

On Generalized Topological Simple Groups

C. Selvi, R.Selvi

Research Scholar, Department of Mathematics, Sriparasakthi College for Women, Courtallam, India

Assistant Professor, Department of Mathematics, Sriparasakthi College for Women, Courtallam, India

ABSTRACT: In this paper, we introduce the concept of generalized topological simple group. A generalized topological simple group has the algebraic structure of group and topological structure of a generalized topological space defined by A.Csasazar[5]. In this paper we discuss and study some basic results and theorems related to $\mathcal{G}$ -topological simple group.

KEYWORDS: $\mathcal{G}$ -topological group, quotient of $\mathcal{G}$ -topological groups, $\mathcal{G}$ -covering, $\mathcal{G}$ -open covering, $\mathcal{G}$ -closed covering.

I. INTRODUCTION

Csaszar[5] introduced the notions of generalized neighbourhood system and generalized topological spaces. He also introduced the notions of associated interior, closure and continuous mappings on generalized neighbourhood system and generalized topological spaces. In 2013 Muard Hussain et.al.[2], introduced the concept of generalized topological group.

In this paper, we introduce concept of generalized topological simple group. Also here we discuss some examples and basic results related to generalized topological simple group ($\mathcal{G}$ -topological simple group). The notion of Quotient $\mathcal{G}$ -topological simple groups are introduced and studied. Through this paper generalized topology is denoted by $\mathcal{G}$ -topology.

II. PRELIMINARIES

Definition: 2.1[2] Let X be any set and let $\mathcal{G} \subseteq P(X)$ be a subfamily of power set of X . Then $\mathcal{G}$ is called a generalized topology if $\emptyset \in \mathcal{G}$ and for any index set I , $\bigcup_{i \in I} O_i \in \mathcal{G}$, $O_i \in \mathcal{G}$, $i \in I$.

Definition : 2.2[2] The elements of $\mathcal{G}$ are called $\mathcal{G}$ -open sets. Similarly, generalized closed set (or) $\mathcal{G}$ -closed, is defined as complement of a $\mathcal{G}$ -open set.

Definition : 2.3[2] Let X and Y be two $\mathcal{G}$ -topological space. A mapping $f: X \rightarrow Y$ is called a $\mathcal{G}$ -continuous on X if for any $\mathcal{G}$ -open set in Y , $f^{-1}(O)$ is $\mathcal{G}$ -open in X .

Definition : 2.4[2] The bijective mapping f is called a $\mathcal{G}$ -homeomorphism from X to Y if both f and f^{-1} are $\mathcal{G}$ -continuous. If there is a $\mathcal{G}$ -homeomorphism between X and Y , then they are said to be $\mathcal{G}$ -homeomorphic. It is denoted by $X \cong_{\mathcal{G}} Y$.

International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Website: www.ijirset.com

Vol. 6, Issue 7, July 2017

Definition : 2.5[2] Collection of all $\mathcal{G}$ -interior points of $A \subset X$ is called $\mathcal{G}$ -interior of A . It denoted by $Int_{\mathcal{G}}(A)$. By definition it obvious that $Int_{\mathcal{G}}(A) \subset A$.

Note: 2.6 (i). $\mathcal{G}$ -interior of A , $Int_{\mathcal{G}}(A)$, is equal to union of all $\mathcal{G}$ -open sets contained in A .

(ii). $\mathcal{G}$ -closure of A as intersection of all $\mathcal{G}$ -closed sets containing A . It is denoted by $Cl_{\mathcal{G}}(A)$.

Definition: 2.7[2] A triple $(G, *, \mathcal{G})$ is said to be an $\mathcal{G}$ -topological group if $(G, *)$ is a group, $(G, \mathcal{G})$ is a generalized topological space and,

(i). The multiplication mapping $m: G \times G \rightarrow G$ defined by $m(x, y) = x * y$, $x, y \in G$ is $\mathcal{G}$ -continuous.

(ii). The inverse mapping $i: G \rightarrow G$ defined by $i(x) = x^{-1}$, $x \in G$, is $\mathcal{G}$ -continuous.

Definition: 2.8[2] Let X be a set and $\Gamma \subset P(X)$. We say that Γ is a covering of X if $X = \bigcup_{\gamma \in \Gamma} \gamma$. If X is a $\mathcal{G}$ -topological space and every element of Γ is $\mathcal{G}$ -open (or $\mathcal{G}$ -closed) then Γ is called $\mathcal{G}$ -open covering (respectively $\mathcal{G}$ -closed covering).

III. GENERALIZED TOPOLOGICAL SIMPLE GROUPS

Definition: 3.1 [18] A group G is called a Simple group if it has no non trivial normal subgroup.

Definition: 3.2 A $\mathcal{G}$ -topological simple group G is a simple group which is also a $\mathcal{G}$ -topological space if the following conditions are satisfied.

(i). The multiplication mapping $m: G \times G \rightarrow G$ defined by $m(x, y) = x * y$, $x, y \in G$ is $\mathcal{G}$ -continuous.

(ii). The inverse mapping $i: G \rightarrow G$ defined by $i(x) = x^{-1}$, $x \in G$ is $\mathcal{G}$ -continuous.

Example: 3.3 Let $G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ be a trivial simple group under multiplication and we define a generalized topology on G by $\mathcal{G} = \left\{ \varphi, \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \right\}$.

Now $G \times G = \left\{ \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \right\}$ and $\mathcal{G} \times \mathcal{G} = \left\{ \varphi, \left\{ \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \right\} \right\}$.

Clearly the multiplication mapping and inverse mapping is $\mathcal{G}$ -continuous. Therefore $(G, \cdot, \mathcal{G})$ is a generalized topological simple group.

Remark: 3.4 $\mathcal{G}$ -Topological simple group $\Rightarrow$ $\mathcal{G}$ -Topological group, but converse need not be true.

Note: 3.5 (i). $GL_n(F) = \{A \in M_n(F) : \det(A) \neq 0\}$.

(ii). $SL_n(F) = \{A \in GL_n(F) : \det(A) = 1\}$.

International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Website: www.ijirset.com

Vol. 6, Issue 7, July 2017

Example: 3.6 $GL_n(F)$ is a $\mathcal{G}$ -topological group. But it is not $\mathcal{G}$ -topological simple group, because $SL_n(F)$ is a normal subgroup of $GL_n(F)$

Example: 3.7 $SL_n(F)$ is a $\mathcal{G}$ -topological simple group.

Example: 3.8 Z_p , (where p is prime) is $\mathcal{G}$ -topological simple group with indiscrete $\mathcal{G}$ -topology.

Proposition: 3.9 Let G be a $\mathcal{G}$ -topological simple group and μ_e be the collection of all $\mathcal{G}$ -open neighbourhoods of e . Then the following hold:

- (i). For every $U \in \mu_e$, there is an element $V \in \mu_e$ such that $V^{-1} \subset U$.
- (ii). For every $U \in \mu_e$ and for every $x \in U$, there is an element $V \in \mu_e$ such that $Vx \subset U$.

Proposition: 3.10 Every $\mathcal{G}$ -topological simple group G has $\mathcal{G}$ -open neighborhood at the identity element e_G consisting of symmetric $\mathcal{G}$ -neighbourhoods.

Proof: For an arbitrary $\mathcal{G}$ -open neighbourhood U of the identity e_G , if $V = U \cap U^{-1}$, then $V = V^{-1}$, the set V is an $\mathcal{G}$ -open neighbourhood of e_G , which implies that V is a symmetric $\mathcal{G}$ -neighbourhood and $V \subset U$.

Proposition: 3.11 Let G be a $\mathcal{G}$ -topological simple group. Every $\mathcal{G}$ -neighbourhood U of e contains an $\mathcal{G}$ -open symmetric neighbourhood V of e such that $VV \subset U$.

Proof: Let U' be the interior of U . Consider the multiplication mapping $\mu: U' \times U' \rightarrow G$. Since μ is $\mathcal{G}$ -continuous, $\mu^{-1}(U')$ is $\mathcal{G}$ -open and contains (e, e) . Hence there are $\mathcal{G}$ -open sets $V_1, V_2 \subset U$ such that $(e, e) \in V_1 \times V_2$, and $V_1 V_2 \subset U$. If we let $V_3 = V_1 \cap V_2$, then $V_3 V_3 \subset U$ and V_3 is an $\mathcal{G}$ -open neighbourhood of e . Finally, let $V = V_3 \cap V_3^{-1}$, which is $\mathcal{G}$ -open, contains e and V is symmetric and satisfies $VV \subset U$.

Corollary: 3.12 Let G be $\mathcal{G}$ -topological simple group. Every $\mathcal{G}$ -neighbourhood of e_G contains an $\mathcal{G}$ -open symmetric neighbourhood V of e_G such that $VV^{-1} \subset U$ and $V^{-1}V \subset U$.

Proof: Since V is a symmetric $\mathcal{G}$ -open neighbourhood, $V = V^{-1}$. Therefore $VV^{-1} \subset U$ and $V^{-1}V \subset U$.

Proposition: 3.13 Let $f: SL_n(F)$ to $SL_n(F)$ be a $\mathcal{G}$ -homeomorphism. Then

- (i). $A \rightarrow A^{-1}$
- (ii). $A \rightarrow \overline{A}$
- (iii). $A \rightarrow A^T$
- (iv). $A \rightarrow A^*$, where A^* is the conjugate transpose.

Proposition: 3.14 Let G be $\mathcal{G}$ -topological simple group. Then the following maps are $\mathcal{G}$ -homeomorphism from G to G for all $g \in G$.

International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Website: www.ijirset.com

Vol. 6, Issue 7, July 2017

(i). The right(left) translation $l_g(r_g)$ of G by g is a $\mathcal{G}$ -homeomorphism of the space G onto itself.

(ii). The inverse map : $x \rightarrow x^{-1}$.

(iii). The inner automorphism map : $x \rightarrow gxg^{-1}$.

Corollary: 3.15 Let G be $\mathcal{G}$ -topological simple group and U be $\mathcal{G}$ -open subset of G , F is closed in G and A be any subset of G . Then

(i). aU, Ua and AU, UA are $\mathcal{G}$ -open in G .

(ii). aF and Fa are $\mathcal{G}$ -closed in G .

Corollary: 3.16 Let μ_e be a collection of all $\mathcal{G}$ -open sets of G at e . Then $\mu_g = \{ Ug : U \in \mu_e \}$ is also a collection of $\mathcal{G}$ -open sets at g .

Corollary: 3.17 Suppose that a subgroup of $\mathcal{G}$ -topological simple group G contains a non-empty $\mathcal{G}$ -open subset of G . Then H is $\mathcal{G}$ -open in G .

Proof: Let U be a $\mathcal{G}$ -open non-empty subset of G with $U \subset H$. For every $a \in H$, By corollary 3.15, the set $l_a(U) = Ua$ $\mathcal{G}$ -open in G , then $H = \cup_{a \in H} Ua$ is $\mathcal{G}$ -open in G .

Proposition: 3.18 Let $f: G \rightarrow H$ be a homomorphism of $\mathcal{G}$ -topological simple groups. If f is $\mathcal{G}$ -continuous at the neutral element e_G of G , then f is $\mathcal{G}$ -continuous.

Proof: Let $x \in G$ be arbitrary and suppose that W is an $\mathcal{G}$ -open neighbourhood of $y = f(x)$ in H . Since the left translation l_y is a $\mathcal{G}$ -homeomorphism of H , there exists an $\mathcal{G}$ -open neighbourhood V of the neutral element e_H in H such that $l_y(V) = yV$ is an $\mathcal{G}$ -open neighbourhood of y . Then $yV \subseteq W$. Since f is $\mathcal{G}$ -continuous at e_G of G , then $f(U) \subset V$, for some $\mathcal{G}$ -open neighbourhood U of e_G in G . since l_x is a $\mathcal{G}$ -homeomorphism of G onto itself, then xU is an $\mathcal{G}$ -open neighbourhood of x in G .

Now we have $f(xU) = f(x)f(U)$

$$= y f(U)$$

$$\subseteq yU$$

$$\subseteq W. \text{ Hence } f \text{ is } \mathcal{G}\text{-continuous at the point } x \in G.$$

Proposition: 3.19 $SL_n(F)$ is an $\mathcal{G}$ -open subset of $M_n(F)$.

Proof: Let A be an element of $M_n(F)$ and let $f(A) = \det(A)$ be a function from $M_n(F)$ to F . Since $SL_n(F)$ contains the matrices of $M_n(F)$ with determinant 1. Then $SL_n(F) = M_n(F) \setminus B$, where B is a matrix with $\det(B) \neq 1$. Since the determinant function is a polynomials and polynomials are $\mathcal{G}$ -continuous, f is $\mathcal{G}$ -continuous. Since $\{0\}$ is $\mathcal{G}$ -closed in F , then $f^{-1}(\{0\})$ is also $\mathcal{G}$ -closed in $M_n(F)$. Hence $SL_n(F)$ is $\mathcal{G}$ -open.

International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Website: www.ijirset.com

Vol. 6, Issue 7, July 2017

Proposition: 3.20 Suppose that G, H and K are $\mathcal{G}$ -topological simple groups and that $\phi: G \rightarrow H$ and $\psi: G \rightarrow K$ are homomorphism Such that $\psi(G) = K$ and $\text{Ker } \psi \subset \text{Ker } \phi$. Then there exists homomorphism $f: K \rightarrow H$ such that $\phi = f \circ \psi$. In addition, for each $\mathcal{G}$ -neighbourhood U of the identity element e_H in H , there exists a $\mathcal{G}$ -neighbourhood V of the identity element e_K in K such that $\psi^{-1}(V) \subset \phi^{-1}(U)$, then f is $\mathcal{G}$ -continuous.

Proof: Suppose U is $\mathcal{G}$ -neighbourhood of e_H in H . By assumption, there exists a $\mathcal{G}$ -neighbourhood V of the identity element e_K in K such that $W = \psi^{-1}(V) \subset \phi^{-1}(U)$.

$$\Rightarrow \phi(W) = \phi(\psi^{-1}(V)) \subset \phi(\phi^{-1}(U))$$

$$\Rightarrow \phi(W) = f(V) \subset U$$

$\Rightarrow f(V) \subset U$. Hence f is $\mathcal{G}$ -continuous at the identity element of K . Therefore f is $\mathcal{G}$ -continuous.

Corollary: 3.21 Let $\phi: G \rightarrow H$ and $\psi: G \rightarrow K$ be $\mathcal{G}$ -continuous homomorphism of a $\mathcal{G}$ -topological simple groups G, H and K Such that $\psi(G) = K$ and $\text{Ker } \psi \subset \text{Ker } \phi$. If the homomorphism ψ is $\mathcal{G}$ -open, then there exists a $\mathcal{G}$ -continuous homomorphism, $f: K \rightarrow H$ such that $\phi = f \circ \psi$.

Proof: The existence of a homomorphism $f: K \rightarrow H$ such that $\phi = f \circ \psi$. Take an arbitrary $\mathcal{G}$ -open set V in H . Then $f^{-1}(V) = \psi(\phi^{-1}(V))$. Since ϕ is $\mathcal{G}$ -continuous and ψ is an $\mathcal{G}$ -open map, $f^{-1}(V)$ is $\mathcal{G}$ -open in K . Therefore f is $\mathcal{G}$ -continuous.

Proposition: 3.22 Let G and H be $\mathcal{G}$ -topological simple groups with neutral element e_G and e_H , respectively, and let p be a $\mathcal{G}$ -continuous homomorphism of G onto H such that, for some non-empty subset U of G , the set $p(U)$ is $\mathcal{G}$ -open in H and the restriction of p to U is an $\mathcal{G}$ -open mapping of U onto $p(U)$. Then the homomorphism p is $\mathcal{G}$ -open.

Proof: It suffices to show that $x \in G$, where W is an $\mathcal{G}$ -open neighbourhood of x in G , then $P(W)$ is a $\mathcal{G}$ -open neighbourhood of $p(x)$ in H . Fix a point y in U , and let l be the left translation of G by yx^{-1} . Then l is a $\mathcal{G}$ -homomorphism of G onto itself such that,

$$\begin{aligned} l_{yx^{-1}}(x) &= yx^{-1}x \\ &= y. \end{aligned}$$

So $V = U \cap l(W)$ is an $\mathcal{G}$ -open neighbourhood of y in U . Then $p(V)$ is $\mathcal{G}$ -open subset of H . consider the left translation h of H by the inverse to $p(yx^{-1})$.

Now clearly, $(h \circ p \circ l)(x) = h(p(l(x)))$

$$\begin{aligned} &= h(p(y)) \\ &= p(xy^{-1})p(y) \\ &= p(xy^{-1}y) \\ &= p(x). \end{aligned}$$

International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Website: www.ijirset.com

Vol. 6, Issue 7, July 2017

Hence $h(p(l(W))) = p(W)$. Clearly h is a $\mathcal{G}$ -homeomorphism of H onto itself. Since $p(V)$ is $\mathcal{G}$ -open in H , $h(p(V))$ is also $\mathcal{G}$ -open in H . Therefore $p(W)$ contains the $\mathcal{G}$ -open neighbourhood $h(p(V))$ of $p(x)$ in H . Hence $p(W)$ is a $\mathcal{G}$ -open neighbourhood of $p(x)$ in H .

Proposition: 3.23 Let G be $\mathcal{G}$ -topological simple group. Then

- (i). If H is a subgroup of G , then $\bar{H}$ also a subgroup of G .
- (ii). If H is a normal subgroup of G , $\bar{H}$ also a normal subgroup of G .

Proof: (i). For subgroup we have to prove the following conditions.

- (a). $\bar{H}$ is non empty.
- (b). $x, y \in \bar{H} \Rightarrow xy \in \bar{H}$
- (c). $x \in \bar{H} \Rightarrow x^{-1} \in \bar{H}$

(a). Since $e \in H \Rightarrow e \in \bar{H}$. So $\bar{H}$ is non empty.

(b). Let $g, h \in \bar{H}$. Let U be an $\mathcal{G}$ -open neighbourhood of gh . Let $\mu : G \times G \rightarrow G$ denote the multiplication map which is $\mathcal{G}$ -continuous. Then $\mu^{-1}(U)$ is $\mathcal{G}$ -open in $G \times G$ and contains (g, h) . so, there are $\mathcal{G}$ -neighbourhood V_1 of g and V_2 of h such that $V_1 \times V_2 \subset \mu^{-1}(U)$. Since $g, h \in \bar{H}$, then there are points $x \in V_1 \cap H \neq \phi$ and $y \in V_2 \cap H \neq \phi$. Since $x, y \in H, xy \in H$ and since $(x, y) \in \mu^{-1}(U)$, then $xy \in U$. Thus $xy \in U \cap H \neq \phi$. Since U is an arbitrary $\mathcal{G}$ -open neighbourhood of gh , then we have $gh \in \bar{H}$.

(c). Now let $i : G \rightarrow G$ denote the inverse map, and W be an $\mathcal{G}$ -open neighbourhood of h^{-1} . Then

$i^{-1}(W) = W^{-1}$ is $\mathcal{G}$ -open and contains h , so there is a point $z \in H \cap W^{-1} \neq \phi$. Then we have $z^{-1} \in H \cap W \neq \phi$. Therefore $h^{-1} \in \bar{H}$.

(ii). Now we have to prove that $g\bar{H}g^{-1} \in \bar{H} \forall g \in G$.

Since H is a normal subgroup of G , $gHg^{-1} \in H \forall g \in G$.

Now $\overline{gHg^{-1}} \subset \bar{H} \forall g \in G$.

$\Rightarrow g\bar{H}g^{-1} \subset \bar{H} \forall g \in G$.

$\Rightarrow g\bar{H}g^{-1} \in \bar{H}, \forall g \in G$. Therefore $\bar{H}$ is a normal subgroup of G .

Corollary: 3.24 Let G be $\mathcal{G}$ -topological simple group and H be the centre of a hausdorff $\mathcal{G}$ -topological simple group G . Then $\bar{H}$ is a subgroup of G .

Corollary: 3.25 Let G be $\mathcal{G}$ -topological simple group $Z(G)$ be the centre of G . Then $\overline{Z(G)}$ is a normal subgroup of G .

International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Website: www.ijirset.com

Vol. 6, Issue 7, July 2017

Proof: proof follows from the above theorem.

Corollary: 3.26 Let G and H be $\mathcal{G}$ -topological simple groups. If $f: G \rightarrow H$ is a homomorphism mapping, then $\overline{\ker f}$ is a normal subgroup of G .

IV. QUOTIENTS ON A $\mathcal{G}$ -TOPOLOGICAL SIMPLE GROUPS

Let G be a $\mathcal{G}$ -topological simple group and H be a normal subgroup of G . Here H is either proper trivial normal subgroup of G or improper trivial normal subgroup of G . Let ϕ be a mapping from $G \rightarrow \frac{G}{H}$ by $\phi(x) = xH, \forall x \in G$. Now we can define a $\mathcal{G}$ -topology on $\frac{G}{H}$, U is $\mathcal{G}$ -open in $\frac{G}{H} \Leftrightarrow \phi^{-1}(U)$ is $\mathcal{G}$ -open in G .

Theorem: 4.1 Let $\frac{G}{H}$ be a $\mathcal{G}$ -topological simple group with quotient $\mathcal{G}$ -topology and $\phi: G \rightarrow \frac{G}{H}$ by $\phi(x) = xH, \forall x \in G$. Then the following statement is hold.

- (i). ϕ is onto.
- (ii). ϕ is $\mathcal{G}$ -continuous.
- (iii). ϕ is $\mathcal{G}$ -open.
- (iv). ϕ is homomorphism.

Proof: (i). Let $gH \in \frac{G}{H}$. Since $\phi(x) = gH, \forall g \in G$. Hence ϕ is onto.

(ii). By definition of quotient $\mathcal{G}$ -topology, U is $\mathcal{G}$ -open in $\frac{G}{H} \Leftrightarrow \phi^{-1}(U)$ is $\mathcal{G}$ -open in G . Hence ϕ is $\mathcal{G}$ -continuous.

(iii). Let U be a $\mathcal{G}$ -open set in G . Then $\phi(U) = UH$ is $\mathcal{G}$ -open in $\frac{G}{H}$.

(iv). Let $x, y \in G, \phi(xy) = xyH$.

$$\begin{aligned} &= xHyH \\ &= \phi(x)\phi(y). \end{aligned}$$

Hence ϕ is a homomorphism.

Theorem: 4.2 Let G be a $\mathcal{G}$ -topological simple group and let H be a proper trivial normal subgroup of G or improper trivial normal subgroup of G . Then $\frac{G}{H}$ with the quotient $\mathcal{G}$ -topology is a $\mathcal{G}$ -topological simple group.

Proof: To show that $\frac{G}{H}$ is a $\mathcal{G}$ -topological simple group. We need to show that the map defined by $m(xH, yH) \rightarrow xy^{-1}H$ is $\mathcal{G}$ -continuous. Let WH be an $\mathcal{G}$ -open neighbourhood of $xy^{-1}H$. By definition of quotient $\mathcal{G}$ -topology, $\phi^{-1}(WH)$ is $\mathcal{G}$ -open in G . Also $xy^{-1}H \in WH$.

International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Website: www.ijirset.com

Vol. 6, Issue 7, July 2017

$$\Rightarrow xy^{-1} \in \phi^{-1}(WH).$$

By assumption, G is a $\mathcal{G}$ -topological simple group, which there exists $\mathcal{G}$ -open sets U and V with $x \in U$ and $y \in V$ such that $UV^{-1} \subset \phi^{-1}(WH)$. Since $xy^{-1} \in UV^{-1}$, $xy^{-1}H \in \phi(UV^{-1})$.

$$\text{Now } UV^{-1} \subset \phi^{-1}(WH).$$

$$\Rightarrow \phi(UV^{-1}) \subset \phi(\phi^{-1}(WH)) = WH.$$

$$\Rightarrow \phi(UV^{-1}) \subset WH.$$

Combining these two facts, $xy^{-1} \in \phi(UV^{-1}) \subset WH$.

Since ϕ is a homomorphism, $\phi(UV^{-1}) = \phi(U)\phi(V^{-1})$.

$\Rightarrow \phi(U)\phi(V^{-1}) \subset WH$. We need to show that $\phi(U)$ and $\phi(V^{-1})$ are $\mathcal{G}$ -neighbourhoods of xH and $y^{-1}H$ respectively. Now $xH \in \phi(U)$ and $y^{-1}H \in \phi(V^{-1})$.

$\Rightarrow x \in U$ and $y^{-1} \in V^{-1}$. Since U and V^{-1} are $\mathcal{G}$ -open in G , $\phi(U)$ and $\phi(V^{-1})$ are $\mathcal{G}$ -open. Hence the theorem.

V. $\mathcal{G}$ -COMPACTNESS ON $\mathcal{G}$ -TOPOLOGICAL SIMPLE GROUP

Proposition: 5.1 If G is a $\mathcal{G}$ -topological simple group and if K_1 and K_2 are $\mathcal{G}$ -compact subsets of G , then K_1K_2 is $\mathcal{G}$ -compact.

Proof: Since K_1, K_2 are $\mathcal{G}$ -compact subsets of G , then $K_1 \times K_2$ is $\mathcal{G}$ -compact in $G \times G$. since the $\mathcal{G}$ -continuous image of a compact set is compact, K_1K_2 is $\mathcal{G}$ -compact.

Definition: 5.2 Let H be a subgroup of a $\mathcal{G}$ -topological simple group G . Then H is called neutral in G if every $\mathcal{G}$ -neighbourhood U of the identity e_G in G , there exists a $\mathcal{G}$ -neighbourhood V of e_G such that $VH \subset HU$.

Proposition: 5.3 Let H be a subgroup of $\mathcal{G}$ -topological simple group G . Suppose that, for every $\mathcal{G}$ -open neighbourhood U of the identity e_G in G , there exists an $\mathcal{G}$ -open neighbourhood V of e_G in G such that $xVx^{-1} \subset U$ whenever $x \in G$. Then H is neutral in G .

Proof: Given a $\mathcal{G}$ -neighbourhood U of e_G in G . Take an $\mathcal{G}$ -open neighbourhood V of e_G satisfying,

$$xVx^{-1} \subset U, \forall x \in G$$

$$\Rightarrow xV \subset Ux, \forall x \in G$$

$$\Rightarrow HV \subset UH, \forall x \in G. \text{ Then } H \text{ is neutral in } G.$$

Corollary: 5.4 Every $\mathcal{G}$ -compact subgroup of $\mathcal{G}$ -topological simple group G is neutral in G .

Proposition: 5.5 Let G be $\mathcal{G}$ -topological simple group, F a $\mathcal{G}$ -closed subset of G , and K is a $\mathcal{G}$ -compact subset of G , such that $F \cap K = \phi$. Then there is an $\mathcal{G}$ -open neighbourhood V of e_G such that $F \cap VK = \phi$.

International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Website: www.ijirset.com

Vol. 6, Issue 7, July 2017

Proof: Let $x \in K$.

$\Rightarrow x \in G \setminus F$ and $G \setminus F$ is $\mathcal{G}$ -open.

$\Rightarrow (G \setminus F)x^{-1}$ is an $\mathcal{G}$ -open neighbourhood of e_G . By proposition 3.11,

there is an $\mathcal{G}$ -open neighbourhood W_x of e_G such that $W_x W_x \subset (G \setminus F)x^{-1}$. Since K is $\mathcal{G}$ -compact and $K \subset \bigcup_{x \in K} W_x x$, so there are finite points $x_1, x_2, \dots, x_n \in K$, such that

$K \subset \bigcup_{i=1}^n W_i x_i$, where $W_i = W_{x_i}$. Now let $V = \bigcap_{i=1}^n W_i$. For any $x \in K, x \in W_i x_i$, for some i . This implies that,

$Vx \subset W_i x \subset W_i W_i x_i \subset G \setminus F$.

$\Rightarrow F \cap Vx = \emptyset$. This true for any $x \in K$, therefore $F \cap VK = \emptyset$.

Proposition: 5.6 Let G be $\mathcal{G}$ -topological simple group, K a $\mathcal{G}$ -compact subset of G , and F is a $\mathcal{G}$ -closed subset of G . Then FK and KF are $\mathcal{G}$ -closed subsets of G .

Proof: If $FK = G$, this is obviously true. Now let $y \in G \setminus FK$. Then $F \cap yK^{-1} = \emptyset$. Since K is $\mathcal{G}$ -compact, yK^{-1} is $\mathcal{G}$ -compact. By above theorem, there is an $\mathcal{G}$ -open neighbourhood V of e_G such that $F \cap VyK^{-1} = \emptyset$. Since Vy is an $\mathcal{G}$ -open neighbourhood of y contained in $G \setminus FK$. Therefore FK is $\mathcal{G}$ -closed.

Proposition: 5.7 Every $\mathcal{G}$ -open subgroup H of $\mathcal{G}$ -topological simple group G is $\mathcal{G}$ -closed in G .

Proof: The family $\gamma = \{Ha: a \in G\}$ of all right cosets of H in G is a disjoint $\mathcal{G}$ -open covering of G . Therefore every element of γ , the complement of the union of all other elements, is $\mathcal{G}$ -closed in G .

Proposition: 5.8 Let G be a $\mathcal{G}$ -compact $\mathcal{G}$ -topological simple group and H be a $\mathcal{G}$ -topological simple group and let $f: G \rightarrow H$ be $\mathcal{G}$ -continuous. Then $f(G)$ is $\mathcal{G}$ -compact.

Proof: Let γ be an $\mathcal{G}$ -open cover for $f(G)$. We know that $\{f^{-1}(U): U \in \gamma\}$ is an $\mathcal{G}$ -open cover of G . Since G is $\mathcal{G}$ -compact, there is a finite $\mathcal{G}$ -subcover. There exists $U_1, U_2, \dots, U_n \in \gamma$ such that

$$G \subseteq f^{-1}(U_1) \cup f^{-1}(U_2) \cup \dots \cup f^{-1}(U_n)$$

$\Rightarrow f(G) \subseteq U_1 \cup U_2 \cup \dots \cup U_n$. Thus an arbitrary $\mathcal{G}$ -open cover of $f(G)$ has a finite $\mathcal{G}$ -subcover.

Proposition: 5.9 Let G be a $\mathcal{G}$ -compact $\mathcal{G}$ -topological simple group and let H be $\mathcal{G}$ -closed subset of G . Then H is $\mathcal{G}$ -compact.

Proof: Let γ be an $\mathcal{G}$ -open cover of H . Since H is $\mathcal{G}$ -closed in G , $G \setminus H$ is $\mathcal{G}$ -open in G . From this we get that $\gamma \cup G \setminus H$ is an $\mathcal{G}$ -open cover for G . Since G is $\mathcal{G}$ -compact, there must be a finite $\mathcal{G}$ -subcover. So there exists $U_1, U_2, \dots, U_n \in \gamma$, such that $G \subset U_1 \cup U_2 \cup \dots \cup U_n \cup G \setminus H$

$\Rightarrow H \subset U_1 \cup U_2 \cup \dots \cup U_n$. An arbitrary $\mathcal{G}$ -open cover of H has a finite $\mathcal{G}$ -subcover. Therefore H is $\mathcal{G}$ -compact.

International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Website: www.ijirset.com

Vol. 6, Issue 7, July 2017

VI.CONCLUSION

In this paper, we explore the notion of $\mathcal{G}$ -topological simple group. The notion is based upon the topological structure of $\mathcal{G}$ -topological space and algebraic structure of simple groups. This idea is opening for new researchers working in $\mathcal{G}$ -topological simple group.

REFERENCES

1. A.V.Arhangel'skii, M.Tkachenko, *Topological Groups and Related Structures*, Atlantis press/world Scientific, Amsterdampairs, 2008.
2. Muard Hussain, Moiz Ud Din Khan, Cenap Ozel, *On generalized topological groups*, Filomat 27:4(2013),567-575
3. Dylan spivak, *Introduction to topological groups*, Math(4301).
4. J. R. Munkres, *Topology, a first course*, Prentice-Hall, Inc., Englewood cliffs, N.J.,1975.
5. A.Csaszar, *generalized topology, generalized continuity*, Acta Math. Hungar. 96(2002)351-357.
6. A.Csaszar, *γ -connected sets*, Acta Math..Hunger.101(2003) 273-279.
7. A.Csaszar, *A separarion axioms for generalized topologies*, Acta Math.Hungar.104(2004) 63-69.
8. A.Csaszar, *Product of generalized topologies*, Acta Math.Hungar.123(2009)127-132.
9. W.K.Min, *Weak continuity on generalized topological spaces*, Acta Math.Hungar. 124(2009)73-81.
10. L.E.De Arruda Saraiva, *Generalized quotient topologies*, Acta Math.Hungar. 132(2011)168-173.
11. R.Shen, *Remarks on products of generalized topologies*, Acta Math.Hungar.124(2009)363-369.
12. Volker Runde, *A Taste of topology*, Springer(2008).
13. Taqdir Husain, *Introduction to Topological groups*, saundres(1966).
14. David Dummit and Richard Foote, *Abstract Algebra(3rd edition)*, Wiley(2003).
15. Morris Kline, *Mathematical Thought from Ancient to modern times*, Oxford University Press(1972).
16. Pierre Ramond, *Group theory: A physicists survey*, Cambridge(2010).
17. Robert Bartle, *The Elements of Integration and Lebesgue Measure*, Wiley(1995).
18. Joseph A. Gallian, *Contemporary Abstract Algebra*, Narosa(fourth edition).