

Some Results on Total Domination Number of Zero Divisor Graph

K. Budadoddi¹, R. Ramakrishna Prasad², A. Mallikarjuna Reddy³

Assistant Professor, Department of Mathematics, Sri Krishnadevaraya University College of Engineering & Technology, Ananthapuramu, India¹

Research Scholar, Department of Mathematics, Sri Krishnadevaraya University, Ananthapuramu, India²

Professor, Department of Mathematics, Sri Krishnadevaraya University, Ananthapuramu India³

ABSTRACT: Graph theory is one of the most developing branches of mathematics with wide applications to various branches of science and technology. A graph can be used to represent almost any physical situation involving discrete objects and a relationship among them. The origin of graph theory started with the problem of Konigsberg bridge problem in 1735 by the great mathematician Leonard Euler. The theory of domination in graphs introduced by C. Berge and O. Ore which is rapidly growing area of research in graph theory today. The study of domination in graphs originated around 1850 with the problem of determining the minimum number of queens that can be placed on an $n \times n$ chess board so as to cover or dominate every square. The concept of zero divisor graphs of a ring was introduced by I. Beck in which the colouring of a commutative ring is mainly discussed. In this work, the set of vertices is taken to be R and two vertices x and y are adjacent if their product should be zero. The investigation of colourings of a commutative ring was then continued by D. D. Anderson and M. Naseer. In this paper, I investigate some results on total domination number of zero divisor graphs.

KEYWORDS: Dominating set, Domination number, Total domination Set, Total domination Number, Zero divisors, Zero divisor graphs.

I. INTRODUCTION

Image Let R be a commutative ring with unity $1 \neq 0$, $Z(R)$ be the set of all zero-divisors of R and $Z(R)^* = Z(R) - \{0\}$ be the set of nonzero zero-divisors of R . The zero-divisor graph of R denoted by $\Gamma(Z(R)^*)$ is usually written $\Gamma(R)$ whose vertices are the elements of the $Z(R)^*$ and distinct vertices $x, y \in Z(R)^*$ are adjacent if and only if $xy = 0$. This definition defined by David F. Anderson and Philip S. Livingston in [3].

The main object of this unit is to study the inter relation of ring theoretic properties of R with graph theoretic properties of $\Gamma(R)$. This study helps illuminate the structure of $Z(R)$. For $x, y \in Z(R)$, define $x \sim y$ if $xy = 0$ or $x = y$. The relation \sim is always reflexive and symmetric, but is usually not transitive. The zero divisor graph $\Gamma(R)$ measures this lack of transitivity in the sense that \sim is transitive if and only if $\Gamma(R)$ is complete.

The idea of a zero-divisor graph of a commutative ring was introduced by I. Beck in [5], where he was mainly interested in colourings. This investigation of colourings of a commutative ring was then continued by D.D. Anderson and M. Naseer in [4]. Their definition was slightly different than the previous definition. They let all elements of R be vertices and has distinct x and y are adjacent if and only if $xy = 0$. They denote their zero-divisor graph of R by $\Gamma_0(R)$. In $\Gamma_0(R)$, the vertex 0 is adjacent to every other vertex, but nonzero zero-divisor are adjacent only to 0 . Note that $\Gamma(R)$ is a sub graph of $\Gamma_0(R)$.

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II. DOMINATION NUMBER OF ZERO DIVISOR GRAPH

We know that a dominating set of a graph $G(V,E)$ is a subset D of $V(G)$ such that every vertex in $V-D$ is adjacent to at least one vertex in D .

The minimum cardinality of a dominating set of G is called the domination number of G and it is denoted by $\gamma(G)$.

III. TOTAL DOMINATION NUMBER OF ZERO DIVISOR GRAPH

Exemplar Let G be a graph without isolated vertices. Then a Total dominating set D is a subset of V such that every vertex of V is adjacent to some vertex in D . The minimum cardinality of a Total dominating set of G is called the Total domination number of G and it is denoted by $\gamma_t(G)$. Total domination number was introduced by E.J.Cockayne, R.W.Dawes and S.T.Hedetniemi in the year 1980 for a survey of a total domination see [7].

Theorem 3.1: If $n=2p$ where p is a prime and $p \geq 3$. Then the Total domination number of non-zero zero divisor graph is 2.

Proof: Let $n=2p$ where p is a prime and $p \geq 3$. Then the vertex set $V = Z(R^*) = \{2, 4, 6, \dots, 2(p-1), p\}$. Edge set $E(G)$ defined by $E(G) = \{(x, y) : xy = 0 \text{ (multiplication modulo } n), \forall x, y \in V\}$.

Decompose the vertex set V into the following disjoint subsets.

1. The set S of non-zero even integers which are less than n .
2. The set M of prime number p .

We know that $D = \{p\}$ is a dominating set, but it is not a total dominating set since $p \in V$ is not adjacent in D . Let us consider another vertex x in S . It is clear that p and x are adjacent.

Let $D = \{(x, p) / x \in S \text{ and } p \in M\}$. The vertex p is adjacent to the vertex x and vice-versa.

\therefore Every vertex in V is adjacent to at least one vertex in D .

$\therefore D$ forms a total dominating set. Continuing like this $D = \{2, 4, 6, \dots, (2p-3)\}$ forms a total dominating set.

$\therefore D = \{x, p\}$ is a minimum total dominating set with cardinality 2.

\therefore The total domination number is 2 when $n=2p$, where p is an odd prime.

$\therefore \gamma_t(\Gamma(Z_n)) = 2$.

Theorem 3.2: If $n=3p$, where p is a prime and $p > 3$. Then the total domination number of non-zero zero divisor graph is 2.

Proof: Let $n=3p$ where p is a prime and $p > 3$. Then the vertex set $V = Z(R^*) = \{3, 6, 9, \dots, (3p-3), p, 2p\}$. Decompose the vertex V in to the following disjoint subsets.

1. The set S of non-zero multiples of 3 which are less than n .
2. The set M of multiples of prime p which are less than n .

The edge set $E(G)$ defined by $E(G) = \{(x, y) : xy = 0 \text{ multiplication modulo } n, \forall x, y \in V\}$. Let us consider two distinct vertices $x, y \in S$, where the vertices of S are only multiples of 3. The product of xy is divisible by 3, but not $3p$. Hence two distinct vertices of S are not adjacent.

Now let us take $x \in S$ and $y \in M$ where M is the multiples of prime p . The product of xy is divisible by $3p$. Hence one vertex in S and other vertex in M are adjacent.

\therefore The vertices of M are adjacent to all the vertices of S and vice-versa.

$D = \{(x, p) / x \in S \text{ and } p \in M\}$ is a dominating set. It is clear that every vertex in V is adjacent to exactly one vertex in D .

$\therefore D = \{x, p\}$ is a minimum total dominating set with cardinality 2.

The total domination number of non-zero zero divisor graph is 2.

$\therefore \gamma_t(\Gamma(Z_n)) = 2$.

Theorem 3.3: If $n = p_1 \times p_2$ where p_1 and p_2 are distinct odd prime numbers and $p_1 < p_2$, then the total domination number of non-zero zero divisor graph is 2.

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Proof: Let $n = p_1 \times p_2$ where p_1 and p_2 are distinct odd prime numbers. Then the vertex set $V = Z(R^*) = \{p_1, 2p_1, 3p_1, \dots, (mp_1 - m), p_2, 2p_2, \dots, (n - 1)p_2\}$. Decompose the vertex V into the following disjoint subsets.

1. The set S of non-zero multiples of p_1 which are less than n .
2. The set M of multiples of prime p_2 which are less than n .

The edge set $E(G)$ defined by $E(G) = \{(x, y) : xy = 0 \pmod{n}, \forall x, y \in V\}$.

Let us consider two distinct vertices $x, y \in S$ where the vertices of S are multiples of p_1 . The product of xy is divisible by p_1 but not $n = p_1 \times p_2$. Hence two distinct vertices of S are not adjacent.

Now let us take $x, y \in M$ where the vertices of M are multiples of prime p_2 . The product of xy is divisible by p_2 but not $n = p_1 \times p_2$. Hence two distinct vertices of M are not adjacent.

Now let us take $x \in S$ and $y \in M$ where M is the multiples of prime p . The product of xy is divisible by $n = p_1 \times p_2$. Hence one vertex in S and other of M are adjacent.

\therefore The vertices of M are adjacent to all the vertices of S and vice-versa.

$D = \{(x, y) / x \in S \text{ and } y \in M\}$ is a dominating set. It is clear that $D = \{(x, y) / x \in S \text{ and } y \in M\}$ forms a total dominating set.

$D = \{x, y\}$ is a minimum total dominating set of $\Gamma(Z_n)$ with cardinality 2.

\therefore Total domination number of $\Gamma(Z_n) = 2$.

$\therefore \gamma_t(\Gamma(Z_n)) = 2$.

Theorem 3.4: If $n = p^2$ where p is an odd prime. Then the total domination number of non-zero zero divisor graph is 2.

Proof: Let $n = p^2$, where p is an odd prime. Then the vertex set of $Z(R^*)$ is $V = Z(R^*) = \{p, 2p, 3p, \dots, p(p - 1)\}$. When we draw the graph we get a complete graph on $p - 1$ vertices. If graph is complete then D forms a total dominating set, D must contain at least two vertices.

$\therefore D = \{p, 2p\}$ forms a total dominating set since every vertex of V is adjacent to at least one vertex of D .

$\therefore D = \{p, 2p\}$ is a minimum total dominating set with cardinality 2. Continuing like this $D = \{p, 2p, 3p, \dots, p(p - 1)\}$ is also a total dominating set.

\therefore The total domination number of $\Gamma(Z_n)$ is 2.

$\therefore \gamma_t(\Gamma(Z_n)) = 2$.

Theorem 3.5: If $n = p^\alpha$ where p is an odd prime and $\alpha > 2$. Then the total domination number of non-zero zero divisor graph is 2.

Proof: Let $n = p^\alpha$ where p is an odd prime and $\alpha > 2$. If $n = p^\alpha$ then $x = p^{\alpha-1}$ is adjacent to every other vertex in $\Gamma(Z_n)$. So $D = \{p^{\alpha-1}\}$ is a minimum dominating set. Let us consider another vertex in $\Gamma(Z_n)$. Now $D = \{p^{\alpha-1}, x\}$ is a total dominating set in every vertex of V is adjacent to at least one vertex of D .

$\therefore D = \{p^{\alpha-1}, x\}$ is a minimum total dominating set with cardinality 2.

Hence the total domination number of $\Gamma(Z_n)$ is 2 when $n = p^\alpha$ where p is an odd prime and $\alpha > 2$.

$\therefore \gamma_t(\Gamma(Z_n)) = 2$.

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