

Integral Solutions of an Infinite Elliptic Cone

$$X^2 = 4Y^2 + 5Z^2$$

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ABSTRACT: The ternary quadratic Diophantine equation representing infinite elliptic cone given by $X^2 = 4Y^2 + 5Z^2$ is analysed for its non-zero distinct integer points on it. Four different patterns of integer points satisfying the cone under consideration are obtained. Also knowing an integer solution satisfying the given cone six triples of integers generated from the given solution are exhibited.

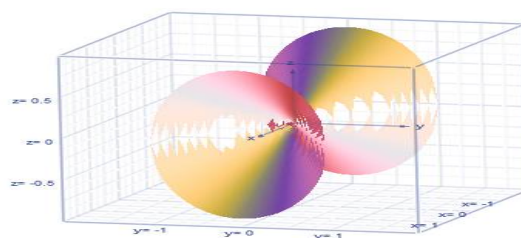
KEYWORDS: Diophantine equation, Integral solution, infinite elliptic cone.

I. INTRODUCTION

Number theory, called the Queen of Mathematics, is a broad and diverse part of Mathematics that developed from the study of the integers. The foundations for Number theory as a discipline were laid by the Greek mathematician Pythagoras and his disciples (known as Pythagoreans). One of the oldest branches of mathematics itself, is the Diophantine equations since its origins can be found in texts of the ancient Babylonians, Chinese, Egyptians, Greeks and so on [7 - 8]. Diophantine problems were first introduced by Diophantus of Alexandria who studied this topic in the third century AD and he was one of the first Mathematicians to introduce symbolism to algebra. The theory of Diophantine equations is a treasure house in which the search for many hidden relations and properties among numbers form a treasure hunt. In fact, Diophantine problems dominated most of the unsolved mathematical problems. Certain Diophantine problems come from physics problems or from immediate Mathematical generalizations and others come from geometry in a variety of ways. Certain Diophantine problems are neither trivial nor difficult to analyze [1 - 6]. In this context one may refer [9 , 10].

In this communication, the ternary quadratic Diophantine equation $X^2 = 4Y^2 + 5Z^2$ is analyzed for various patterns of integer solutions.

Pictorial representation of the equation:



International Journal of Innovative Research in Science, Engineering and Technology

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II. METHOD OF ANALYSIS

The quadratic Diophantine equation with three unknowns to be solved is

$$X^2 = 4Y^2 + 5Z^2 \quad (1)$$

We present below different patterns of integral solutions of (1):

Pattern 1:

Consider (1) as

$$X^2 = 5Z^2 + (2Y)^2 \quad (2)$$

The solutions are found to be

$$\begin{aligned} X &= 5m^2 + n^2 \\ 2Y &= 5m^2 - n^2 \\ Z &= 2mn \end{aligned}$$

Since our interest centers on finding integral solutions, we consider the following cases.

Case (i): Both m and n are odd.

(i.e.), Take $m = 2p + 1$, $n = 2q + 1$, $p, q \in \mathbf{Z}$,

The solutions of (1) are

$$\begin{aligned} X &= 5(2p + 1)^2 + (2q + 1)^2 \\ Y &= 10p^2 - 2q^2 + 10p - 2q + 2 \\ Z &= 2(2p + 1)(2q + 1) \end{aligned}$$

Case (ii): Both m and n are even.

(i.e.), take $m = 2p$, $n = 2q$, $p, q \in \mathbf{Z}$.

Hence, the solutions of (1) are given by

$$\begin{aligned} X &= 5(2p)^2 + (2q)^2 \\ Y &= 10p^2 - 2q^2 \\ Z &= 8pq \end{aligned}$$

In general,

$$X = 2^n(5p^2 + q^2), \quad Y = 2^{n-1}(5p^2 - q^2), \quad Z = 2^{n+1}(pq), \quad n \in \mathbf{N}.$$

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Pattern 2:

Consider (1) as:

$$X^2 - 4Y^2 = 5Z^2 \quad (3)$$

which can be written in the form of ratio as

$$\frac{X + 2Y}{5Z} = \frac{Z}{X - 2Y} = \frac{p}{q} \text{ (say), } q \neq 0$$

Expressing this as a system of simultaneous equations

$$qX + 2qY - 5pZ = 0 \quad (4)$$

$$pX - 2pY - qZ = 0 \quad (5)$$

and solving (4) and (5) we get

$$X = -(10p^2 + 2q^2)$$

$$Y = -5p^2 + q^2$$

$$Z = -4pq$$

In general

$$X = -2^n(5p^2 + q^2),$$

$$Y = -2^{n-1}(5p^2 - q^2),$$

$$Z = -2^{n+1}(pq), \quad n \in \mathbb{N}.$$

Pattern 3:

(3) can also be written as:

$$\frac{X + 2Y}{Z} = \frac{5Z}{X - 2Y} = \frac{p}{q} \text{ (say), } q \neq 0$$

Proceeding as above, we obtain

$$X = -(10q^2 + 2p^2)$$

$$Y = -p^2 + 5q^2$$

$$Z = -4pq$$

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In general,

$$X = -2^n(p^2 + 5q^2),$$

$$Y = -2^{n-1}(p^2 - 5q^2),$$

$$Z = -2^{n+1}(pq), \quad n \in \mathbb{N}.$$

Pattern 4:

Consider (1) as

$$X^2 - 5Z^2 = 4Y^2 \tag{6}$$

Assume Y as

$$Y = a^2 - 5b^2 \tag{7}$$

Write 4 as

$$4 = p^2 - 5q^2 \tag{8}$$

In view of (7) and (8), (6) is written in the factorizable form as

$$(X + \sqrt{5}Z)(X - \sqrt{5}Z) = (p + \sqrt{5}q)(p - \sqrt{5}q)(a + \sqrt{5}b)^2(a - \sqrt{5}b)^2$$

Define $X + \sqrt{5}Z = (p + \sqrt{5}q)(a + \sqrt{5}b)^2$

Equating the rational and irrational parts, we get

$$X = p(a^2 + 5b^2) + 10abq, \quad Z = q(a^2 + 5b^2) + 2abp \tag{9}$$

Thus, (7) and (9) represent the general solution of (1).

To find the value of p, q we proceed as follows:

Let (p_0, q_0) be the initial solution of $p^2 - 5q^2 = 4$.

Employing the linear transformations $p_1 = p_0 + 9h, q_1 = q_0 + 4h$ repeatedly, the other values of p and q satisfying the equation (8) are given by

$$p = -161p_0 + 360q_0,$$

$$q = -72p_0 + 161q_0$$

In addition, we have an another set of values of p, q satisfying the equation (8) given by

$$p = 9p_0 - 20q_0,$$

$$q = 4p_0 - 9q_0$$

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A few numerical examples are presented below:

	Algebraic expression for 4	Solutions
Case I	$(18 + 8\sqrt{5})(18 - 8\sqrt{5})$	$X = 18(a^2 + 5b^2) + 80ab$ $Y = a^2 - 5b^2$ $Z = 8(a^2 + 5b^2) + 36ab$
Case II	$(47 + 21\sqrt{5})(47 - 21\sqrt{5})$	$X = 47(a^2 + 5b^2) + 210ab$ $Y = a^2 - 5b^2$ $Z = 21(a^2 + 5b^2) + 94ab$
Case III	$(123 + 55\sqrt{5})(123 - 55\sqrt{5})$	$X = 123(a^2 + 5b^2) + 550ab$ $Y = a^2 - 5b^2$ $Z = 55(a^2 + 5b^2) + 246ab$
Case IV	$(322 + 144\sqrt{5})(322 - 144\sqrt{5})$	$X = 322(a^2 + 5b^2) + 1440ab$ $Y = a^2 - 5b^2$ $Z = 144(a^2 + 5b^2) + 644ab$

Generation of solutions:

Suppose that (X_0, Y_0, Z_0) represents an initial solution to (1).

The following triples also satisfy (1).

Choice 1: Let (X_1, Y_1, Z_1) be the second solution of (1) where

$$X_1 = 5h - X_0, \quad Y_1 = Y_0 + h, \quad Z_1 = Z_0 + 2h \tag{10}$$

in which h is an arbitrary non- zero integer to be determined.

Substituting (10) in (1), we get $h = 10X_0 + 8Y_0 + 20Z_0$.

Hence the matrix representation of the solution (X_1, Y_1, Z_1) is,

$$(X_1 \ Y_1 \ Z_1)^T = M^1(X_0 \ Y_0 \ Z_0)^T$$

where $M^1 = \begin{pmatrix} 49 & 40 & 100 \\ 10 & 9 & 20 \\ 20 & 16 & 41 \end{pmatrix}$

Proceeding in a similar manner, the general solution $(X_{n+1}, Y_{n+1}, Z_{n+1})$ in the matrix form is given by

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$(X_{n+1} \ Y_{n+1} \ Z_{n+1})^T = M^{n+1}(X_0 \ Y_0 \ Z_0)^T$, T denotes the transpose of the matrix,

$$\text{with } M^{n+1} = \begin{pmatrix} Y_n & 2X_n & 5X_n \\ \frac{X_n}{2} & \frac{Y_{n+5}}{6} & \frac{5Y_{n-5}}{12} \\ X_n & \frac{Y_{n-1}}{3} & \frac{5Y_{n+1}}{6} \end{pmatrix}$$

where (X_n, Y_n) is the general solution of the Pellian $Y^2 = 6X^2 + 1$ with $X_0 = 20, Y_0 = 49$.

Choice 2: The second solution (X_1, Y_1, Z_1) of (1) is taken as

$$X_1 = X_0 + 2h$$

$$Y_1 = Y_0$$

$$Z_1 = Z_0 + h$$

Substituting in (1), we get $h = 4X_0 - 10Z_0$.

To get a sequence of solutions, consider the matrix representation of solution (X_1, Z_1)

$$\text{(i.e.) } \begin{pmatrix} X_1 \\ Z_1 \end{pmatrix} = A \begin{pmatrix} X_0 \\ Z_0 \end{pmatrix}$$

where $A = \begin{pmatrix} 9 & -20 \\ 4 & -9 \end{pmatrix}$.

In general, the solution (X_n, Z_n) is given by

$$(X_n \ Z_n)^T = A^n(X_0 \ Z_0)^T.$$

It is well known that A^n is represented by the formula

$$A^n = \frac{\alpha^n}{\alpha - \beta} [A - \beta I] + \frac{\beta^n}{\beta - \alpha} [A - \alpha I]$$

where α, β are the eigen values of A and I is the unit matrix of order 2.

On simplifying, we get

$$X_n = (5 + 4(-1)^n)X_0 - 10(1 + (-1)^n)Z_0, \quad Z_n = 2(1 + (-1)^n)X_0 - (4 + 5(-1)^n)Z_0 \quad (11)$$

Therefore equation (11) along with $Y_n = Y_0$ represent the general solution of (1).

Choice 3: The second solution (X_1, Y_1, Z_1) of (1) is taken as

$$X_1 = 9X_0$$

$$Y_1 = h - 9Y_0$$

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$$Z_1 = h - 9Z_0$$

Substituting in (1), we get $h = 8Y_0 + 10Z_0$.

Following the procedure as above, the general solution is given by

$$X_n = 9^n X_0$$

$$Y_n = 9^{n-1}[(4 + 5(-1)^n)Y_0 + (5 - 5(-1)^n)Z_0]$$

$$Z_n = 9^{n-1}[(4 - 4(-1)^n)Y_0 + (5 + 4(-1)^n)Z_0]$$

Choice 4: The second solution (X_1, Y_1, Z_1) of (1) is taken as

$$X_1 = X_0 + 38h,$$

$$Y_1 = Y_0.$$

$$Z_1 = Z_0 + 17h.$$

Substituting in (1), we get $h = 76X_0 - 170Z_0$.

Following the procedure as above, the general solution is given by

$$X_n = [1444(1 - (-1)^n) + 1]X_0 + [3230((-1)^n - 1)]Z_0$$

$$Y_n = Y_0$$

$$Z_n = [646(1 - (-1)^n)]X_0 + [1444((-1)^n - 1) + (-1)^n]Z_0$$

Choice 5: The second solution (X_1, Y_1, Z_1) of (1) is taken as

$$X_1 = -5X_0 + 3h,$$

$$Y_1 = 5Y_0 + h,$$

$$Z_1 = 5Z_0$$

Substituting in (1), we get $h = 6X_0 + 8Y_0$.

Following the procedure as above, the general solution is given by

$$X_n = \left(\frac{5^{2n} + 1}{2}\right)X_0 + (5^{2n} - 1)Y_0$$

$$Y_n = \left(\frac{5^{2n} - 1}{4}\right)X_0 + \left(\frac{5^{2n} + 1}{2}\right)Y_0$$

$$Z_n = 5^n Z_0$$

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Choice 6:

The second solution (X_1, Y_1, Z_1) of (1) is taken as

$$X_1 = -4X_0 + 3h,$$

$$Y_1 = 4Y_0,$$

$$Z_1 = 4Z_0 + h$$

Substituting in (1), we get $h = 6X_0 + 10Z_0$.

Following the procedure as above, the general solution is given by

$$X_n = \left(\frac{\alpha^n + \beta^n}{2}\right)X_0 + 5\left(\frac{\alpha^n - \beta^n}{2\sqrt{5}}\right)Z_0$$

$$Y_n = 4^n Y_0$$

$$Z_n = \left(\frac{\alpha^n - \beta^n}{2\sqrt{5}}\right)X_0 + \left(\frac{\alpha^n + \beta^n}{2}\right)Z_0$$

where $\alpha = 14 + 6\sqrt{5}$, $\beta = 14 - 6\sqrt{5}$

III. CONCLUSION

in this paper, we have presented four different patterns of non-zero distinct integer solutions of the infinite elliptic cone given by $X^2 = 4Y^2 + 5Z^2$. One may search for other patterns of non-zero distinct integer solutions for the same equation. Since the ternary quadratic Diophantine equations are rich in variety, one may search for other choices of Diophantine equations to find their corresponding integer solutions.

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ISSN(Online) : 2319-8753
ISSN (Print) : 2347-6710

International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 5, Issue 10, October 2016

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