

# Some Properties of Graph of Mobius Function for '0'

K. K. Srimitra<sup>1</sup>, Shaik Sajana<sup>2</sup>, D. Bharathi<sup>3</sup>

Research Scholar, Department of Mathematics, S.V. University, Tirupati, Andhra Pradesh, India<sup>1,2</sup>

Professor, Department of Mathematics, S. V. University, Tirupati, Andhra Pradesh, India<sup>3</sup>

**ABSTRACT:** In this paper, we consider the graph of Mobius function for zero,  $G(\mu_n^{(0)})$ . For an integer  $n \geq 1$ , the graph of Mobius function for zero is a graph with vertex set  $\{1, 2, 3, \dots, n\}$  and an edge, between two vertices  $a, b$  if the Mobius function value,  $\mu(ab) = 0$ . We study the connectedness of this graph based on the number of vertices. We proved that it has vertex induced sub graph for any positive integer  $k$ ,  $k \leq n$ . Also studied and noted the values of some parameters viz. Radius, Diameter and girth of the graph with illustrations.

**KEYWORDS:** Diameter, Disconnected, Girth, Induced sub graph, Mobius function, Mobius graph, Radius, Tree.

## I. INTRODUCTION

The Mobius function in [8] is a well known function in Number theory and it is defined as  $\mu(1) = 1$  and if  $n > 1$ , we write  $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \dots p_k^{\alpha_k}$ , then,  $\mu(n) = \begin{cases} (-1)^k, & \text{if } \alpha_1 = \alpha_2 = \dots = \alpha_k = 1 \\ 0, & \text{otherwise} \end{cases}$ . Here we consider the graph  $G(V, E)$  by defining on the powerful arithmetic function called Mobius function. The graph of Mobius function for 0 was introduced by Vasumathi [10] and is denoted by  $G(\mu_n^{(0)})$ . The vertex set is the set of first  $n$  natural numbers and two vertices  $a, b$  are adjacent if and only if  $\mu(ab) = 0$ . The graph considered in this paper is simple graph. A graph is connected if there exists a path between every pair of vertices. From [9], the eccentricity of a vertex  $u$  in a graph is the maximum distance from  $u$  to any other vertex. The minimum eccentricity among all the vertices is called the Radius of the graph and the maximum eccentricity is called the Diameter of the graph. If the eccentricity of any vertex is not defined then the Radius and Diameter of the graph are infinite. And the Center of the graph is the set of vertices whose eccentricity equal to the Radius. From [3], a tree is a connected acyclic graph. The length of the shortest cycle in a graph is called girth of the graph. If a graph has no cycle then girth is infinite. Let  $G(V, E)$  be a graph and  $V_1$  be the subset of  $V$ . Then the sub graph of  $G$  whose vertex set is  $V_1$  and whose edge set is the set of edges of  $E$  whose both ends are in  $V_1$  is called the vertex induced sub graph and is denoted by  $\langle V_1 \rangle$ .

The concepts of Number theory were applied in Graph theory to design a graph is introduced in 1980 by Nathanson [6]. He used the theory of congruences for defining the adjacency between two vertices of a graph. Later in 1994, Vasumathi [10] introduced a new class of graph called Mobius graph by applying the Mobius function on the adjacency of two vertices in a graph and in [7], Srimitra, Sajana and Bharathi studied this graph and discussed the adjacency, degree of the vertices of this graph. In recent years, many researchers of graph theory are working on the parameters of the graph like Radius, Diameter for different graphs. A. Saha [1], B. Bollobas, O. Riordan [2], F. F. Dragon [4], S. C. Li, Y. Y. Wu, L. L. Sun [4] were discussed the eccentricity, Radius and Diameter for different graphs.

## II. MAIN THEOREMS

Now we discuss the vertex inducedness of sub graph of a graph of Mobius function for 0 by the following theorem.

**Theorem 1:-**  $G(\mu_k^{(0)})$  is a vertex induced sub graph of  $G(\mu_n^{(0)})$ , where  $k \leq n$ .

**Proof:** - Consider the graph of Mobius function for '0',  $G(\mu_n^{(0)})$

Let  $u, v$  be two adjacent vertices in  $G(\mu_k^{(0)})$ ,  $k \leq n$

## International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Website: [www.ijirset.com](http://www.ijirset.com)

Vol. 6, Issue 8, August 2017

Then,  $\mu(uv) = 0, u, v \leq k$

For  $k \leq n, \mu(uv) = 0, u, v \leq k \leq n$

Implies that,  $\mu(uv) = 0, u, v \leq n$

Therefore,  $u, v$  are adjacent vertices in  $G(\mu_n^{(0)})$

i.e., every edge in  $G(\mu_k^{(0)})$  is an edge in  $G(\mu_n^{(0)}), k \leq n$  between the same vertices in both the graphs.

Let  $V_1 = V(G(\mu_k^{(0)})) = \{1, 2, 3, \dots, k\}$  and  $V_2 = V(G(\mu_n^{(0)})) = \{1, 2, 3, \dots, k, k+1, \dots, n\}$

Implies,  $V_1 \subseteq V_2$

Hence,  $G(\mu_k^{(0)})$  is a vertex induced sub graph of  $G(\mu_n^{(0)}), \forall k \leq n$ . ■

It can be illustrated by Figure 3 and Figure 4. It is clear that in Figure 3, the graph with  $K=15$  vertices is vertex induced sub graph of the graph in Figure 4 with  $n=20$  vertices.

Now, we will discuss the connectedness, bipartition of the graph of Mobius function for '0' based on the number of vertices  $n$  by the following theorems.

**Theorem 2:-** A graph  $G(\mu_n^{(0)})$  is disconnected if and only if  $n \leq 3$ .

**Proof:** - Let  $G(\mu_n^{(0)})$  be a disconnected graph.

Now assume  $n \geq 4$

Since  $\mu(4) = 0$ , then 4 is adjacent to all other vertices of  $G(\mu_n^{(0)})$

So, every vertex is connected to the vertex 4.

Then, every pair of vertices has a path.

Therefore,  $G(\mu_n^{(0)})$  is connected. This is contradiction to the data.

Therefore,  $n \geq 4$  is wrong. And hence,  $n \leq 3$ .

Conversely, let  $n \leq 3$ .

Since,  $\mu(2), \mu(3)$  and  $\mu(6)$  are non zero,

So, there is no edge between the vertices 1, 2 and 3.

That is, the vertices 1, 2 and 3 are isolated vertices in  $G(\mu_3^{(0)})$

Since, from Theorem -1,  $G(\mu_1^{(0)})$  and  $G(\mu_2^{(0)})$  are vertex induced sub graphs of  $G(\mu_3^{(0)})$

We have, 1 and 2 are isolated vertices in  $G(\mu_1^{(0)}), G(\mu_2^{(0)})$

Therefore,  $G(\mu_n^{(0)}), n \leq 3$  are disconnected. ■

**Theorem 3:-** A graph  $G(\mu_n^{(0)})$  is a tree if  $n = 4$  or  $n = 5$ .

**Proof:** - Let  $n < 4$ ,

Then by the Theorem -2, the graph  $G(\mu_n^{(0)})$  is disconnected.

Therefore, the graph  $G(\mu_n^{(0)}), n < 4$  is not a tree.

Let  $n > 5$ ,

Then, 6 is the vertex of  $G(\mu_n^{(0)}), n > 5$

Since,  $\mu(2.6) = \mu(12) = 0$  So, 2 and 6 adjacent in  $G(\mu_n^{(0)}), n > 5$

Also since,  $\{4\}$  is the Independent dominating set of  $G(\mu_n^{(0)})$ ,

So, the vertex 4 is adjacent to both the vertices 2 and 6.

Then, the vertices 2, 4 and 6 are adjacent to each other.

Therefore, (2, 4, 6) is a cycle in  $G(\mu_n^{(0)}), n > 5$

Hence,  $G(\mu_n^{(0)}), n > 5$  is not a tree.

Now, if  $n = 4$  or  $n = 5$ ,

Then the vertices 1, 2 and 3 are adjacent to 4 only in  $G(\mu_4^{(0)})$  and the vertices 1, 2, 3 and

5 are adjacent to 4 only in  $G(\mu_5^{(0)})$ .

Therefore,  $G(\mu_4^{(0)}), G(\mu_5^{(0)})$  have no cycles. And also from Theorem -2, they are connected.

Thus,  $G(\mu_4^{(0)}), G(\mu_5^{(0)})$  are connected and acyclic graphs.

# International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Website: [www.ijirset.com](http://www.ijirset.com)

Vol. 6, Issue 8, August 2017

Hence,  $G(\mu_n^{(0)})$ ,  $n = 4$  or  $n = 5$  is a tree. ■

**Theorem 4:-** The graph  $G(\mu_n^{(0)})$ ,  $n \geq 6$  is neither a tree nor a bipartite graph.

**Proof:** - Let  $G(\mu_n^{(0)})$ ,  $n \geq 6$  be a graph

Since,  $n \geq 6$ , then 2, 4, 6 are the vertices of  $G(\mu_n^{(0)})$ ,  $n \geq 6$ . And also since,  $\mu(4) = 0$

So, the vertex 4 is adjacent to the vertices 2, 6.

There exists two edges (2, 4) and (4, 6).

Now,  $\mu(2.6) = \mu(12) = 0$ . Then, there is an edge between 2, 6.

Therefore, the vertices 2, 4, 6 are adjacent to each other.

i.e., the vertices 2, 4, 6 form a triangle.

Then there is a cycle in the graph  $G(\mu_n^{(0)})$ ,  $n \geq 6$

Therefore, this graph is not a tree.

Further, the cycle (2, 4, 6, 2) has length 3, which is an odd number

Thus, the graph  $G(\mu_n^{(0)})$ ,  $n \geq 6$  has an odd cycle.

By the known theorem, "A graph G is bipartite, if and only if it has no odd cycles"

So that, the graph  $G(\mu_n^{(0)})$ ,  $n \geq 6$  is not a bipartite graph.

Hence the graph  $G(\mu_n^{(0)})$ ,  $n \geq 6$  is neither a tree nor a bipartite graph. ■



Fig 1: The graph of Mobius function with 2 vertices,  $G(\mu_2^{(0)})$

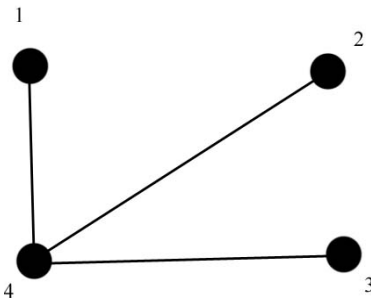


Fig 2: The graph of Mobius function with 4 vertices,  $G(\mu_4^{(0)})$

From Figure 1, we can observe that the graph with  $n=2$  vertices is disconnected and from the Figure 2, the graph with  $n=4$  vertices is a tree.

By using the above theorems, now we can calculate the values of some parameters viz. Radius, Diameter and girth of these graphs from the theorems as below.

**Theorem 5:-** For a graph  $G(\mu_n^{(0)})$ , the radius,  $Rad(G(\mu_n^{(0)})) = \begin{cases} \infty, & n \leq 3 \\ 1, & n > 3 \end{cases}$

**Proof:** - Let us consider the graph  $G(\mu_n^{(0)})$  with  $n$  vertices.

**Case I:** - Let  $n \leq 3$

For  $n \leq 3$ , from the Theorem – 2, the graph  $G(\mu_n^{(0)})$  is disconnected and has the isolated vertices 1, 2, 3.

So, the eccentricity of 1, 2, 3 are not defined.

Therefore,  $Rad(G(\mu_n^{(0)})) = \infty$  for  $n \leq 3$

**Case II:** - Let  $n > 3$

Every graph  $G(\mu_n^{(0)})$ ,  $n > 3$  is connected and 4 is one of the vertices of this graph.

Since,  $\mu(4) = \mu(2^2) = 0$

Then, the vertex 4 has an edge with all the other vertices in the graph.

# International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Website: [www.ijirset.com](http://www.ijirset.com)

Vol. 6, Issue 8, August 2017

Therefore, the eccentricity of the vertex 4 is 1, which is the least value of all eccentricities in the graph.

For, let  $u, v$  be any vertices such that  $u \neq v$  and  $\mu(v) = 0$ .

Then, there exists a path  $u - v - w$  from  $u$  to another vertex  $w$  with the distance  $d(u, w) = 2$

So, the eccentricity of  $u, e(u) = 2$

Therefore,  $e(v) = 1$  where  $\mu(v) = 0$ , is the least eccentricity of all the vertices of the graph  $G(\mu_n^{(0)}), n > 3$

Therefore, Radius of the graph is 1.

i.e.,  $\text{Rad}(G(\mu_n^{(0)})) = 1, n > 3$

From the above two cases,  $\text{Rad}(G(\mu_n^{(0)})) = \begin{cases} \infty, & n \leq 3 \\ 1, & n > 3 \end{cases}$  ■

**Theorem 6:-** For a graph  $G(\mu_n^{(0)})$ , the diameter,  $\text{Diam}(G(\mu_n^{(0)})) = \begin{cases} \infty, & n \leq 3 \\ 2, & n > 3 \end{cases}$

**Proof:-** Let  $G(\mu_n^{(0)})$  be a graph of Mobius function for 0 with  $n$  vertices.

Case I:- Let  $n \leq 3$

Then, the graph  $G(\mu_n^{(0)}), n \leq 3$  is disconnected graph.

Therefore,  $\text{Diam}(G(\mu_n^{(0)})) = \infty, n \leq 3$

Case II:- Let  $n > 3$

For the graph  $G(\mu_n^{(0)}), n > 3$ ,

Let  $u$  be any vertex such that  $\mu(u) \neq 0$

Let  $v$  be any other vertex of  $G(\mu_n^{(0)}), n > 3$  such that the distance from  $u$  to  $v$  is the eccentricity of  $u$ . i.e.,  $e(u) = d(u, v)$

Since,  $n > 3$ , then 4 is the vertex of  $G(\mu_n^{(0)}), n > 3$

Also since,  $\mu(4) = 0$ , then  $\{4\}$  is a Independent dominating set of  $G(\mu_n^{(0)}), n > 3$  i.e., the vertex 4 is adjacent with all other vertices.

Then, there exists two edges  $(u, 4)$  and  $(4, v)$

We have,  $u - 4 - v$  is a path from  $u$  to  $v$  and has the length 2.

Therefore,  $d(u, v) = 2$

i.e.,  $e(u) = d(u, v) = 2$

Since,  $u$  is an arbitrary vertex, then the eccentricity of any vertex  $u$  such that  $\mu(u) \neq 0$  is 2.

Let  $w$  be a vertex of  $G(\mu_n^{(0)}), n > 3$  such that  $\mu(w) = 0$

Since,  $\mu(w) = 0$ , then  $\{w\}$  is a Independent dominating set.

i.e.,  $w$  is adjacent with all other vertices.

So,  $e(w) = 1$

Therefore, the maximum of eccentricities of all the vertices is 2.

The diameter of  $G(\mu_n^{(0)}), n > 3$  is 2.

i.e.,  $\text{Diam}(G(\mu_n^{(0)})) = 2, n > 3$

From the above two cases,  $\text{Diam}(G(\mu_n^{(0)})) = \begin{cases} \infty & n \leq 3 \\ 2, & n > 3 \end{cases}$  ■

**Theorem 7:-** For a graph  $G(\mu_n^{(0)})$ , the girth,  $gr(G(\mu_n^{(0)})) = \begin{cases} \infty, & \text{if } n < 6 \\ 3, & \text{if } n \geq 6 \end{cases}$

**Proof:-** Consider the graph of Mobius function for zero,  $G(\mu_n^{(0)})$

Case 1:- Let  $n \leq 3$

Then, the graph  $G(\mu_n^{(0)}), n \leq 3$  is disconnected and all the vertices

# International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Website: [www.ijirset.com](http://www.ijirset.com)

Vol. 6, Issue 8, August 2017

are isolated vertices

Therefore, for  $n \leq 3$ ,  $gr(G(\mu_n^{(0)})) = \infty$

Case 2:- Let  $3 < n \leq 5$

i.e., if  $n = 4$  or  $n = 5$

Then, by the Theorem -3, the graph  $G(\mu_n^{(0)})$ ,  $n = 4$  or  $n = 5$  is a tree.

Then we have these two graphs have no cycles.

Therefore,  $gr(G(\mu_n^{(0)})) = \infty$  for  $3 < n \leq 5$

From the these two cases we can say that,  $gr(G(\mu_n^{(0)})) = \infty$  for  $n < 6$ .

Case 3:- Let  $n \geq 6$

Then, 2, 4, 6 are the vertices of the graph  $G(\mu_n^{(0)})$ ,  $n \geq 6$

Since,  $\mu(2,4) = \mu(4,6) = \mu(6,2) = 0$ ,

So, there exists three edges (2, 4), (4, 6), (6, 2) in  $G(\mu_n^{(0)})$ ,  $n \geq 6$ , which forms a triangle. I.e., there exists a cycle of length 3, which is the least length in  $G(\mu_n^{(0)})$ .

Because, we considered the graphs as simple graphs, so these have no cycles of length 2 (i.e., parallel edges)

Therefore,  $gr(G(\mu_n^{(0)})) = 3$  for  $n \geq 6$

From the above all cases, we have, the girth is given by

$$gr(G(\mu_n^{(0)})) = \begin{cases} \infty, & \text{if } n < 6 \\ 3, & \text{if } n \geq 6 \end{cases} \blacksquare$$

In Figure 3 and Figure 4, since  $n > 6$  then for the graphs  $G(\mu_{15}^{(0)})$  and  $G(\mu_{20}^{(0)})$  the radius is 1, diameter is 2 and the girth is 3. Also in Figure 1, for the graph  $G(\mu_2^{(0)})$ ,  $n=2$  then radius, diameter and girth are all infinite.

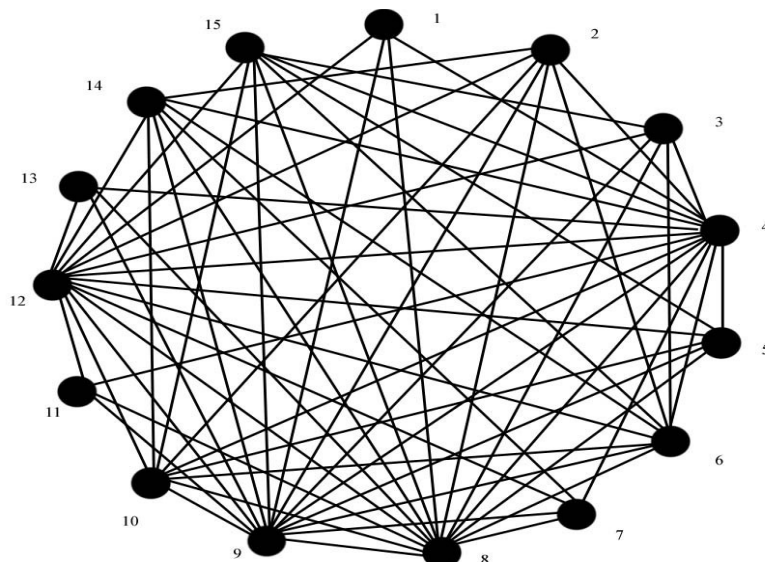


Fig 3: Graph of Mobius function for 0 with 15 vertices,  $G(\mu_{15}^{(0)})$

# International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Website: [www.ijirset.com](http://www.ijirset.com)

Vol. 6, Issue 8, August 2017

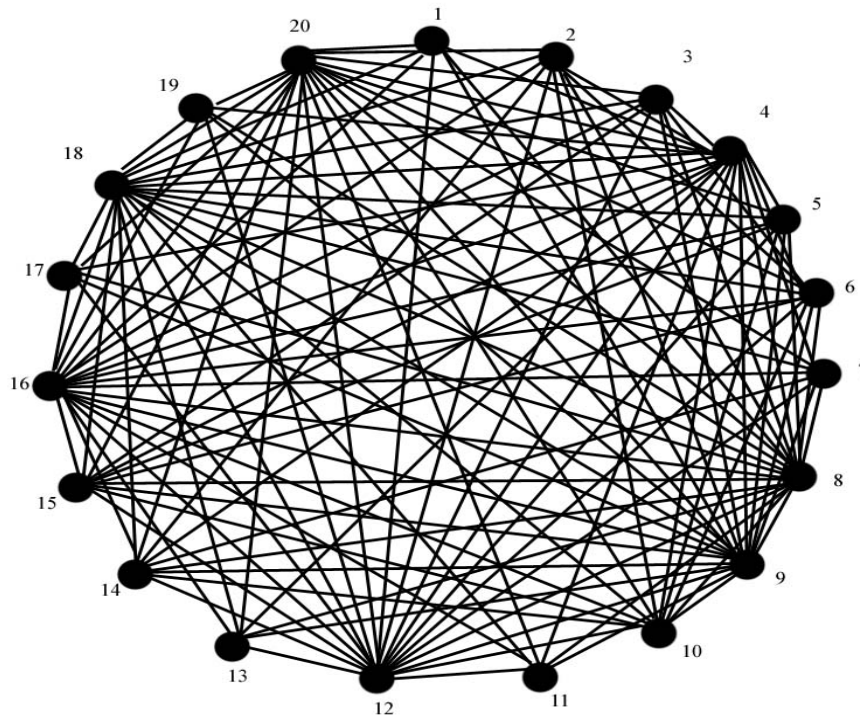


Fig 4: Graph of Mobius function for 0 with 20 vertices,  $G(\mu_{20}^{(0)})$

### III. CONCLUSION

In addition to the Degree of the vertices and the adjacency between two vertices studied by Srimitra [7] for the graph of Mobius function for '0', in this paper, we discussed some properties and parameters of this graph. First we studied the connectedness: in that the graph has only isolated vertices if the number of vertices  $n < 4$ , the graph is a tree when the number of vertices are 4 or 5 and otherwise the graph is neither a tree nor a bipartite. Next, we proved that every sub graph of the graph  $G(\mu_n^{(0)})$  is an induced sub graph of  $G(\mu_n^{(0)})$ . Also we calculated some parameters Radius, Diameter and Girth of this graph and then concluded that, since the graph  $G(\mu_n^{(0)})$ ,  $n \leq 3$  has only isolated vertices, the values of Radius, Diameter and Girth are infinite, also, if  $n=4$  or 5, the graph is a tree, then girth is infinite. In all other cases i.e., if  $n > 3$ , then the Radius is 1, Diameter is 2 and if  $n > 5$ , then the Girth is 3. Also, we concluded these results by giving some illustrations of graphs as the Figures.

### REFERENCES

- [1] Anita Saha, "Computation of Average Distance, Radius and Centre of a Circular-Arc Graph in Parallel", Journal of physical sciences, 2006.
- [2] Bela Bollobas and Oliver Riordan, "The Diameter of a Scale-Free Random Graph", Combinatorial 24(1), 2004, 5-34.
- [3] Bondy, J. A. and Murthy, U. S. R, "Graph Theory with Applications", Macmillan, London, 1976.
- [4] Feodor F. Dragan, "Almost diameter of a house-hole-free Graph in linear time via LexBFS", Discrete Applied Mathematics 95, 223-239, 1999.
- [5] Li, S. C., Wu, Y. Y. and Sun, L. L., "On the minimum eccentric distance sum of bipartite Graphs with some given parameters" Journal of Mathematical Analysis and Applications, 430, 1149-1162, 2015.
- [6] Nathanson Melvyn. B, "Connected components of Arithmetic Graphs", Monatshefte fur Mathematik, 89, 219-222, 1980.
- [7] Srimitra, K. K., Shaik Sajana, Bharathi, D., "Adjacency and Degree of Graph of Mobius function", i - Manager's Journal on Mathematics, 5(1), 31 - 38, 2016.
- [8] Tom M. Apostol, "Introduction to Analytic Number theory", Narosa publishing house, 1998.
- [9] The web site <http://math.fau.edu/locke/Center.htm>.
- [10] Vasumathi, N, "Graphs on Numbers", Ph.D. Thesis, Sri Venkateswara University, Tirupati, India, 1994.