ be the vertex of $S(W_n)$ which is adjacent to u_0 and u_i .

Let w_i be the vertex subdividing the edge $u_i u_{i+1}, 1 \leq i \leq n-2$ and let w_{n-1} be the vertex subdividing the edge $u_{n-1} u_1$. Let D be a minimum paired dominating set of the cycle

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$(u_1, w_1, u_2, w_2, u_3, \dots, u_{n-1}, w_{n-1}, u_n)$, so that $|D_0| = 2 \left\lceil \frac{n-1}{2} \right\rceil$. Now, $D = D_0 \cup \{u_n, v_n\}$ is a paired dominating set of

$S(W_n)$ and hence $\gamma_{pr}(S(W_n)) \leq 2 + 2 \left\lceil \frac{n-1}{2} \right\rceil$

Now, let D be any paired dominating set of $S(W_n)$.

Case i. $u_n \notin D$.

Since D is a paired dominating set, we may assume without loss of generality that $v_i, u_i \in D$. Now, for each $i, 2 \leq i \leq n-1$, at least one of the vertices u_i, v_i is in D and since D is a paired dominating set, it follows that $|D| \geq 2n - 2$.

Case ii. $u_n \in D$.

Since D is a paired dominating set, we may assume without loss of generality that $v_1 \in D$. Now, $D - \{u_n, v_1\}$ is a paired dominating set of the path $(w_1, u_2, w_2, u_3, \dots, u_{n-1}, w_{n-1})$. Hence it follows that, $|D| \geq 2 + 2 \left\lceil \frac{n-1}{2} \right\rceil$

Hence $\gamma_{pr}(S(W_n)) \geq 2 + 2 \left\lceil \frac{n-1}{2} \right\rceil$. Thus $\gamma_{pr}(S(W_n)) = 2 + 2 \left\lceil \frac{n-1}{2} \right\rceil$

Theorem 2.8. For the complete bipartite graph $K_{m,n}$ with $m \leq n$, we have $\gamma_{pr}(S(K_{m,n})) = 2n$.

Proof. Let $V_1 = \{u_1, u_2, \dots, u_m\}$ and $V_2 = \{v_1, v_2, \dots, v_n\}$ be a bipartition of $K_{m,n}$.

Let $n = m + r$ where $r \geq 0$. Let $w_{ij} (1 \leq i \leq m, 1 \leq j \leq n)$ be the vertex of $S(K_{m,n})$ which is adjacent to u_i and v_j .

Let $S = \left(\bigcup_{j=1}^m \{v_j, w_{ij}\} \right) \cup \left(\bigcup_{j=m+1}^r \{v_j, w_{jm}\} \right)$. Clearly S is a paired dominating set of $S(K_{m,n})$ and hence

$\gamma_{pr}(S(K_{m,n})) \leq 2n$. Also, any paired dominating set of $S(K_{m,n})$ contains at least n vertices from $K_{m,n}$ and at least n vertices from $V(S(K_{m,n})) - V(K_{m,n})$, so that $\gamma_{pr}(S(K_{m,n})) \geq 2n$. Thus $\gamma_{pr}(S(K_{m,n})) = 2n$.

Theorem 2.9. For the complete graph K_n we have $\gamma_{pr}(S(K_n)) = 2(n-1)$.

Proof. It follows from Theorem 2.1 that $\gamma_{pr}(S(K_n)) \leq 2(n-1)$. Now, let $V(K_n) = \{v_1, v_2, \dots, v_n\}$ and let

$W = V(S(K_n)) - V(K_n) = \{w_1, w_2, \dots, w_r\}$ where $r = \binom{n}{2}$

Let S be any paired dominating set of $S(K_n)$ with $|S| = 2k, k \geq 1$. Then S contains k vertices of $V(K_n)$ and k vertices of $W = V(S(K_n)) - V(K_n)$. We may assume without loss of generality that $S = \{v_1, w_1, v_2, w_2, \dots, v_k, w_k\}$ where $k \leq n$. Consider the first pair (v_1, w_1) of vertices in S . This pair of vertices dominate $(n-1)$ vertices in W . Now, the second pair (v_2, w_2) dominates $(n-2)$ vertices in W other than the vertices dominated by (v_1, w_1) . In general the pair (v_i, w_i) dominates $n-i$ vertices of W , other than the vertices dominated by $\{v_1, w_1, \dots, v_{i-1}, w_{i-1}\}$. Since S dominates all the

$\binom{n}{2}$ vertices of W , it follows that $\frac{n(n-1)}{2} = nk - \frac{k(k-1)}{2}$ Hence

$k^2 + k(n-2) + (n^2 - n) = 0$. Solving this equation for k , we get $k = n$ or $k = n-1$.

Hence $|S| \geq 2(n-1)$ so that $\gamma_{pr}(S(K_n)) \geq 2(n-1)$. Thus $\gamma_{pr}(S(K_n)) = 2(n-1)$.

III.CONCLUSION

We have found paired domination number of the subdivision of some standard graphs. Using the proof technique we can find paired domination number of the subdivision of some other graphs also.

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REFERENCES

- [1] S. Arumugam and J. Paulraj Joseph, "Domination in subdivision graphs", Journal of the Indian Math. Soc.,62(1-4), pp 274 – 282, 1996,.
- [2] C. Berge, The Theory of graphs and its applications, Methuren, London (1962)
- [3] E.J. Cockayne and S.T. Hedetniemi, Towards a theory of domination in graphs, Networks, pp 241 – 261,1977
- [4] F. Harary, Graph Theory, Addison -Wesley, Reading Mass,1972.
- [5] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, Inc., New York, 1997.
- [6] T. W. Haynes, S. T. Hedetniemi and P. J. Slater. Domination in Graphs - Advanced Topics, Marcel Dekker, Inc., New York, 1997.
- [7] T. W. Haynes and P. J. Slater, Paired-domination in graphs, Networks , pp 199 – 206,1998
- [8] Mustapha Chellai and Teresa W. Haynes, Total and Paired- Domination Number of a Tree, AKCE. J. Graphs. Combin.,pp 69 – 75,2004.
- [9] H.B. Walikar, B.D. Acharya and E. Sampathkumar, Recent Development in the theory of domination in graphs, In MRI Lecture Notes in Math., Mehta Research Instit., Allahabad,1, 1979.
- [10] O. Ore, Theory of graphs, Amer. Math. Soc. Colloq. Pub., Providence, 274 – 276,1962.