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A Study of Ground State Correlation for Even Nuclei

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ABSTRACT

To a certain extent, a way out from the explosion of dimensionality, may consist in using a better single-particle space.

Instead of parametrizing fields a_x^+ by space-spin-isospin points \mathbf{x} , one can use a parametrization by the shell-model orbitals $\phi_i(\mathbf{x})$ that are active near the Fermi surface of a given nucleus, i.e., by fields

$$a_i^+ = \int dx \phi_i(x) a_x^+$$

When a complete set of orbitals is used, the descriptions in terms of creation operators a_i^+ and a_x^+ are equivalent.

However, one can also attempt a drastic reduction of the set a_i^+ to a finite number, $i=1 \dots M$, of "most important" orbitals, similarly as we have been previously using finite sets of the space-spin-isospin points instead of continuous variables.

The reduction is now not a mere question of discretizing continuous fields, but involves a serious limitation of the Hilbert space. In quantum mechanics one can always split the Hilbert space into two

subspaces, $|\Psi\rangle = P|\Psi\rangle + Q|\Psi\rangle$, where P and Q are projection operators such that $P + Q = 1$.

Then, the Schrödinger equation $H|\Psi\rangle = E|\Psi\rangle$ is strictly equivalent to the following 2×2 matrix of equations,

$$\begin{pmatrix} PHP & PHQ \\ QHP & QHQ \end{pmatrix} \begin{pmatrix} P|\Psi\rangle \\ Q|\Psi\rangle \end{pmatrix} = E \begin{pmatrix} P|\Psi\rangle \\ Q|\Psi\rangle \end{pmatrix} .$$

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Using the second equation, one can now formally express the "excluded" component, $|\Psi_Q\rangle \equiv Q|\Psi\rangle$, of the wave function by the "kept" component, $|\Psi_P\rangle \equiv P|\Psi\rangle$, i.e.,

$$|\Psi_Q\rangle = \frac{1}{E - QH} QH|\Psi_P\rangle,$$

and put it back into the first equation. This gives the Schrödinger equation reduced to the "kept" Hilbert space,

$$H_{\text{eff}}|\Psi_P\rangle = E|\Psi_P\rangle,$$

where the effective Hamiltonian H_{eff} is given by the Bloch-Horowitz equation Blo58,

$$H_{\text{eff}} = H + H \frac{1}{E - QH} QH .$$

The main question is, of course, whether the Bloch-Horowitz effective interaction, $V_{\text{eff}} = H_{\text{eff}} - T$, can be replaced by a simple phenomenological interaction, and used to describe real systems. In particular, when a two-body, energy-independent interaction is postulated in a very small phase space, one obtains the shell model, which is successfully used since many years in nuclear structure physics.

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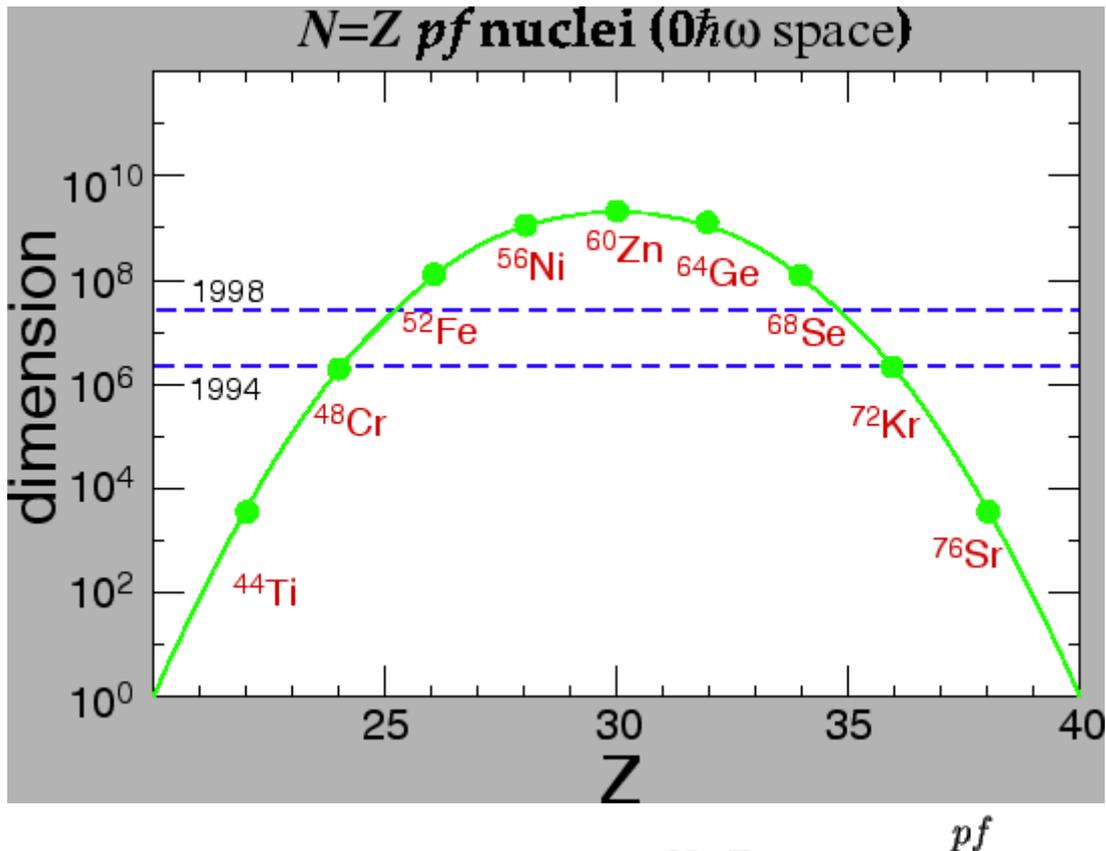


Figure 1: Dimension of the shell-model space for calculations of $N = Z$ nuclei within the pf space. (Picture courtesy: W. Nazarewicz, ORNL/University of Tennessee/Warsaw University.) From <http://www-highspin.phys.utk.edu/~witek/>.

In order to illustrate the dimensions of the shell-model Hilbert space, in Fig. 1 we show the numbers of many-fermion states that are obtained when states in $N = Z$ medium heavy nuclei are described within the pf space (20 s.p. states for protons and 20 for neutrons). Currently, complete solutions for the pf space become available, i.e., dimensions of the order of 10^9 can effectively be treated. Progress in this domain closely follows the progress in size and speed of computers, i.e., one order of magnitude is gained in about every two-three years.

KEYWORDS: ground state, even, nuclei, study, neutrons, shell, model, fermion, protons, heavy

I. INTRODUCTION

The Hartree-Fock (HF) approach relies on assuming that the ground state of a many-fermion system can be uniquely characterized by the one-body density matrix. There are many ways of deriving the HF equations; the simplest one is to use the variational principle together with the following approximation of the two-body density matrix :

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$$\rho_{x'y'xy} = \rho_{x'x}\rho_{y'y} - \rho_{x'y}\rho_{y'x}.$$

This equation expresses the two-body density matrix by the one-body density matrix, and hence the total energy becomes a functional of the one-body density matrix only,[1,2]

$$\begin{aligned} E_{\text{HF}} &= T_{xy}\rho_{yx} + \frac{1}{4}G_{xyx'y'}(\rho_{x'x}\rho_{y'y} - \rho_{x'y}\rho_{y'x}) \\ &= T_{xy}\rho_{yx} + \frac{1}{2}\Gamma_{xx'}\rho_{x'x} = \frac{1}{2}(T_{xy} + h_{xy})\rho_{yx}, \end{aligned}$$

for

$$\begin{aligned} \Gamma_{xx'} &= G_{xyx'y'}\rho_{y'y} && \Leftarrow \text{HF potential,} \\ h_{xy} &= T_{xy} + \Gamma_{xy} && \Leftarrow \text{HF Hamiltonian.} \end{aligned}$$

By minimizing the HF energy with respect to the one-body density matrix, one obtains

$$h_{xy}\rho_{yz} - \rho_{xy}h_{yz} = 0 \quad \Leftarrow \text{HF equation,}$$

which is usually solved by finding the HF s.p. orbitals that diagonalize the HF Hamiltonian

$$\int dy h_{xy}\phi_i(y) = \epsilon_i\phi_i(x),$$

and then constructing the one-body density matrix from these orbitals:

$$\rho_{xy} = \sum_{i \in \text{occ}} \phi_i(x)\phi_i^*(y).$$

Equations above guarantee that the HF condition is fulfilled (because h_{xy} and ρ_{xy} are then diagonal in the common basis), so the HF solution is found whenever, for a given set of occupied orbitals, $i \in \text{occ}$, the density matrix self-consistently reproduces the HF potential. From Eq. it is clear that not the real interaction $V_{xyx'y'}$, but the effective

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interaction $G_{xyx'y'}$, must be used in the HF method. Indeed, when the density-matrix is inserted in the expression for the HF energy one recovers the action of the effective interaction on the two-body product wave [3,4]functions. It is now obvious that the determination of the effective interaction must be coupled to the solution of the HF equations, and performed self-consistently. Namely, for a given effective interaction one solves the HF equations, and the obtained HF orbitals are in turn used in the Bethe-Goldstone equation to find effective interaction. Such a doubly self-consistent procedure is called the Brueckner-Hartree-Fock method.[5,6]

Modern understanding of the HF approximation is not directly based on the variational method applied to Slater determinants. Certainly, the basic approximation for the two-body density matrix is an exact result for a Slater determinant, but the key element of the approach is expression which states that the ground-state energy can be approximated by a functional of the one-body density matrix.

II. DISCUSSIONS

Representation of many-fermion states by density matrices and the HF approximation of the two-body density matrix allow us to give a precise definition of what one really means by conserved and broken symmetries in many-body systems. Moreover, it also links the spontaneous symmetry breaking mechanism to a description of correlations.[7,8]

Consider a unitary symmetry operator \hat{P} such that

$$\hat{P}^+ a_x \hat{P} = P_{xy} a_y \quad , \quad \hat{P}^+ a_x^+ \hat{P} = P_{xy}^* a_y^+$$

and

$$P_{xx'}^+ T_{x'y'} P_{y'y} = T_{xy} \quad , \quad P_{zz'}^+ P_{tt'}^+ G_{z't't'x'y'} P_{x'x} P_{y'y} = G_{ztxy}.$$

$$[\hat{H}, \hat{P}]$$

These are equivalent to the symmetry condition $[\hat{H}, \hat{P}] = 0$ obeyed by Hamiltonian. Symmetry operator \hat{P} acts in the fermion Fock space by mixing elementary fields a_y^+ with the integral kernel P_{xy} (remember that the sum-

$$\sum dy$$

integral is implied for every repeated index). All the most interesting symmetries act in this way - they can be represented as exponents of one-body symmetry generators, i.e., \hat{P} can be any one of the following:

- 1 translational symmetry,

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$$\hat{P} = \exp \left(i \mathbf{r}_0 \cdot \hat{\mathbf{P}} \right),$$

- where $\hat{\mathbf{P}} = \sum_{i=1}^A \mathbf{p}_i$ is the total linear momentum operator, and \mathbf{r}_0 is the shift vector.
 2 egintex2html_wrap_inlinetex2html_wrap_inline.[9,10]
 rotational symmetry,

$$\hat{P} = \exp \left(i \boldsymbol{\alpha}_0 \cdot \hat{\mathbf{I}} \right),$$

- where $\hat{\mathbf{I}} = \sum_{i=1}^A \mathbf{j}_i$ is the total angular momentum operator, and $\boldsymbol{\alpha}_0$ is the rotation angle.
 3 egintex2html_wrap_inlinetex2html_wrap_inline.
 isospin symmetry,

$$\hat{P} = \exp \left(i \vec{\alpha}_0 \circ \hat{\mathbf{T}} \right),$$

- where $\hat{\mathbf{T}} = \frac{1}{2} \sum_{i=1}^A \vec{\tau}_i$ is the total isospin operator, and $\vec{\alpha}_0$ is the iso-rotation angle.
 4 egintex2html_wrap_inlinetex2html_wrap_inline.
 particle-number symmetry,

$$\hat{P} = \exp \left(i \phi_0 \hat{N} \right),$$

- where $\hat{N} = \sum dx a_x^+ a_x$ is the total particle number operator, and ϕ_0 is the gauge angle.
 5 egintex2html_wrap_inlinetex2html_wrap_inline.
 inversion (parity) symmetry,

$$\hat{P} = \prod_{i=1}^A \hat{\pi}_i,$$

- where $\hat{\pi}_i$ is the inversion operator for the i th particle.
 6 egintex2html_wrap_inlinetex2html_wrap_inline.[11,12]

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time-reversal symmetry.

$$\hat{P} = \exp(-i\pi\hat{S}_y) \hat{K},$$

where $\hat{S}_y = \frac{1}{2} \sum_{i=1}^A \sigma_{iy}$ is the y component of the total spin operator, and \hat{K} is the complex conjugation operator in spatial representation.

There can also be terms in the Hamiltonian that *explicitly* break some of the above symmetries (e.g., the Coulomb interaction explicitly breaks the isospin symmetry), but we disregard them for simplicity.

Let us begin with the simplest case, namely, let \hat{P} be the parity symmetry. In this case, the integral kernel

reads $P_{xy} \equiv \delta(\mathbf{x} + \mathbf{y})$, and is, of course, independent of spin and isospin. For a parity-invariant interaction, the

exact energy of an arbitrary state $|\Psi\rangle$, depends only on the scalar parts (in this case, the parity invariant parts) of the one- and two-body density matrices, i.e.,

$$E = T_{xy} \rho_{yx}^{(+)} + \frac{1}{4} G_{xyzt} \rho_{ztxy}^{(+)},$$

for

$$\rho_{yx}^{(\pm)} = \frac{1}{2} \left(P_{yy'} \rho_{y'x'} P_{x'x}^{\pm} \pm \rho_{yx} \right),$$

$$\rho_{ztxy}^{(\pm)} = \frac{1}{2} \left(P_{zz'} P_{tt'} \rho_{z't'x'y'} P_{x'x}^{\pm} P_{y'y}^{\pm} \pm \rho_{ztxy} \right).$$

Within the HF approximation we may have two classes of solutions:[10]

- symmetry-conserving solution:

$$\begin{aligned} \rho_{xy} &= \rho_{xy}^{(+)} \\ \rho_{ztxy}^{(+)} &= \rho_{zx}^{(+)} \rho_{ty}^{(+)} - \rho_{zy}^{(+)} \rho_{tx}^{(+)} \end{aligned}$$

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- symmetry-breaking solution:

$$\rho_{xy} = \rho_{xy}^{(+)} + \rho_{xy}^{(-)},$$

$$\rho_{ztxy}^{(+)} = \rho_{zx}^{(+)} \rho_{ty}^{(+)} - \rho_{zy}^{(+)} \rho_{tx}^{(+)} + \rho_{zx}^{(-)} \rho_{ty}^{(-)} - \rho_{zy}^{(-)} \rho_{tx}^{(-)}.$$

In the case of the broken symmetry, neither of the density matrices is invariant with respect to the symmetry operator.

However, the symmetry breaking part of the one-body density matrix $\rho_{xy}^{(-)}$ enters the HF energy only through the two-body interaction energy. Moreover, the symmetry-projected two-body density matrix *does not obey* the HF condition. In other words, the symmetry-breaking part of the one-body density matrix gives a *correlation* term of the two-body density matrix. Symmetry breaking is, therefore, a reflection of correlations beyond HF, taken into account with respect to the symmetry-conserving HF method.

One can also say that the symmetry-breaking part $\rho_{xy}^{(-)}$ constitutes an additional set of variational parameters, which become allowed when a larger class of the one-body density matrices (beyond symmetry conservation) is considered. As in every variational procedure, a larger variational class may lead (sometimes) to lower energies. Whether it does, depends on the specific case, and in particular on the type of the two-body interaction. It is obvious, that one can gain energy by breaking symmetry only if the appropriate correlation energy is negative, i.e., when the last two terms of the

two-body density matrix, $\rho_{zx}^{(-)} \rho_{ty}^{(-)} - \rho_{zy}^{(-)} \rho_{tx}^{(-)}$, give a negative contribution when averaged with the two-body effective interaction G_{xyzt} .

Within such an approach to the symmetry breaking, one does not, in fact, break any symmetry of the exact wave

function. Indeed, the density matrices, $\rho_{xy}^{(+)}$ and $\rho_{ztxy}^{(+)}$ that are "active" in the total energy do conserve the symmetry. We should also use these density matrices to calculate all other observables for the symmetry-broken (correlated) solution of the HF equations.

Let us now give results of an analogous analysis for the case of deformed nuclei, i.e., for the case of broken rotational symmetry. For axial shapes we then have the following density matrices, [8,9]

$$\rho_{xy} = \sum_J \rho_{xy}^{(J)},$$

$$\rho_{ztxy}^{(0)} = \sum_J \left(\rho_{zx}^{(J)} \times \rho_{ty}^{(J)} \right)_0 - \sum_J \left(\rho_{zy}^{(J)} \times \rho_{tx}^{(J)} \right)_0,$$

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and the total HF energy,

$$E = T_{xy}\rho_{yx}^{(0)} + \frac{1}{4}G_{xyzt}\rho_{ztxy}^{(0)},$$

that depends only on the scalar ($J=0$) parts of the density matrices. On the other hand, the broken-symmetry one-body density matrix is the sum of components $\rho_{xy}^{(J)}$ that transform as irreducible rotational tensors of rank J . In the scalar two-body density matrix (92), these components are coupled to $J=0$, and every such a term defines the multipole correlation energy of rank J . It is now obvious that the broken-symmetry solution becomes the ground state for interactions that have appropriately strong multipole-multipole terms (see Refs. Dob88, Wer94 for numerical results in heavy nuclei).

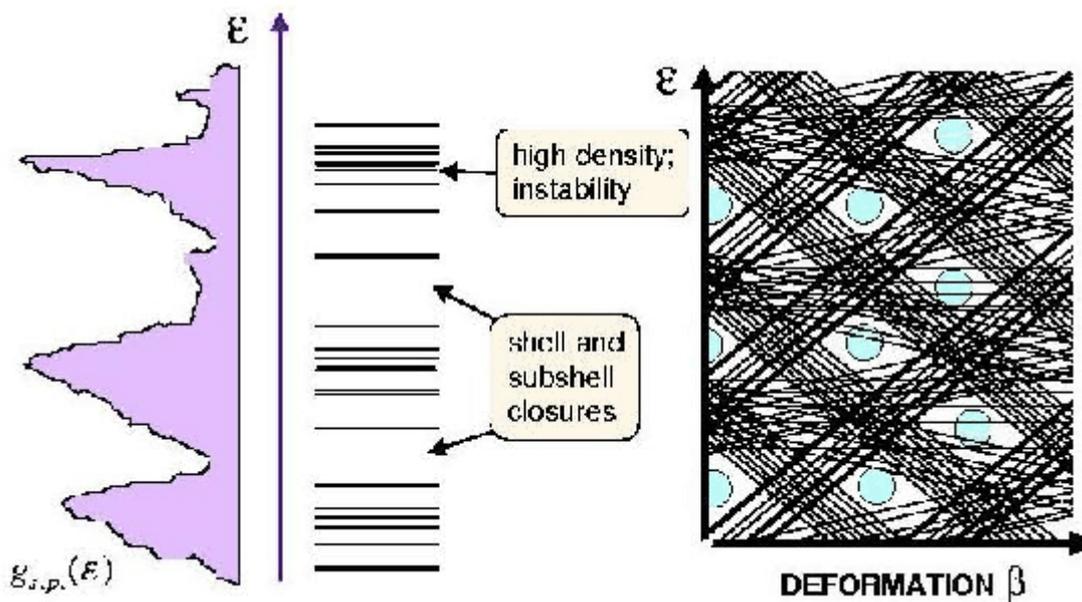


Figure 2: Schematic illustration of the s.p. level density (left), corresponding to the s.p. spectrum of a deformed nucleus (centre). The right panel shows the evolution of the spectrum with nuclear deformation. (Picture courtesy: W. Nazarewicz, ORNL/University of Tennessee/Warsaw University.) From <http://www-highspin.phys.utk.edu/~witek/>.

Without going into detailed discussion of the multipole-multipole decomposition of effective interactions, we may easily tell in which nuclei the rotational symmetry is broken and deformation appears. A schematic diagram presented in the right panel shows the evolution of the s.p. spectrum with nuclear deformation, i.e., the dependence of

eigenvalues of the mean-field Hamiltonian having the shape characterized by the deformation parameter β . In such a spectrum, some s.p. levels go down, and other go up in energy, and at specific deformations there appear in the spectrum larger or smaller gaps. When the particles are filling the lowest levels up to certain energy (prescribed by the

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number of particles), the last occupied level may appear either below or above the gap. This leads respectively to a decrease or an increase of the total energy. The overall density of s.p. levels at the Fermi surface determines, therefore, the total energy of the system. In other words, a system having a given number of particles adopts the shape at which the last occupied level is below a large gap. Therefore, nuclei that correspond to magic particle numbers are spherical (large gaps appear at spherical shape) and the rotational symmetry is conserved, while nuclei with particle numbers between the magic gaps (the so-called open-shell nuclei) choose non-zero deformed ground states corresponding to broken rotational symmetry.

III. RESULTS

Approximation of the many-body energy by a functional of the one-body density matrix can be further simplified in the coordinate representation. Namely, it appears that the HF density matrix influences the energy mostly through the local density $\rho_{xx}, \rho_{yy}, \rho_{xy}, \rho_{yx}$. This observation defines the local density approximation (LDA).

Neglecting for simplicity the spin-isospin degrees of freedom, we can write the interaction energy in the form[10]

$$E_{\text{int}} = \frac{1}{2} \int d^3x d^3y d^3x' d^3y' \tilde{G}_{xyx'y'} (\rho_{x'x} \rho_{y'y} - \rho_{x'y} \rho_{y'x}).$$

For local effective interaction, the non-antisymmetrized matrix element $\tilde{G}_{xyx'y'}$ is given by the

potential $G(\mathbf{x}, \mathbf{y})$,

$$\tilde{G}_{xyx'y'} = \delta(\mathbf{x} - \mathbf{x}') \delta(\mathbf{y} - \mathbf{y}') G(\mathbf{x}, \mathbf{y}),$$

and the interaction energy reads

$$E_{\text{int}} = \frac{1}{2} \int d^3x d^3y G(\mathbf{x}, \mathbf{y}) (\rho_{xx} \rho_{yy} - \rho_{xy} \rho_{yx}).$$

The first term (direct) depends only on the local density matrix (equal arguments), while the second term (exchange) involves the full one-body density matrix. Therefore, the local density plays a special role due to locality of the effective interaction.

It is therefore convenient to represent the one-body density matrix in total and relative coordinates, i.e.,

$$\rho_{xy} = \rho(\mathbf{R}, \mathbf{r}),$$

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where

$$\mathbf{R} = \frac{1}{2}(\mathbf{x} + \mathbf{y}) \quad \text{and} \quad \mathbf{r} = \mathbf{x} - \mathbf{y}.$$

$$\rho(\mathbf{R}) = \rho_{\mathbf{x}\mathbf{x}} = \rho(\mathbf{R}, \mathbf{0})$$

Denoting the local density by single argument, and noting that by translational

$$G(\mathbf{x}, \mathbf{y}) = G(\mathbf{x} - \mathbf{y})$$

invariance the potential depends only on the relative coordinate, we have

$$E^{\text{int}} = \frac{1}{2} \int d^3 \mathbf{R} d^3 \mathbf{r} G(\mathbf{r}) \left[\rho(\mathbf{R} + \frac{1}{2}\mathbf{r}) \rho(\mathbf{R} - \frac{1}{2}\mathbf{r}) - \rho(\mathbf{R}, \mathbf{r}) \rho(\mathbf{R}, -\mathbf{r}) \right].$$

$$E^{\text{int}} = E_{\text{dir}}^{\text{int}} + E_{\text{exch}}^{\text{int}}$$

We see that direct and exchange terms, have markedly different dependence on the density matrix, and thus have to be treated separately.

In the direct term, we can use the fact that the range of the effective force is smaller than the typical distance at which the density changes. Indeed, the nuclear density is almost constant inside the nucleus, and then falls down to zero within the region called the nuclear surface, which has a typical width of about 3fm. Hence, within the range of interaction, and for the purpose of evaluating the direct interaction energy, the density can be approximated by the quadratic expansion,

$$\rho(\mathbf{R} \pm \frac{1}{2}\mathbf{r}) = \rho(\mathbf{R}) \pm \frac{1}{2}r^i \nabla_i \rho(\mathbf{R}) + \frac{1}{8}r^i r^j \nabla_i \nabla_j \rho(\mathbf{R}) + \dots$$

and

$$\rho(\mathbf{R} + \frac{1}{2}\mathbf{r}) \rho(\mathbf{R} - \frac{1}{2}\mathbf{r}) = \rho^2(\mathbf{R}) + \frac{1}{4}r^i r^j \left(\rho(\mathbf{R}) \nabla_i \nabla_j \rho(\mathbf{R}) - [\nabla_i \rho(\mathbf{R})][\nabla_j \rho(\mathbf{R})] \right) + \dots,$$

$$\nabla_i \frac{\partial}{\partial R^i}$$

where = . When inserted into this expansion gives [for scalar

$$G(\mathbf{r}) = G(|\mathbf{r}|) = G(r)$$

interactions] the **direct** interaction energy:

$$E_{\text{dir}}^{\text{int}} = \frac{1}{2} \int d^3 \mathbf{R} \left[G_0 \rho^2 + \frac{1}{4} G_2 \left(\rho \Delta \rho - (\nabla \rho)^2 \right) \right] + \dots,$$

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where coupling constants G_0 and G_2 are given by the moments of the interaction:

$$G_0 = 4\pi \int dr r^2 G(r) \quad \text{and} \quad G_2 = \frac{4}{3}\pi \int dr r^4 G(r).$$

In the exchange term, the situation is entirely different, because here the range of interaction matters in the non-local, relative direction \mathbf{r} . In order to get a feeling what are the properties of the one-body density matrix in this direction, we can calculate it for infinite matter,

$$\rho_{xy} = \int_{|\mathbf{k}| < k_F} d^3\mathbf{k} \frac{\exp(i\mathbf{k} \cdot \mathbf{x})}{\sqrt{8\pi^3}} \frac{\exp(-i\mathbf{k} \cdot \mathbf{y})}{\sqrt{8\pi^3}},$$

$$|\mathbf{k}| < k_F$$

where the s.p. wave functions (plane waves) are integrated within the Fermi sphere of momenta

Obviously, ρ_{xy} depends only on the relative coordinate, i.e.,

$$\rho(\mathbf{R}, \mathbf{r}) = \frac{1}{2\pi^2 r} \int_0^{k_F} dk k \sin(kr) = \frac{k_F^3}{6\pi^2} \left[3 \frac{\sin(k_F r) - k_F r \cos(k_F r)}{(k_F r)^3} \right] = \frac{k_F^3}{6\pi^2} \left[3 \frac{j_1(k_F r)}{k_F r} \right].$$

$$r \simeq 4.4934/k_F \simeq 3$$

Function in square parentheses equals 1 at $r=0$, and has the first zero at $r \simeq 4.4934/k_F$ fm, i.e., in the non-local direction the density varies on the same scale as it does in the local direction.

Therefore, the quadratic expansion of the density matrix in the relative variable

$$\rho(\mathbf{R}, \pm\mathbf{r}) = \rho(\mathbf{R}) \pm r^i \partial_i \rho(\mathbf{R}, \mathbf{r}) + \frac{1}{2} r^i r^j \partial_i \partial_j \rho(\mathbf{R}, \mathbf{r}) + \dots,$$

$$\partial_i \partial / \partial r^i$$

where derivatives $\partial_i = \partial / \partial r^i$ are always calculated at $r^i=0$, is, in principle, sufficient for the evaluation of the exchange interaction energy. However, we can improve it by introducing three universal functions

$$r = |\mathbf{r}| \quad \pi_0(r) \quad \pi_1(r) \quad \pi_2(r)$$

of π_0 , π_1 , π_2 , and π_3 , which vanish at large r , i.e., we define the LDA by:

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$$\rho(\mathbf{R}, \pm \mathbf{r}) = \pi_0(r)\rho(\mathbf{R}) \pm \pi_1(r)r^i \partial_i \rho(\mathbf{R}, \mathbf{r}) + \frac{1}{2}\pi_2(r)r^i r^j \partial_i \partial_j \rho(\mathbf{R}, \mathbf{r}) + \dots$$

Since for small \mathbf{r} , must be compatible with the Taylor expansion the auxiliary functions must fulfill conditions at $\mathbf{r}=0$,

$$\pi_0(0) = \pi_1(0) = \pi_2(0) = 1, \quad \pi_0'(0) = \pi_1'(0) = 0, \quad \text{and} \quad \pi_0''(0) = 0.$$

In order to conserve the local-gauge-invariance properties of the interaction energy Dob95e, we also require that

$$\pi_1^2(r) = \pi_0(r)\pi_2(r).$$

The auxiliary functions $\pi_0(r)$ and $\pi_2(r)$ can be calculated *a posteriori*, to give the best possible approximation of a given density matrix $\rho(\mathbf{R}, \mathbf{r})$. However, they can also be estimated *a priori* by making momentum expansion around the Fermi momentum k_F . This gives the density-matrix expansion (DME) of Ref. Neg72, in which

$$\pi_0(r) = \frac{6j_1(k_F r) + 21j_3(k_F r)}{2k_F r} \quad \text{and} \quad \pi_2(r) = \frac{105j_3(k_F r)}{(k_F r)^3},$$

where $j_n(k_F r)$ are the spherical Bessel functions.

The term depending on the non-local density in the exchange integral now reads

$$\rho(\mathbf{R}, \mathbf{r})\rho(\mathbf{R}, -\mathbf{r}) = \pi_0^2(r)\rho^2(\mathbf{R}) + \pi_0(r)\pi_2(r)r^i r^j \left(\rho(\mathbf{R})\partial_i \partial_j \rho(\mathbf{R}, \mathbf{r}) - [\partial_i \rho(\mathbf{R}, \mathbf{r})][\partial_j \rho(\mathbf{R}, \mathbf{r})] \right)$$

and gives the **exchange** interaction energy:

$$E_{\text{exch}}^{\text{int}} = -\frac{1}{2} \int d^3 \mathbf{R} \left[G_0' \rho^2 + \frac{1}{4} G_2' \left(\rho \Delta \rho - 4(\rho \tau - \mathbf{j}^2) \right) \right] + \dots,$$

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where coupling constants G'_0 and G'_2 are given by the following integrals of the interaction:

$$G'_0 = 4\pi \int dr r^2 \pi_0^2(r) G(r) \quad \text{and} \quad G'_2 = \frac{4}{3}\pi \int dr r^4 \pi_0(r) \pi_2(r) G(r).$$

The exchange interaction energy also depends on densities π_j and τ that we define below. It is obvious that when the pure Taylor expansion is used to approximate the density in the non-local direction, i.e.,

$$\pi_0(r) = \pi_2(r) = 1$$

for π_j , the direct and exchange coupling constants are

$$G'_0 = G_0 \quad G'_2 = G_2$$

equal, and .

Altogether, quadratic approximations to the one-body density matrix allow expressing the direct and exchange interaction energies as integrals of local energy density. Such energy density depends on the local density, on derivatives of the local density, and on several other densities that represent properties of the one-body density matrix in the non-local direction.

We should stress that the validity of the LDA depends on different scales involved in properties of nuclei. Namely, the scale of distances characterizing the ground-state one-body density matrix is significantly larger than the range of effective forces. Therefore, the LDA may apply only to selected, low-energy phenomena where the spatial structure of the density matrix is not very much affected.

Moreover, we see that the low-energy nuclear properties may depend on an extremely restricted set of properties of effective interactions. Within the LDA, only a few numbers [the coupling constants determine the energy density. This is entirely in the spirit of the effective field theory; separation of scales results in a transmission of a very limited information from one scale to another. Once this information (in our case - the coupling constants) is either evaluated, or fit to data, properties of the system can be properly calculated at the larger scale.

We also see that the coupling constants can be evaluated by assuming any effective interaction that has a *smaller* range than the physical range. In doing so, we can even go down to zero range, and nothing will change, provided we fix the parameters of the zero-range force so as to properly describe the moments of the force, and thus properly reproduce the coupling constants.[11]

We can now proceed to the real world by putting back into our description the spin and isospin degrees of freedom. Based on the results above, we can first construct the most general set of local densities, with derivatives up to the second order taken into account, and then build the local energy density. The complete such construction has been performed only very recently Per03; it involves the full proton-neutron mixing and treats both the particle-hole and particle-particle channels of interaction.

We begin by writing the one-body density matrix with all variables shown explicitly,

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$$\rho_{\mathbf{x}\sigma\tau, \mathbf{y}\sigma'\tau'} = \langle \Psi | a_{\mathbf{y}\sigma'\tau'}^+ a_{\mathbf{x}\sigma\tau} | \Psi \rangle,$$

and we define the densities in total and relative coordinates as

$$\rho(\mathbf{R}, \mathbf{r}, \sigma\tau, \sigma'\tau') = \rho_{\mathbf{x}\sigma\tau, \mathbf{y}\sigma'\tau'}.$$

The spin-isospin components can now be separated,

$$\begin{aligned} \rho(\mathbf{R}, \mathbf{r}, \sigma\tau, \sigma'\tau') &= \frac{1}{4}\rho_0(\mathbf{R}, \mathbf{r})\delta_{\sigma\sigma'}\delta_{\tau\tau'} + \frac{1}{4}\delta_{\sigma\sigma'}\vec{\rho}(\mathbf{R}, \mathbf{r}) \circ \vec{\tau}_{\tau\tau'} \\ &+ \frac{1}{4}\mathbf{s}_0(\mathbf{R}, \mathbf{r}) \cdot \boldsymbol{\sigma}_{\sigma\sigma'}\delta_{\tau\tau'} + \frac{1}{4}\vec{\mathbf{s}}(\mathbf{R}, \mathbf{r}) \cdot \boldsymbol{\sigma}_{\sigma\sigma'} \circ \vec{\tau}_{\tau\tau'}, \end{aligned}$$

where $\boldsymbol{\sigma}$ and $\vec{\tau}$ are the spin Pauli matrices. The scalar-isoscalar $\rho_0(\mathbf{R}, \mathbf{r})$, scalar-isovector $\vec{\rho}(\mathbf{R}, \mathbf{r})$, vector-isoscalar $\mathbf{s}_0(\mathbf{R}, \mathbf{r})$, and vector-isovector $\vec{\mathbf{s}}(\mathbf{R}, \mathbf{r})$ densities can be obtained in a standard way by taking appropriate traces with the Pauli matrices. All necessary local densities can now be obtained by calculating at $\mathbf{r}=0$ the derivatives in the total $\nabla = \partial/\partial\mathbf{R}$ and relative $\partial = \partial/\partial\mathbf{r}$ coordinates, up to the second order.

Without the proton-neutron mixing, which we neglect from now on in order to simplify the presentation, only the third components of isovectors are non-zero, and we can use the notation

$$\rho_1(\mathbf{R}, \mathbf{r}) \equiv \vec{\rho}_3(\mathbf{R}, \mathbf{r}) \quad \text{and} \quad \mathbf{s}_1(\mathbf{R}, \mathbf{r}) \equiv \vec{\mathbf{s}}_3(\mathbf{R}, \mathbf{r}).$$

The list of all required local densities then reads Eng75:

$$\begin{aligned} \text{Matter:} \quad & \rho_t(\mathbf{R}) = \rho_t(\mathbf{R}, 0), \\ \text{Current:} \quad & \mathbf{j}_t(\mathbf{R}) = [k\rho_t(\mathbf{R}, \mathbf{r})]_{\mathbf{r}=0}, \\ \text{Kinetic:} \quad & \tau_t(\mathbf{R}) = [(k^2 - \frac{1}{4}K^2)\rho_t(\mathbf{R}, \mathbf{r})]_{\mathbf{r}=0}, \\ \text{Spin:} \quad & \mathbf{s}_t(\mathbf{R}) = \mathbf{s}_t(\mathbf{R}, 0), \end{aligned}$$

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$$\text{Spin-current: } J_t^{ij}(\mathbf{R}) = [k^i s_t^j(\mathbf{R}, \mathbf{r})]_{\mathbf{r}=0},$$

$$\text{Spin-kinetic: } T_t(\mathbf{R}) = [(k^2 - \frac{1}{4}K^2)\rho_t(\mathbf{R}, \mathbf{r})]_{\mathbf{r}=0},$$

where

$$\mathbf{k} = -i\partial = -i\partial/\partial\mathbf{r} = \frac{1}{2i}(\nabla^x - \nabla^y) \quad \text{and} \quad \mathbf{K} = -i\nabla = -i\partial/\partial\mathbf{R} = -i(\nabla^x + \nabla^y),$$

are momentum operators in the relative and total coordinate, and index $t=0, 1$ distinguishes between the isoscalar and isovector components. The kinetic densities are usually defined in terms of the derivatives acting on

the \mathbf{x} and \mathbf{y} coordinates i.e., $(k^2 - \frac{1}{4}K^2) = \nabla^x \cdot \nabla^y$. There is also one density depending on $\mathbf{K} \otimes \mathbf{k}$ (tensor-kinetic density) Flo75,Per03, which we do not discuss here because it appears only for tensor interactions. Since the Pauli matrices $\boldsymbol{\sigma}$ and momenta \mathbf{k} are time-odd operators, we see that

densities $\rho_t(\mathbf{R})$, $\tau_t(\mathbf{R})$, and $J_t^{ij}(\mathbf{R})$ are time-even, and densities $\mathbf{j}_t(\mathbf{R})$, $\mathbf{s}_t(\mathbf{R})$, and $T_t(\mathbf{R})$ are time-odd.

For an arbitrary central finite-range local potential with the full spin-isospin dependence

$$G(\mathbf{x}, \mathbf{y}) = W(\mathbf{x}, \mathbf{y}) + B(\mathbf{x}, \mathbf{y})P_\sigma - H(\mathbf{x}, \mathbf{y})P_\tau - M(\mathbf{x}, \mathbf{y})P_\sigma P_\tau,$$

we can now repeat the derivation of the LDA functional, in each spin-isospin channel. As a result, we obtain the interaction energy (direct and exchange terms combined) in the form

$$E^{\text{int}} = \sum_{t=0,1} \int d^3\mathbf{R} [C_t^\rho \rho_t^2 + C_t^{\Delta\rho} \rho_t \Delta\rho_t + C_t^\tau (\rho_t \tau_t - \mathbf{j}_t^2) + C_t^s \mathbf{s}_t^2 + C_t^{\Delta s} \mathbf{s}_t \cdot \Delta\mathbf{s}_t + C_t^\tau (\mathbf{s}_t \cdot \mathbf{T}_t - J_t^{\leftrightarrow 2})],$$

$$J_t^{\leftrightarrow 2} = \sum_{ij} J^{ij} J_{ij}$$

where J^{ij} are simple moments of potentials, i.e., J_{ij} . The energy density depends on six isoscalar and six isovector coupling constants that are

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$$8 \begin{pmatrix} C_0^\rho \\ C_1^\rho \\ C_0^s \\ C_1^s \end{pmatrix} = \begin{pmatrix} 4 & -1 & 2 & -2 \\ 0 & -1 & 0 & -2 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} W_0 + M'_0 \\ M_0 + W'_0 \\ B_0 + H'_0 \\ H_0 + B'_0 \end{pmatrix}$$

and

$$32 \begin{pmatrix} C_0^{\Delta\rho} \\ C_1^{\Delta\rho} \\ C_0^\tau \\ C_1^\tau \\ C_0^{\Delta s} \\ C_1^{\Delta s} \\ C_0^T \\ C_1^T \end{pmatrix} = \begin{pmatrix} 8 & -1 & 4 & -2 & -4 & 2 & -2 & 4 \\ 0 & -1 & 0 & -2 & -4 & 0 & -2 & 0 \\ 0 & 4 & 0 & 8 & 0 & -8 & 0 & -16 \\ 0 & 4 & 0 & 8 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & 2 & -2 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 4 & 0 & 0 & 0 & -8 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} W_2 \\ W'_2 \\ B_2 \\ B'_2 \\ H_2 \\ H'_2 \\ M_2 \\ M'_2 \end{pmatrix}$$

where (for $X \equiv W, B, H,$ or M)

$$X_0 = 4\pi \int dr r^2 X(r) \quad \text{and} \quad X_2 = \frac{4}{3}\pi \int dr r^4 X(r),$$

$$X'_0 = 4\pi \int dr r^2 \pi_0^2(r) X(r) \quad \text{and} \quad X'_2 = \frac{4}{3}\pi \int dr r^4 \pi_0(r) \pi_2(r) X(r).$$

$$\pi_0(r) = \pi_2(r) = 1$$

Again we see, that for $X_0 = X'_0$ and $X_2 = X'_2$, the direct and exchange coupling constants are

$$X_0 = X'_0 \quad X_2 = X'_2$$

equal, and hence only six coupling constants in energy density are independent. This requires that the so-called time-odd coupling constants are linear combinations of the so-called time-even coupling constants Dob95e:

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$$3 \begin{pmatrix} C_0^s \\ C_1^s \end{pmatrix} = \begin{pmatrix} -2 & -3 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} C_0^\rho \\ C_1^\rho \end{pmatrix}$$

and

$$24 \begin{pmatrix} C_0^{\Delta s} \\ C_1^{\Delta s} \\ C_0^T \\ C_1^T \end{pmatrix} = \begin{pmatrix} -12 & -12 & 3 & 9 \\ -4 & -4 & 3 & -3 \\ 16 & 48 & -4 & 12 \\ 16 & -16 & 4 & -12 \end{pmatrix} \begin{pmatrix} C_0^{\Delta \rho} \\ C_1^{\Delta \rho} \\ C_0^T \\ C_1^T \end{pmatrix}$$

It is well known that the local energy density is also obtained for the Skyrme zero-range momentum-dependent interaction Sky56, Sky59, Vau72. Without density-dependent and spin-orbit terms, this interaction reads

$$G(\mathbf{x}, \mathbf{y}) = t_0 (1 + x_0 P_\sigma) \delta(\mathbf{x} - \mathbf{y}) + \frac{1}{2} t_1 (1 + x_1 P_\sigma) [\hat{\mathbf{k}}^2 \delta(\mathbf{x} - \mathbf{y}) + \delta(\mathbf{x} - \mathbf{y}) \hat{\mathbf{k}}^2] + t_2 (1 + x_2 P_\sigma) \hat{\mathbf{k}}' \cdot \delta(\mathbf{x} - \mathbf{y}) \hat{\mathbf{k}},$$

where $\hat{\mathbf{k}}' = i\boldsymbol{\partial}$ acts to the left, and $\hat{\mathbf{k}} = -i\boldsymbol{\partial}$ acts to the right. For this interaction, the interaction energy has exactly the form with coupling constants Eng75, Dob95 that depend on parameters t_0 , x_0 , t_1 , x_1 , t_2 , x_2 , and

$$8 \begin{pmatrix} C_0^\rho \\ C_1^\rho \\ C_0^s \\ C_1^s \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ -1 & -2 \\ -1 & 2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} t_0 \\ t_0 x_0 \end{pmatrix}$$

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and

$$64 \begin{pmatrix} C_0^{\Delta\rho} \\ C_1^{\Delta\rho} \\ C_0^\tau \\ C_1^\tau \\ C_0^{\Delta s} \\ C_1^{\Delta s} \\ C_0^T \\ C_1^T \end{pmatrix} = \begin{pmatrix} -9 & 0 & 5 & 4 \\ 3 & 6 & 1 & 2 \\ 12 & 0 & 20 & 16 \\ -4 & -8 & 4 & 8 \\ 3 & -6 & 1 & 2 \\ 3 & 0 & 1 & 0 \\ -4 & 8 & 4 & 8 \\ -4 & 0 & 4 & 0 \end{pmatrix} \begin{pmatrix} t_1 \\ t_1 x_1 \\ t_2 \\ t_2 x_2 \end{pmatrix}$$

$$\pi_0(r) = \pi_2(r) = 1$$

For $\pi_0(r) = \pi_2(r) = 1$, the Skyrme interaction exactly reproduces the LDA of the finite-range interaction provided the Skyrme parameters are given by

$$t_0 = W_0 + M_0 \quad , \quad t_0 x_0 = B_0 + H_0,$$

$$t_1 = -W_2 - M_2 \quad , \quad t_1 x_1 = -B_2 - H_2,$$

$$t_2 = W_2 - M_2 \quad , \quad t_2 x_2 = B_2 - H_2.$$

Coupling constants of the Skyrme functional fulfill constraints . When the better approximation of the non-local density

matrix is used, i.e., for $\pi_0(r) \neq 1$ or $\pi_2(r) \neq 1$ the Skyrme interaction cannot reproduce the LDA energy density. However, it is enough to release constraints and treat all the twelve coupling constants as independent parameters, to recover the full freedom of the LDA local energy density.

Again we explicitly see that (exactly in the spirit of the effective field theory), the zero-range interaction can reproduce the same properties of nuclear systems as does the real effective interaction, provided the coupling constants in the energy density are either adjusted to data, or calculated from the real finite-range interaction. It is also clear that the zero-range interaction cannot be treated literally - it is significant only as a "generator" of the proper energy density, while all physical results depend only on this energy density, and not on the interaction itself. In particular, it is incorrect to look for exact eigenstates of the system interacting with the zero-range interaction; we know that for such an interaction the ground state does not exist because of the collapse. However, even for the finite-range effective interaction (for which the ground state does, in principle, exist) the exact ground state is irrelevant, because the interaction has been built to act only in the space of Slater determinants

Of course, there is nothing magic or fundamental in the LDA to the energy density. It just reflects the fact that the nuclear one-body density matrix varies on a larger scale of distances than does the nuclear effective interaction. Validity of this approximation depends on the fundamental assumption that the total energy can be described as a

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functional of the one-body density matrix. The fact that we assumed a local effective interaction is not crucial - for non-local interactions the direct term becomes more complicated, but the LDA .. However, effective interactions must, in fact, also depend on energy so the presented derivation of the LDA is not complete. One usually goes beyond the local energy density derived from approximations to one-body density, and one includes also terms that depend on local densities in a more complicated way, cf. the density-dependent term of the Gogny interaction

IV. CONCLUSIONS

Some people say: the LDA is just fitting of parameters - it is enough to have many parameters to fit anything one wants. This point of view simply disregards the success of the LDA in nuclear phenomenology. The effective field theory point of view is, in my opinion, more interesting, and potentially more fruitful. It regards the success of phenomenological LDA as indication that scales between quark-gluon QCD interactions and low-energy nuclear phenomena are indeed very well separated, and hence few numbers only are enough to define latter in terms of the former. The challenge of course remains: to look for derivations of these few numbers by decent fundamental theory, and to adjust these numbers to data and look for phenomena where the adjustments fail.

Upon variation of the energy with respect to local densities, one obtains the HF equation in spatial coordinates,

$$h_{\alpha} \psi_{i,\alpha}(\mathbf{r}\sigma) = \epsilon_{i,\alpha} \psi_{i,\alpha}(\mathbf{r}\sigma),$$

where i numbers the neutron ($\alpha = n$) and proton ($\alpha = p$) orbitals, and

$$h_n = -\frac{\hbar^2}{2m} \Delta + \Gamma_0^{\text{even}} + \Gamma_0^{\text{odd}} + \Gamma_1^{\text{even}} + \Gamma_1^{\text{odd}},$$

$$h_p = -\frac{\hbar^2}{2m} \Delta + \Gamma_0^{\text{even}} + \Gamma_0^{\text{odd}} - \Gamma_1^{\text{even}} - \Gamma_1^{\text{odd}}.$$

The isoscalar ($t=0$) and isovector ($t=1$) time-even and time-odd mean fields read

$$\Gamma_t^{\text{even}} = -\nabla \cdot M_t(\mathbf{r}) \nabla + U_t(\mathbf{r}) + \frac{1}{2i} \left(\nabla \sigma \cdot \vec{B}_t(\mathbf{r}) + \vec{B}_t(\mathbf{r}) \cdot \nabla \sigma \right),$$

$$\Gamma_t^{\text{odd}} = -\nabla \cdot (\sigma \cdot C_t(\mathbf{r})) \nabla + \sigma \cdot \Sigma_t(\mathbf{r}) + \frac{1}{2i} \left(\nabla \cdot I_t(\mathbf{r}) + I_t(\mathbf{r}) \cdot \nabla \right),$$

where we defined the following mean-field potentials as functions of densities

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$$U_t = 2C_t^{\rho} \rho_t + 2C_t^{\Delta\rho} \Delta\rho_t + C_t^{\tau} \tau_t,$$

$$\Sigma_t = 2C_t^s s_t + 2C_t^{\Delta s} \Delta s_t + C_t^T T_t,$$

$$M_t = C_t^{\tau} \rho_t,$$

$$C_t = C_t^T s_t,$$

$$\overset{\leftrightarrow}{B}_t = 2C_t^J \overset{\leftrightarrow}{J}_t,$$

$$I_t = 2C_t^j j_t.$$

Since neither in the effective interactions, nor in the energy density we showed the spin-orbit, tensor, or density-dependent terms, such contributions are not shown in the mean fields above. The mean-field Hamiltonian resulting from the LDA is simply given by local one-body potentials, with a complete dependence on spin, and by momentum-dependent terms that have the form of generalized effective-mass and spin-momentum couplings.[12]

REFERENCES

1. Glashow, S. (1959). "The renormalizability of vector meson interactions." Nucl. Phys. **10**, 107.
2. ^ Salam, A.; Ward, J. C. (1959). "Weak and electromagnetic interactions". Nuovo Cimento. **11** (4): 568–577. Bibcode:1959NCim...11..568S. doi:10.1007/BF02726525. S2CID 15889731.
3. ^ Weinberg, S (1967). "A Model of Leptons" (PDF). Phys. Rev. Lett. **19** (21): 1264–66. Bibcode:1967PhRvL..19.1264W. doi:10.1103/PhysRevLett.19.1264. Archived from the original (PDF) on 2012-01-12.
4. ^ S. Bais (2005). The Equations: Icons of knowledge. p. 84. ISBN 0-674-01967-9.
5. ^ "The Nobel Prize in Physics 1979". The Nobel Foundation. Retrieved 2008-12-16.
6. ^ Salam, A.; Ward, J.C. (November 1964). "Electromagnetic and weak interactions". Physics Letters. **13** (2): 168–171. doi:10.1016/0031-9163(64)90711-5.
7. ^ Lee, T.D. (1981). Particle Physics and Introduction to Field Theory.
8. ^ Englert, F.; Brout, R. (1964). "Broken symmetry and the mass of gauge vector mesons". Physical Review Letters. **13** (9): 321–323. Bibcode:1964PhRvL..13..321E. doi:10.1103/PhysRevLett.13.321.
9. ^ Higgs, P.W. (1964). "Broken symmetries and the masses of gauge bosons". Physical Review Letters. **13** (16): 508–509. Bibcode:1964PhRvL..13..508H. doi:10.1103/PhysRevLett.13.508.
10. ^ Guralnik, G.S.; Hagen, C.R.; Kibble, T.W.B. (1964). "Global conservation laws and massless particles". Physical Review Letters. **13** (20): 585–587. Bibcode:1964PhRvL..13..585G. doi:10.1103/PhysRevLett.13.585.
11. ^ Guralnik, G.S. (2009). "The history of the Guralnik, Hagen, and Kibble development of the theory of spontaneous symmetry breaking and gauge particles". International Journal of Modern Physics A. **24** (14): 2601–2627. arXiv:0907.3466. Bibcode:2009IJMPA..24.2601G. doi:10.1142/S0217751X09045431. S2CID 16298371.
12. ^ D'Onofrio, Michela; Rummukainen, Kari (2010). "Standard model cross-over on the lattice". Phys. Rev. D. **93** (2): 025003. arXiv:1508.07161. Bibcode:2016PhRvD..93b5003D. doi:10.1103/PhysRevD.93.025003. hdl:10138/159845. S2CID 119261776.