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Fixed Point Theorems of Fuzzy Function in Complete Metric Spaces

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ABSTRACT: In this paper we have proved some fixed point theorems of fuzzy function in complete metric spaces. The presented results generalize the results proved in various spaces and extend of Bose and Mukherjee's theorems of common fixed points of a pair of multivalued mappings and a sequence of single valued mappings.

In this paper we establish some fixed point theorems of fuzzy mapping in complete metric spaces.

KEYWORDS: Fuzzy mappings, fixed point, membership function, metric linear space

I. INTRODUCTION

Heilpern [8] introduced the concept of fuzzy mappings and proved fixed point theorem for fuzzy contraction mappings. Since then, many authors have generalized Bose and Sahani's results in different directions. Also Bose and Mukherjee (see [2], [3]) considered common fixed points of a pair of multivalued mappings and a sequence of single valued mappings. We present some theorems which are generalized to ordered metric space setting and generalizations of the main theorem of several authors

II. PRELIMINARIES

Definition 2.1.

Let (X, d) be a metric linear space and $F(X)$, the collection of all fuzzy sets in X . Let $A \in F(X)$ and $\alpha \in [0, 1]$. The α -level set of A , denoted by A_α , and $A_\alpha = \{x : A(x) \geq \alpha\}$ if $\alpha \in (0, 1]$, $A_0 = \overline{\{x : A(x) > 0\}}$ where \bar{A} stands for the (non fuzzy) closure of A .

Definition 2.2.

A fuzzy subset A of X is an approximate quantity iff its $\sup_{x \in X} A(x) = 1$. From the collection $F(X)$, a sub collection of approximate quantities is denoted as $W(X)$.

The distance between two approximate quantities is defined by the following scheme.

Let $A, B \in W(X)$, and $\alpha \in [0, 1]$;

$D_\alpha(A, B) = \inf_{x \in A_\alpha, y \in B_\alpha} d(x, y)$; and D_α is a non-decreasing function of α .

$H_\alpha(A, B) = \text{dist}(A_\alpha, B_\alpha)$;

$H(A, B) = \sup_\alpha D_\alpha(A, B)$; Where in the dist is in the sense of Hausdorff distance.

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The function D_α is called a α –distance (induced by d),

H_α is called a α -distance (induced by dist), and H a distance between A and B .

Let $A, B \in W(X)$. An approximate quantity A is more accurate than B , denoted by $A \subset B$ iff $A(x) \leq B(x)$, for each $x \in X$. It is clear that \subset is a partial order relation determined on the family $W(X)$.

Let Y be an arbitrary set and X any metric linear space. F is called fuzzy mapping iff F is a mapping from the set Y into $W(X)$, that is $F(y) \in W(X)$ for each $y \in Y$ and the function value $F(y, x)$ stands for the grade of membership of x in $F(y)$.

Let $A \in F(Y)$, $B \in F(X)$. Then the fuzzy set $F(A)$ in $F(X)$ is defined by

$F(A)(x) = \sup_{y \in Y} (F(y, x) \wedge A(y))$, $x \in X$ and the fuzzy set $F^{-1}(B)$ in $F(Y)$ is defined by

$F^{-1}(B)(y) = \sup_{x \in X} (F(y, x) \wedge B(x))$, $y \in Y$

Lemma 2.1. [8]. Let $x \in X$, $A \in W(x)$ and $\{x\}$ a fuzzy set with membership function equal to a characteristic function of $\{x\}$. If $\{x\} \subset A$, then $D_\alpha(x, A) = 0$ for each $\alpha \in [0, 1]$.

Lemma 2.2. [8]: $D_\alpha(x, A) \leq d(x, y) + D_\alpha(y, A)$ for each $x, y \in X$.

Lemma 2.3. [8]: If $\{x_0\} \subset A$, then $D_\alpha(x_0, B) \leq H_\alpha(A, B)$ for each $B \in W(x)$.

III. MAIN RESULTS

Theorem 3.1. Let X be a complete metric linear space and T be fuzzy mapping from X to $W(x)$ satisfying

$H(T(x), T(y)) + k[D_\alpha(x, T(x)) + D_\alpha(y, T(y))] \leq aD_\alpha(x, T(y)) + bD_\alpha(y, T(x)) + 2cd(x, y)$ for all $x, y \in X$; where $a, b, c, k \geq 0$ and $2b + 2c < 1 + 2k, a + b + c < 1$ and $1 + k > b$. Then there exists a point $z \in X$ such that $\{z\} \subset T(z)$.

Proof.

Take $x_0 \in X$. Let $\{x_1\} \subset T(x_0)$, choose x_2 such that $\{x_2\} \subset T(x_1)$, and

$d(x_1, x_2) \leq H_1(T(x_0), T(x_1))$.

Continuing this process we construct a sequence $\{x_n\}$ such that $\{x_{n+1}\} \subset T(x_n)$ and

$d(x_{n+1}, x_{n+2}) \leq H_1(T(x_n), T(x_{n+1}))$ $n = 0, 1, 2, \dots$

$d(x_n, x_{n+1}) \leq H_1(T(x_n), T(x_{n-1}))$

$\leq H(T(x_n), T(x_{n-1}))$

$\leq aD_\alpha(x_n, T(x_{n-1})) + bD_\alpha(x_{n-1}, T(x_n)) + 2cd(x_n, x_{n-1}) - k[D_\alpha(x_n, T(x_n)) + D_\alpha(x_{n-1}, T(x_{n-1}))]$

$\leq ad(x_n, x_n) + bd(x_{n-1}, x_{n+1}) + 2cd(x_n, x_{n-1}) - k[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)]$

$\leq bd(x_{n-1}, x_n) + bd(x_n, x_{n+1}) + cd(x_n, x_{n-1}) - k[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)]$

$\leq Vd(x_{n-1}, x_n)$ for each $n = 0, 1, 2, \dots$

Where $V = \left(\frac{b+2c-k}{1-b+k} \right) < 1$

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Hence $\{x_n\}$ is a Cauchy sequence in X. It has a limit in X. Call it z. In the view of Lemmas (2.2) - (2.3)

$$\begin{aligned} D_0(z, T(z)) &\leq d(z, x_{n+2}) + H_0(T(z), T(x_{n+1})) \\ &\leq d(z, x_{n+2}) + a D_0(z, T(x_{n+1})) + b D_0(x_{n+1}, T(z)) + 2cd(z, x_{n+1}) - k[D_0(z, T(z)) + D_0(x_{n+1}, T(x_{n+1}))] \\ &\leq d(z, x_{n+2}) + ad(z, x_{n+2}) + bd(x_{n+1}, z) + b D_0(z, T(z)) + 2cd(z, x_{n+1}) - k[D_0(z, T(z)) + d(x_{n+1}, x_{n+2})] \end{aligned}$$

Making $n \rightarrow \infty$ we get $D_0(z, T(z)) \leq JD_0(z, T(z))$, Where $J = b - k$ and $J < 1$, which gives $D_0(z, T(z)) = 0$.

Hence $z \in T(z)$.

Theorem 3.2. Let X be a complete metric linear space and T be fuzzy mapping from X to $W(x)$ satisfying

$$H(T(x), T(y)) \leq a [D_\alpha(x, T(y)) + D_\alpha(y, T(x))] + 2b d(x, y) \text{ for all } x, y \in X; \text{ where } a, b \geq 0 \text{ and } 2a + 2b < 1 \text{ and } a < 1.$$

Then there exists a point $z \in X$ such that $\{z\} \subset T(z)$.

Proof.

Take $x_0 \in X$. Let $\{x_1\} \subset T(x_0)$, choose x_2 such that $\{x_2\} \subset T(x_1)$, and

$$d(x_1, x_2) \leq H_1(T(x_0), T(x_1))$$

Continuing this process we can construct a sequence $\{x_n\}$ such that

$$\{x_{n+1}\} \subset T(x_n), \{x_{n+2}\} \subset T(x_{n+1}), \text{ and}$$

$$d(x_{n+1}, x_{n+2}) \leq H_1(T(x_n), T(x_{n+1})) \quad n = 0, 1, 2, \dots$$

$$\begin{aligned} d(x_n, x_{n+1}) &\leq H_1(T(x_n), T(x_{n-1})) \\ &\leq H(T(x_n), T(x_{n-1})) \\ &\leq a [D_\alpha(x_n, T(x_{n-1})) + D_\alpha(x_{n-1}, T(x_n))] + 2b d(x_n, x_{n-1}) \\ &\leq a [d(x_n, x_n) + d(x_{n-1}, x_{n+1})] + 2bd(x_n, x_{n-1}) \\ &\leq ad(x_{n-1}, x_n) + ad(x_n, x_{n+1}) + 2bd(x_n, x_{n-1}) \end{aligned}$$

$$\leq Cd(x_{n-1}, x_n) \text{ for each } n = 0, 1, 2, \dots$$

$$\text{Where } C = \left(\frac{a+2b}{1-a} \right) < 1.$$

Hence $\{x_n\}$ is a Cauchy sequence in X.

It has a limit in X. Call it z. In the view of Lemmas (2.2) - (2.3) we get

$$\begin{aligned} D_0(z, T(z)) &\leq d(z, x_{n+2}) + H_0(T(z), T(x_{n+1})) \\ &\leq d(z, x_{n+2}) + a [D_0(z, T(x_{n+1})) + D_0(x_{n+1}, T(z))] + 2b d(z, x_{n+1}) \\ &\leq d(z, x_{n+2}) + a [d(z, x_{n+2}) + d(x_{n+1}, z) + D_0(z, T(z))] + 2bd(z, x_{n+1}) \end{aligned}$$

Making $n \rightarrow \infty$ we get $D_0(z, T(z)) \leq a D_0(z, T(z))$. Which gives $D_0(z, T(z)) = 0$.

Hence $z \in T(z)$

Theorem 3.3. Let X be a complete metric linear space and T be fuzzy mapping from X to $W(x)$ satisfying

$$H(T(x), T(y)) + k [D_\alpha(x, T(x)) + D_\alpha(y, T(y))] \leq a D_\alpha(x, T(y)) + b D_\alpha(y, T(x)) + c d(x, y) + d D_\alpha(x, T(x)) + e D_\alpha(y, T(y))$$

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for all $x, y \in C$; where $a, b, c, d, e, k \geq 0$ and $2b + c + e + d - 2k < 1, a + b + c + d + e < 1$ and $1 + k > b + d$.
Then there exists a point $z \in X$ such that $\{z\} \subset T(z)$.

Proof.

Take $x_0 \in X$. Let $\{x_1\} \subset T(x_0)$, choose x_2 such that $\{x_2\} \subset T(x_1)$, and

$$d(x_1, x_2) \leq H_1(T(x_0), T(x_1)).$$

Continuing this process we construct a sequence $\{x_n\}$ such that $\{x_{n+1}\} \subset T(x_n)$ and

$$d(x_{n+1}, x_{n+2}) \leq H_1(T(x_n), T(x_{n+1})) \quad n = 0, 1, 2, \dots$$

$$d(x_n, x_{n+1}) \leq H_1(T(x_n), T(x_{n-1}))$$

$$\leq H(T(x_n), T(x_{n-1}))$$

$$\leq a D_\alpha(x_n, T(x_{n-1})) + b D_\alpha(x_{n-1}, T(x_n)) + c d(x_n, x_{n-1}) + d D_\alpha(x_n, T(x_n))$$

$$+ e D_\alpha(x_{n-1}, T(x_{n-1})) - k [D_\alpha(x_n, T(x_n)) + D_\alpha(x_{n-1}, T(x_{n-1}))]$$

$$\leq a d(x_n, x_n) + b d(x_{n-1}, x_{n+1}) + c d(x_n, x_{n-1}) + d d(x_n, x_{n+1}) + e d(x_{n-1}, x_n) \\ - k [d(x_n, x_{n+1}) + d(x_{n-1}, x_n)]$$

$$\leq b d(x_{n-1}, x_n) + b d(x_n, x_{n+1}) + c d(x_n, x_{n-1}) + d d(x_n, x_{n+1}) + e d(x_{n-1}, x_n) \\ - k [d(x_n, x_{n+1}) + d(x_{n-1}, x_n)]$$

$$\leq Q d(x_{n-1}, x_n) \text{ for each } n = 0, 1, 2, \dots$$

$$\text{Where } Q = \left(\frac{b+c+e-k}{1-b-d+k} \right) < 1$$

Hence $\{x_n\}$ is a Cauchy sequence in X . It has a limit in X . Call it z . In the view of Lemmas (2.2) - (2.3)

$$D_0(z, T(z)) \leq d(z, x_{n+2}) + H_0(T(z), T(x_{n+1})) \\ \leq d(z, x_{n+2}) + a D_0(z, T(x_{n+1})) + b D_0(x_{n+1}, T(z)) + c d(z, x_{n+1}) \\ + d D_0(z, T(z)) + e D_0(x_{n+1}, T(x_{n+1})) - k [D_0(z, T(z)) + D_0(x_{n+1}, T(x_{n+1}))] \\ \leq d(z, x_{n+2}) + a d(z, x_{n+2}) + b d(x_{n+1}, z) + b D_0(z, T(z)) + c d(z, x_{n+1}) \\ + d D_0(z, T(z)) + e D_0(x_{n+1}, x_{n+2}) - k [D_0 d(z, T(z)) + d(x_{n+1}, x_{n+2})]$$

Making $n \rightarrow \infty$ we get

$$D_0(z, T(z)) \leq M D_0(z, T(z)), \text{ where } M = b + d - k \text{ and } M < 1. \text{ Which gives } D_0(z, T(z)) = 0.$$

Hence $z \in T(z)$.

Theorem 3.4. Let X be a complete metric linear space and T be a fuzzy mapping from X to $W(x)$ satisfying

$$H(T(x), T(y)) \leq a [D_\alpha(x, T(y)) + D_\alpha(y, T(x))] + b [D_\alpha(x, T(x)) + D_\alpha(y, T(y))] + c d(x, y) \text{ for all } x, y \in X;$$

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where $a, b \geq 0$ and $2a + 2b + c < 1$ and $a < 1 - b$ Then there exists a point $z \in X$ such that $\{z\} \subset T(z)$.

Proof.

Take $x_0 \in X$. Let $\{x_1\} \subset T(x_0)$, choose x_2 such that $\{x_2\} \subset T(x_1)$, and $d(x_1, x_2) \leq H_1(T(x_0), T(x_1))$

Continuing this process we can construct a sequence $\{x_n\}$ such that

$\{x_{n+1}\} \subset T(x_n), \{x_{n+2}\} \subset T(x_{n+1})$, and

$d(x_{n+1}, x_{n+2}) \leq H_1(T(x_n), T(x_{n+1}))$ $n = 0, 1, 2, \dots$

$$\begin{aligned} d(x_n, x_{n+1}) &\leq H_1(T(x_n), T(x_{n-1})) \\ &\leq H(T(x_n), T(x_{n-1})) \\ &\leq a [D_\alpha(x_n, T(x_{n-1})) + D_\alpha(x_{n-1}, T(x_n))] + b [D_\alpha(x_n, T(x_n)) + D_\alpha(x_{n-1}, T(x_{n-1}))] + c d(x_n, x_{n-1}) \\ &\leq a [d(x_n, x_n) + d(x_{n-1}, x_{n+1})] + b [d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] + c d(x_n, x_{n-1}) \\ &\leq a d(x_{n-1}, x_n) + a d(x_n, x_{n+1}) + b d(x_n, x_{n+1}) + b d(x_{n-1}, x_n) + c d(x_n, x_{n-1}) \\ &\leq N d(x_{n-1}, x_n) \text{ for each } n = 0, 1, 2, \dots \end{aligned}$$

Where $N = \frac{a+b+c}{1-a-b} < 1$. Hence $\{x_n\}$ is a Cauchy sequence in X .

It has a limit in X . Call it z . In the view of Lemmas (2.2) - (2.3) we get

$$\begin{aligned} D_0(z, T(z)) &\leq d(z, x_{n+2}) + H_0(T(z), T(x_{n+1})) \\ &\leq d(z, x_{n+2}) + a [D_0(z, T(x_{n+1})) + D_0(x_{n+1}, T(z))] + b [D_0(z, T(z)) + D_0(x_{n+1}, T(x_{n+1}))] + c d(z, x_{n+1}) \\ &\leq d(z, x_{n+2}) + a [d(z, x_{n+2}) + d(x_{n+1}, z) + D_0(z, T(z))] + b [D_0(z, T(z)) + d(x_{n+1}, x_{n+2})] + c d(z, x_{n+1}) \end{aligned}$$

Making $n \rightarrow \infty$ we get

$$D_0(z, T(z)) \leq S D_0(z, T(z)) \text{ where } S = a + b \text{ and } S < 1. \text{ Which gives } D_0(z, T(z)) = 0.$$

Hence $z \in T(z)$

IV. CONCLUSION

We have proved some fixed point theorems of fuzzy function in complete metric spaces. The presented results generalize the results proved in various spaces and extend some results from the literature.

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