Roman and Strong Roman Domination Numbers in Graphs

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ABSTRACT: Let $f : V \rightarrow \{0, 1, 2\}$ be a function having the property that for every vertex $v \in V$ with $f(v) = 0$, there exists a neighbourhood $u \in N(v)$ with $f(u) = 2$. Such a function is called a Roman dominating function or just an RDF. The weight of an RDF is the value $\sum_{u \in V} f(u)$. The minimum weight of an RDF on $G$ is called the Roman domination number of $G$ and is denoted by $\gamma_R(G)$. The function $f : V \rightarrow \{0, 1, 2, 3\}$ is named a Strong Roman dominating function on a graph $G$ whenever for each vertex $u$ with $f(u) = 0$ there exists an adjacent vertex $v$ to $u$ such that $f(v) = 3$. Also for each vertex $u$ with $f(u) = 1$ there should be at least an adjacent vertex $v$ to $u$ weighted 2. The weight of an SRDF is the value $\sum_{u \in V} f(u)$. Strong Roman domination number of graph $G$ which we denoted by $\gamma_{SR}(G)$ is minimum weight of an SRDF on $G$. In this paper, we show that

$$\frac{1}{2} \gamma_{SR}(G) \leq \gamma_R(G) \leq \frac{2}{3} \gamma_{SR}(G).$$

And we study the Roman and Strong Roman domination of the trees and we show that $\gamma_R(T) = \frac{2}{3} \gamma_{SR}(T)$.

KEYWORDS: Roman domination, Strong Roman domination, Tree.

I. INTRODUCTION

For notations and graph theoretic terminologies not defined herein, we refer [3, 4, 9]. Let $G = (V, E)$ be a finite simple graph with the vertex set $V = V(G)$ and the edge set $E = E(G)$. Let $|V(G)| = n(G) = n$ be the order of graph $G$. For any vertex $v \in V$, the open neighbourhood of $v$ is the set $N(v) = \{u \in V \mid uv \in E\}$ and the closed neighbourhood of $v$ is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighbourhood of $S$ is the set $N[S] = \bigcup_{v \in S} N(v)$ and the closed neighbourhood of $S$ is the set $N[S] = N[S] \cup S$. The degree of a vertex $v \in V$ is equals to $|N(v)|$ and denoted by $\deg(v)$. The maximum and minimum degree of a graph $G$ are...
denoted by $\Delta(G)$ and $\delta(G)$, respectively. Let $S \subseteq V$ be any subset of vertices of $G$. Then the induced sub-graph $G[S]$ is the graph whose set is $S$ and whose edge set consists of all of the edges in $E$ that have both endpoints in $S$. The induced sub-graph $G[S]$ may also be called the sub-graph induced in $G$ by $S$. A walk in a graph $G$ is a finite sequence $W = v_0 e_1 v_1 e_2 v_2 \ldots e_n v_n$ whose terms are alternatively vertices and edges such that for $1 \leq i \leq n$ the ends of $e_i$ are $v_{i-1}$ and $v_i$. The length of a walk is the number of edges occur in the walk. The distance between two vertices $u$ and $v$ in graph $G$ is the minimum length of a path joining them. The diameter of a graph $G$ is the maximum distance between vertices of $G$, denoted $\text{diam}(G)$. The minimum diameter of graph $G$ is called a diagonal in $G$. A non-trivial graph, is a graph $G$ of order at least 2. A walk consisting of $k + 1$ distinct vertices $v_0 v_1 \ldots v_k$ is a path, and if $v_0 = v_k$ then these vertices form a cycle. A graph $G$ is connected if for every pair of vertices $v$ and $x$ in $V(G)$, there is a $v - x$ path. Otherwise, $G$ is disconnected. A tree $T$ is a connected graph with no cycles. Every tree $T$ with $n$ vertices has $m = n - 1$ edges. A Star $K_{1,n}$ is a tree with a vertex of degree $n$ and $n$ vertices of degree 1. Observe that a star is a type of tree. A leaf is a vertex with degree 1 and a Support vertex is an adjacent vertex to a leaf. A Strong support vertex is a vertex which adjacent to at least two leaves.

In this paper, all support vertices and all leaves of tree $T$ are denoted by $S(T)$ and $L(T)$, respectively. A rooted tree $T$ distinguishes one vertex $r$ called the root. For each vertex $v \neq r$ of $T$ the parent of $v$ is the neighbourhood of $v$ on the unique $(r, v)$-path, while a child of $v$ is any other neighbourhood of $v$. A simple graph $G$, is said to be complete if each pair of distinct vertices is joined by an edge. Generally $K_n$ denotes a complete graph on $n$ vertices. The complement $\overline{G}$ of a graph $G$ has $V(\overline{G}) = V(G)$ and $\{u, v\} \in E(G)$ if and only if $\{u, v\} \notin E(G)$. Thus, the complement of a complete graph is the empty graph.

A set $S \subseteq V$ is a dominating set if $N[S] = V$. The domination number is the minimum cardinality of a dominating set in $G$ and denoted by $\gamma(G)$, see [2].

A Roman dominating function on a graph $G = (V, E)$ is a function $f : V \rightarrow \{0, 1, 2\}$ satisfying the condition that every vertex $u$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ for which $f(v) = 2$. The weight of a Roman dominating function is the value $f(V) = \sum_{u \in V} f(u)$. The minimum weight of a Roman dominating function on a graph $G$ is called the Roman domination number of $G$ which is denoted by $\gamma_R(G)$, see [1, 10, 11].

Other notations and terminologies not explicitly given here could be find in [3, 4, 5, 6, 12]. K. Selvakumaran et al. in [7] introduced Strong Roman domination in 2016. A Strong Roman dominating function on a graph $G = (V, E)$ is a function $f : V \rightarrow \{0, 1, 2, 3\}$ satisfying the condition that every vertex $u$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ for which $f(v) = 3$ and also every vertex $u$ for which $f(u) = 1$ is adjacent to at least one vertex $v$ for which $f(v) = 2$. The weight of a Strong Roman dominating function is the value $f(V) = \sum_{u \in V} f(u)$. The minimum weight of a Strong Roman dominating function on a graph $G$ is called
the Strong Roman domination number of $G$ which is denoted by $\gamma_{SR}(G)$. A Strong Roman dominating function can be displayed with $f = (V_0', V_1, V_2', V_3)$, or $f = (V_0', V_1, V_2', V_3')$ to refer to $f$, where $V_i = \{v \in V(G) \mid f(v) = i\}, (i = 0, 1, 2, 3)$. Clearly in a Strong Roman dominating function of weight $\gamma_{SR}(G)$, no vertex needs to be assigned the value 1. Therefore, we can assume that for each Strong Roman domination function, $V_1 = \phi$. In this paper, we show that

$$\frac{1}{2} \gamma_{SR}(G) \leq \gamma_R(G) \leq \frac{2}{3} \gamma_{SR}(G)$$

and characterize the Lower bound. Also, we characterize all trees $T$ with $\gamma_R(T) = \frac{2}{3} \gamma_{SR}(T)$.

In this paper we quote the following Theorems from [7, 8].

**Theorem 1 [8]:** For any graph $G$, $2 \gamma(G) \leq \gamma_{SR}(G) \leq 3 \gamma(G)$.

**Theorem 2 [7]:** For any graph $G$, $\gamma_R(G) \leq \gamma_{SR}(G)$.

**Theorem 3 [7]:** Let $f = (V_0, V_1, V_2, V_3)$ be any $\gamma_{SR}(G)$ - function. Then

a) $G[V_2]$ has maximum degree 0.

b) No edge of $G$ joins $V_2$ and $V_3$.

c) No edge of $G$ joins $V_1$ and $V_3$.

II. MAIN RESULTS

First, we present the Lower and Upper bound on Roman domination number on Strong Roman domination number.

**Theorem 4:** For any graph $G$,

$$\frac{1}{2} \gamma_{SR}(G) \leq \gamma_R(G) \leq \frac{2}{3} \gamma_{SR}(G)$$

**Proof:** Let $f = (V_0, \phi, V_2, V_3)$ be a $\gamma_{SR}(G)$ - function. In this case, the function $f' = (V_0', V_1', V_2')$, where $V_0' = V_0$, $V_1' = V_2$ and $V_2' = V_3$ is an RDF on graph $G$. Therefore

$$\gamma_R(G) \leq f'(V)$$

$$= |V_1'| + 2|V_2'|$$

$$= |V_2| + 2|V_3|$$

$$= 2|V_2| + 3|V_3| - (|V_2| + |V_3|)$$

$$= \gamma_{SR}(G) - (|V_2| + |V_3|).$$

(1)
Since $V_2 \cup V_3$ is dominating set on graph $G$, we have $\gamma(G) \leq |V_2| + |V_3|$ and thus from (1) we have
\[
\gamma_R(G) \leq \gamma_{SR}(G) - (|V_2| + |V_3|) 
\leq \gamma_{SR}(G) - \gamma(G).
\]
(2)

Using Theorem 1, $\gamma_{SR}(G) \leq 3 \gamma(G)$ implies $\frac{1}{3} \gamma_{SR}(G) \leq \gamma(G)$. Therefore from (2) we have
\[
\gamma_R(G) \leq \gamma_{SR}(G) - \frac{1}{3} \gamma(G) 
\leq \frac{2}{3} \gamma_{SR}(G).
\]

Hence the Upper bound exists. Now, to prove the existence of Lower bound, assume that $g = (V_0, V_1, V_2)$ is a $\gamma_R(G)$ – function. In this case, the function $g' = (V_0', \phi, V_2', V_3')$, where $V_0' = V_0$, $V_2' = V_1$ and $V_3' = V_2$ is an SRDF on graph $G$. Therefore
\[
\gamma_R(G) \leq g'(V) 
= 2|V_2'| + 3|V_3'| 
= 2|V_1| + 3|V_2| 
\leq 2|V_1| + 4|V_2| 
= 2 \gamma_{SR}(G).
\]

So, $\gamma_{SR}(G) \leq 2 \gamma_R(G)$ and thus $\frac{1}{2} \gamma_{SR}(G) \leq \gamma_R(G)$.

Hence the proof follows. □

Note that, the Upper bound in Theorem 4 can be viewed as an extension of the Upper bound in Theorem 2 for all graphs. Also, the boundaries presented in Theorem 4 are sharp. For sharpness of Lower and Upper bound we consider the trivial graph $K_n$ and path $P_n$, respectively. In next theorem we characterize the boundaries with $\frac{1}{2} \gamma_{SR}(G) = \gamma_R(G)$.

**Theorem 5:** Let $G$ be a graph of order $n$. Then $\frac{1}{2} \gamma_{SR}(G) = \gamma_R(G)$ if and only if $G = K_n$.

**Proof:** Suppose that $f = (V_0, V_1, V_2)$ is a $\gamma_R(G)$ – function, such that $V_2$ is maximum. In this case, the function $f' = (V_0', \phi, V_2', V_3')$, where $V_0' = V_0$, $V_2' = V_1$ and $V_3' = V_2$ is an SRDF on graph $G$. Therefore
From this above relation it follows that
\[ 2|V_1| + 3|V_2| = 2|V_1| + 4|V_2| \]
therefore \(|V_2| = 0\). Since we chose \(f\) such that \(V_2\) is maximum, we have \(V_1\) is an independent set and thus \(G = K_n\).

The converse part is obvious. □

### III. CHARACTERIZATION OF THE TREES WITH \(\gamma_R(T) = \frac{2}{3}\gamma_{SR}(T)\)

First, we begin with following supporting Lemmas.

**Lemma 6:** For any graph \(G\),
\[
\gamma_R(G) = \frac{2}{3}\gamma_{SR}(G),
\]
if and only if \(\gamma_{SR}(G) = 3\gamma(G)\) and \(\gamma_R(G) = 2\gamma(G)\).

**Proof:** Suppose that \(\gamma_R(G) = \frac{2}{3}\gamma_{SR}(G)\) and \(f = (V_0, \phi, V_2, V_3)\) is a \(\gamma_{SR}(G)\)–function. In this case, the function \(g = (V'_0, V'_1, V'_2)\), where \(V'_0 = V_0\), \(V'_1 = V_2\) and \(V'_2 = V_3\) is an SRDF on graph \(G\). Therefore
\[
\frac{2}{3}\gamma_{SR}(G) = \gamma_R(G)
\]
\[
\leq g(V)
\]
\[
= |V'_1| + 2|V'_2|
\]
\[
= |V_2| + 2|V_3|
\]
\[
\leq \gamma_{SR}(G) - |V_2| - |V_3|
\]
\[
\leq \gamma_{SR}(G) - \gamma(G).
\]
So, \(\frac{2}{3}\gamma_{SR}(G) \leq \gamma_{SR}(G) - \gamma(G)\) and thus \(3\gamma(G) \leq \gamma_{SR}(G)\). From Theorem 1, we have \(\gamma_{SR}(G) \leq 3\gamma(G)\). Hence \(\gamma_{SR}(G) = 3\gamma(G)\), also
\[ \gamma_R(G) = \frac{2}{3} \gamma_{SR}(G) \]
\[ = \frac{2}{3} (3 \gamma(G)) \]
\[ = 2 \gamma(G). \]

The converse part is obvious. \( \square \)

**Lemma 7:** Let for any graph \( G \) with \( \gamma_R(G) = \frac{2}{3} \gamma_{SR}(G) \), the function \( f = (V_0, \phi, V_2, V_3) \) be a \( \gamma_{SR}(G) \)-function. Then \( V_2 = \phi \).

**Proof:** Let \( f' = (V_0', V_1', V_2') \), where \( V_0' = V_0, V_1' = V_3 \) and \( V_2' = V_3 \) be an RDF on graph \( G \). Therefore
\[ \frac{2}{3} \gamma_{SR}(G) = \gamma_R(G) \]
\[ \leq f'(V) \]
\[ = |V_1'| + 2|V_2'| \]
\[ = |V_2| + 2|V_3| \]
\[ \leq \gamma_{SR}(G) - |V_2| - |V_3| \]

Thus
\[ |V_2| + 2|V_3| \leq \frac{1}{3} \gamma_{SR}(G) \]
\[ = \frac{2|V_2| + 3|V_3|}{3} \]
\[ = \frac{2}{3} |V_2| + |V_3|. \]

So, \( |V_2| \leq \frac{2}{3} |V_2| \) and hence \( |V_2| = 0 \). Then \( V_2 = \phi \). \( \square \)

Our aim is to characterize all trees with Upper bound where equality holds in Theorem 4. Let \( F \) be a family of trees \( T \) which is obtained from a sequence of trees \( T_1, T_2, \ldots, T_j \), \( (j \geq 1) \) such that \( T_1 \) is a Star \( K_{1,r} \), \( (r \geq 1) \) and if \( j \geq 2 \), then \( T_j \) can be obtained from \( T_{j-1} \) with one of the five Operations \( \Phi_1, \Phi_2, \Phi_3, \Phi_4 \) and \( \Phi_5 \) as shown in figures 1, 2, 3, 4 and 5, respectively.

**Operation \( \Phi_1 \):** Let \( u \) be a Strong support vertex of \( T_i \). In this case, \( T_{i+1} \) is obtained from \( T_i \) by adding a new vertex \( v \) and adding the edge \( uv \). (see Figure 1)

**Operation \( \Phi_2 \):** Let \( u \in V(T_i) \) and \( \gamma_R(T_i - u) \geq \gamma_R(T_i) \). In this case, \( T_{i+1} \) is obtained from \( T_i \) by adding a Star \( K_{1,2} \) with central vertex \( v \) and adding the edge \( uv \). (see Figure 2)
Operation $\varphi_1$: Let $u \in V(T_i)$. In this case, $T_{i+1}$ is obtained from $T_i$ by adding a Star $K_{1,2}$ with central vertex $v$ and leaf $w$ and adding the edge $u w$. (see Figure 3)

Operation $\varphi_2$: Let $u \in V(T_i)$. In this case, $T_{i+1}$ is obtained from $T_i$ by adding a Star $K_{1,3}$ with central vertex $v$ and leaf $w$ and adding the edge $u w$. (see Figure 4)

Operation $\varphi_3$: Let $u \in V(T_i)$ and $\gamma_R(T_i - u) \geq \gamma_R(T_i)$. In this case, $T_{i+1}$ is obtained from $T_i$ by adding a Path $P_3$ with central vertex $v$ and adding the edge $u v$. (see Figure 5)
In the following, we show that each tree $T$ of family $\mathcal{F}$ satisfy the condition $\gamma_R(T_i) = \frac{2}{3} \gamma_{SR}(T_i)$. 

**Lemma 8:** If $\gamma_R(T_i) = \frac{2}{3} \gamma_{SR}(T_i)$ and $T_{i+1}$ is obtained from $T_i$ with Operation $\varphi_1$, then $\gamma_R(T_{i+1}) = \frac{2}{3} \gamma_{SR}(T_{i+1})$.

**Proof:** Suppose that $f$ is a $\gamma_{SR}(T_i)$-function. Then, since $u$ is a strong support vertex, we have $f(u) = 3$. We can extend the function $f$ to an SRDF on tree $T_{i+1}$ by assigning the weight 0 to the vertex $v$. Therefore $\gamma_{SR}(T_{i+1}) \leq f(V)$

$$= \gamma_{SR}(T_i).$$

Always $\gamma_R(T_i) \leq \gamma_{SR}(T_{i+1})$. Hence $\gamma_{SR}(T_{i+1}) = \gamma_{SR}(T_i)$. Assume the same way, $g$ is a $\gamma_R(T_i)$-function. Since $u$ is a strong support vertex, we get $g(u) = 2$ and thus we can extend the function $g$ to an RDF on tree $T_{i+1}$ by assigning the weight 0 to the vertex $v$. Therefore $\gamma_{SR}(T_{i+1}) \leq g(V)$

$$= \gamma_R(T_i).$$

Since always $\gamma_R(T_i) \leq \gamma_{SR}(T_{i+1})$, we have $\gamma_{SR}(T_i) = \gamma_{SR}(T_{i+1})$.

So,$$
\gamma_R(T_{i+1}) = \gamma_R(T_i)
= \frac{2}{3} \gamma_{SR}(T_i)
= \frac{2}{3} \gamma_{SR}(T_{i+1}).
$$

Hence $\gamma_R(T_{i+1}) = \frac{2}{3} \gamma_{SR}(T_{i+1})$. □

**Lemma 9:** If $\gamma_R(T_i) = \frac{2}{3} \gamma_{SR}(T_{i+1})$ and $T_{i+1}$ is obtained from $T_i$ with Operation $\varphi_2$, then $\gamma_R(T_{i+1}) = \frac{2}{3} \gamma_{SR}(T_{i+1})$.

**Proof:** Suppose that $u \in V(T_i)$ and $\gamma_R(T_i - u) \geq \gamma_R(T_i)$. Let $v$ be a vertex of Star $K_{1,2}$ relative to Operation $\varphi_2$. If $D$ is a dominating set for tree $T_i$, then $D \cup \{v\}$ is a dominating set for tree $T_{i+1}$. Therefore

$$\gamma(T_{i+1}) \leq \gamma(T_i) + 1. \quad (1)$$

Now, suppose that $f$ is a $\gamma_{SR}(T_{i+1})$-function. Since $f|_{T_i}$ is an SRDF on tree $T_i$, we have
Otherwise, \( f \big|_{T_i - u} \) is an SRDF on tree \( T_i - u \). Thus
\[
\gamma_{SR}(T_i) = \frac{3}{2} \gamma_R(T_i) \\
\leq \frac{3}{2} \gamma_R(T_i - u) \\
\leq \gamma_{SR}(T_i - u) \\
\leq f(V_{T_i - u}) \\
\leq f(V) - 3 \\
= \gamma_{SR}(T_{i+1}) - 3.
\]
Therefore, in this case, \( \gamma_{SR}(T_i) \leq \gamma_{SR}(T_{i+1}) - 3 \).

Any \( \gamma_{SR}(T_i) - function \) can be extended to an SRDF on tree \( T_{i+1} \) by assigning the weight 3 to the vertex \( v \) and the weight 0 to all its neighbouring vertices. Therefore \( \gamma_{SR}(T_{i+1}) \leq \gamma_{SR}(T_i) + 3 \) which implies that \( \gamma_{SR}(T_{i+1}) = \gamma_{SR}(T_i) + 3 \). Now, from (1) and Lemma 6 we have
\[
\gamma_{SR}(T_{i+1}) = \gamma_{SR}(T_i) + 3 \\
= 3 \gamma(T_i) + 3 \\
= 3 \gamma(T_{i+1}).
\]

Any \( \gamma_R(T_i) - function \) can be extended to an RDF on tree \( T_{i+1} \) by assigning the weight 2 to the vertex \( v \) and 0 to all its neighbouring vertices. Therefore \( \gamma_R(T_{i+1}) \leq \gamma_R(T_i) + 2 \). Also, if \( g \) is a \( \gamma_R(T_{i+1}) - function \), then since \( v \) is a Strong support vertex, we have \( g(v) = 2 \). If \( f(u) \neq 0 \), then \( g \big|_{T_i} \) is an RDF on tree \( T_i \). Thus
\[
\gamma_R(T_i) \leq g(V_{T_i}) \\
= g(V) - 2 \\
= \gamma_R(T_{i+1}) - 2.
\]
And in case \( g(u) = 0 \), then \( g \big|_{T_i - u} \) is an RDF on tree \( T_i - u \). Therefore
\[ \gamma_R(T_i) \leq \gamma_R(T_i - u) \leq g(V|_{T_i - u}) = g(V) - 2 = \gamma_R(T_{i+1}) - 2. \]

Thus in each case, \( \gamma_R(T_i) \leq \gamma_R(T_{i+1}) - 2 \) and since we showed that \( \gamma_R(T_{i+1}) \leq \gamma_R(T_i) + 2 \), so,

\[ \gamma_R(T_{i+1}) = \gamma_R(T_i) + 2. \]

Now, from (1) and Lemma 6 we have

\[ \gamma_R(T_{i+1}) = \gamma_R(T_i) + 3 = 2\gamma(T_i) + 2 = 2\gamma(T_{i+1}). \]

Hence \( \gamma_{SR}(T_{i+1}) = 3\gamma(T_{i+1}) \) and \( \gamma_R(T_{i+1}) = 2\gamma(T_{i+1}) \). Thus Lemma 6 implying that

\[ \gamma_R(T_{i+1}) = \frac{2}{3}\gamma(T_{i+1}). \]

**Lemma 10:** If \( \gamma_R(T_i) = \frac{2}{3}\gamma(T_i) \) and \( T_{i+1} \) is obtained from \( T_i \) with Operation \( \varphi_3 \), then

\[ \gamma_R(T_{i+1}) = \frac{2}{3}\gamma(T_{i+1}). \]

**Proof:** Suppose that \( u \in V(T_i) \) and \( v \) is the central vertex and \( w \) and \( x \) are the leaves of Star \( K_{1,2} \), respectively. If \( D \) is a \( \gamma(T_i) \)-set, then \( D \cup \{v\} \) is a dominating set on tree \( T_{i+1} \). Therefore

\[ \gamma(T_{i+1}) \leq \gamma(T_i) + 1. \quad (1) \]

Suppose that \( f \) is a \( \gamma_{SR}(T_{i+1}) \)-function. Clearly, we can assume that \( f(v) = 3 \) and \( f(x) = f(w) = 0 \).

Therefore \( f|_{T_i} \) is an SRDF on tree \( T_i \) and thus

\[ \gamma_{SR}(T_i) \leq f(V|_{T_i}) = f(V) - 3 = \gamma_{SR}(T_{i+1}) - 3. \]

Any \( \gamma_{SR}(T_i) \)-function can be extended to an SRDF on tree \( T_{i+1} \) by assigning the weight 3 to the vertex \( v \) and the weight 0 to all its neighbouring vertices. Therefore

\[ \gamma_{SR}(T_{i+1}) \leq \gamma_{SR}(T_i) + 3. \]

And thus based on previous

\[ \gamma_{SR}(T_{i+1}) = \gamma_{SR}(T_i) + 3 = \gamma_{SR}(T_{i+1}). \]
Now, from (1) and Lemma 6 we have
\[ \gamma_{SR}(T_{i+1}) = 3\gamma(T_{i+1}) \]
\[ \leq 3\gamma(T_i) + 3 \]
\[ \leq \gamma_{SR}(T_i) + 3 \]
\[ = \gamma_{SR}(T_{i+1}). \]

Therefore \( \gamma_{SR}(T_{i+1}) = 3\gamma(T_{i+1}) \). Now, to prove the problem based on Lemma 6, we have its enough to show that \( \gamma_{R}(T_{i+1}) = 2\gamma(T_{i+1}) \). Let \( g \) be a \( \gamma_{R}(T_{i+1}) \)-function. In this case, we can assume that \( g(v) = 2 \) and \( g(x) = g(w) = 0 \). Therefore \( g|_{T_i} \) is an RDF on tree \( T_i \). And thus
\[ \gamma_{R}(T_i) \leq g(V |_{T_i}) \]
\[ = g(V) - 2 \]
\[ = \gamma_{R}(T_{i+1}) - 2. \]

Any \( \gamma_{R}(T_i) \)-function can be extended to an RDF on tree \( T_{i+1} \) by assigning the weight 2 to the vertex \( v \) and the weight 0 to all its neighbouring vertices. Therefore \( \gamma_{R}(T_{i+1}) \leq \gamma_{R}(T_i) + 2 \).

Hence \( \gamma_{R}(T_{i+1}) = \gamma_{R}(T_i) + 2 \). Now, from (1) and Lemma 6 we have
\[ \gamma_{R}(T_{i+1}) \leq 2\gamma(T_{i+1}) \]
\[ \leq 2\gamma(T_i) + 2 \]
\[ = \gamma_{R}(T_i) + 2 \]
\[ = \gamma_{R}(T_{i+1}). \]

So,
\[ \gamma_{R}(T_{i+1}) = 2\gamma(T_{i+1}). \]

Hence the proof. \( \square \)

**Lemma 11:** If \( \gamma_{R}(T_i) = \frac{2}{3}\gamma_{SR}(T_i) \) and \( T_{i+1} \) is obtained from \( T_i \) with Operation \( \varphi_4 \), then
\[ \gamma_{R}(T_{i+1}) = \frac{2}{3}\gamma_{SR}(T_{i+1}). \]

**Proof:** Suppose that \( u \) is a vertex of tree \( T_i \) and \( w \) is a leaf of Star \( K_{1,2} \) with central vertex \( v \) relative to Operation \( \varphi_4 \). If \( D \) is a dominating set on tree \( T_i \), then \( D \cup \{v\} \) is a dominating set on tree \( T_{i+1} \). Therefore
\[ \gamma(T_{i+1}) \leq \gamma(T_{i}) + 1. \]  \( (1) \)

Any \( \gamma_{SR}(T_i) \)-function can be extended to an SRDF on tree \( T_{i+1} \) by assigning the weight 3 to the vertex \( v \) and the weight 0 to all its neighbouring vertices. Therefore
\[ \gamma_{SR}(T_{i+1}) \leq \gamma_{SR}(T_i) + 3. \]
Suppose that \( f \) is a \( \gamma_{SR}(T_{i+1}) \) function. Since \( v \) is a Strong support vertex, we have \( f(v) = 3 \). Also, we can assume that \( f(w) = 0 \). In this case, \( f|_{T_i} \) is an SRDF on tree \( T_i \). Thus
\[
\gamma_{SR}(T_i) \leq f(V|_{T_i}) = f(V) - 3 = \gamma_{SR}(T_{i+1}) - 3.
\]
Therefore \( \gamma_{SR}(T_{i+1}) = \gamma_{SR}(T_i) + 3 \). Now, from (1) and Lemma 6 we have
\[
\gamma_{SR}(T_{i+1}) = 3\gamma(T_{i+1}) \leq 3\gamma(T_i) + 3 \\
\leq \gamma_{SR}(T_i) + 3 \\
= \gamma_{SR}(T_{i+1}).
\]
Hence \( \gamma_{SR}(T_{i+1}) = 3\gamma(T_{i+1}) \). Now, suppose that \( f \) is a \( \gamma_{R}(T_{i+1}) \) function. Since \( v \) is a Strong support vertex, we can suppose that \( f(v) = 3 \) and each of its neighbouring vertices has the weight 0. Therefore \( f|_{T_i} \) is an RDF on tree \( T_i \). Thus \( \gamma_{R}(T_i) \leq \gamma_{R}(T_{i+1}) - 2 \). Any \( \gamma_{R}(T_i) \) function can be extended to an RDF by assigning the weight 2 to the vertex \( v \) and the weight 0 to all its neighbouring vertices. Therefore \( \gamma_{R}(T_{i+1}) \leq \gamma_{R}(T_i) + 2 \). So,
\[
\gamma_{R}(T_{i+1}) = \gamma_{R}(T_i) + 2.
\]
Hence based on Lemma 6 we have \( \gamma_{R}(T_{i+1}) = \frac{2}{3}\gamma_{SR}(T_{i+1}). \)

**Lemma 12:** If \( \gamma_{R}(T_i) = \frac{2}{3}\gamma_{SR}(T_i) \) and \( T_{i+1} \) is obtained from \( T_i \) with Operation \( \varphi_5 \), then
\[
\gamma_{R}(T_{i+1}) = \frac{2}{3}\gamma_{SR}(T_{i+1}).
\]

**Proof:** Let \( u \in V(T_i) \) and \( P_5 = xywvz \) be a Path with five vertices relative to Operation \( \varphi_5 \). If \( D \) is a \( \gamma(T_i) \) set, then \( D \cup \{y, w\} \) is a dominating set on tree \( T_{i+1} \). Therefore
\[
\gamma(T_{i+1}) \leq \gamma(T_i) + 2. \tag{1}
\]
Assume that \( f \) is a \( \gamma_{R}(T_{i+1}) \) function. If \( f(v) \neq 2 \), then \( f|_{T_i} \) is an RDF on tree \( T_i \).
Since \( f(V(P_5)) \geq 4 \), we have
\[
\gamma_{R}(T_i) \leq f(V|_{T_i}) = f(V) - 4 = \gamma_{R}(T_{i+1}) - 4.
\]
Now, suppose that $f(v) = 2$. In this case, we can assume that $f(u) = 0$ and therefore $f|_{T_i-u}$ is an RDF on tree $T_i - u$. Based on assumption we have

$$\gamma_R(T_i) \leq \gamma_R(T_i - u)$$

$$\leq f(V|_{T_i-u})$$

$$= f(V) - 4$$

$$= \gamma_R(T_{i+1}) - 4.$$ 

Thus in each case, $\gamma_R(T_i) \leq \gamma_R(T_{i+1}) - 4$. Any $\gamma_R(T_i)$ function can be extended to an RDF on tree $T_{i+1}$ by assigning the weight 2 to y and w and the weight 0 to x, v and z. Therefore $\gamma_R(T_{i+1}) \leq \gamma_R(T_i) + 4$. Then same as before $\gamma_R(T_{i+1}) = \gamma_R(T_i) + 4$. Now, from (1) and Lemma 6 we have

$$\gamma_R(T_{i+1}) \leq 2\gamma(T_{i+1})$$

$$\leq 2\gamma(T_i) + 4$$

$$= \gamma_R(T_i) + 4$$

$$= \gamma_R(T_{i+1}).$$

So, we conclude that $\gamma_R(T_{i+1}) = 2\gamma(T_{i+1})$. Based on Lemma 6, it is enough to show that $\gamma_{SR}(T_{i+1}) = 3\gamma(T_{i+1})$. If $g$ is a $\gamma_{SR}(T_i)$ function, then the function $g$ can be extended to an SRDF on tree $T_{i+1}$ by assigning the weight 3 to y and w and the weight 0 to x, v and z. Therefore $\gamma_{SR}(T_{i+1}) \leq \gamma_{SR}(T_i) + 6$. Now, suppose that $f$ is a $\gamma_{SR}(T_{i+1})$ function. If $f(u) \neq 0$, then $f|_{T_i}$ is an SRDF on tree $T_i$ and also $f(V(P_i)) \geq 6$. Therefore

$$\gamma_{SR}(T_i) \leq g(V|_{T_i})$$

$$= g(V) - 6$$

$$= \gamma_{SR}(T_{i+1}) - 6.$$ 

Otherwise, $g(u) = 0$. Then $g|_{T_i-u}$ is an SRDF on tree $T_i - u$. Based on assumption $\gamma_R(T_{i-u}) \geq \gamma_R(T_i)$. So,
\[ \gamma_{SR}(T_i) = \frac{3}{2} \gamma_R(T_i) \]
\[ \leq \frac{3}{2} \gamma_R(T_i - u) \]
\[ \leq \gamma_{SR}(T_i - u) \]
\[ \leq g(V | T_i - u) \]
\[ \leq g(V) - 6 \]
\[ = \gamma_{SR}(T_i) - 6. \]

Therefore in each case, \( \gamma_{SR}(T_i) \leq \gamma_{SR}(T_i) - 6 \) and since we showed that earlier \( \gamma_{SR}(T_{i+1}) \leq \gamma_{SR}(T_i) + 6 \).

Hence \( \gamma_{SR}(T_{i+1}) = \gamma_{SR}(T_i) + 6 \). Now, from (1), Lemma 6 and Theorem 1 we have

\[ \gamma_{SR}(T_{i+1}) \leq 3 \gamma(T_{i+1}) \]
\[ \leq 3 \gamma(T_i) + 6 \]
\[ \leq \gamma_{SR}(T_i) + 6 \]
\[ = \gamma_{SR}(T_{i+1}). \]

Therefore \( \gamma_{SR}(T_{i+1}) = 3 \gamma(T_{i+1}) \) and thus the problem is solved. \( \square \)

In this section we provide a constructive characterization of trees with \( \gamma_R(T) = \frac{2}{3} \gamma_{SR}(T) \).

**Theorem 13:** The tree \( T \) satisfies the condition \( \gamma_R(T) = \frac{2}{3} \gamma_{SR}(T) \) if and only if \( T \in \mathcal{F} \).

**Proof:** First we use induction on the order \( n \geq 3 \) of the tree \( T \) in which \( \gamma_R(T) = \frac{2}{3} \gamma_{SR}(T) \). If \( \text{diam}(T) = 2 \), then \( T \) is a Star. Therefore \( T \in \mathcal{F} \). Now, assume that \( \text{diam}(T) = 3 \). Thus \( T \) is a Double Star \( (S_{r,s}) \) on \( 1 \leq s \leq r \).

Let \( s = 1 \). In this case, \( T \) includes a vertex \( v \) of degree 2 which has exactly an adjacent leaf. The function in which the adjacent leaves to \( v \) have the weight 3 and all the other vertices have the weight 0 is a \( \gamma_{SR}(T) - \text{function} \).

Therefore \( \gamma_{SR}(T) = 5 \). Also, the function in which the adjacent leaves to \( v \) have the weight 2, the non-adjacent leaves to \( v \) have the weight 3 and all the other vertices have the weight 0 is a \( \gamma_R(T) - \text{function} \). Therefore \( \gamma_R(T) = 3 \). Hence \( \gamma_R(T) \neq \frac{2}{3} \gamma_{SR}(T) \) which is a contradiction. Thus \( s \geq 2 \).

Let \( u \) be a non-adjacent leaf to \( v \). Then \( T \) is a Double Star with central vertices \( u \) and \( v \) where \( \text{deg}(u) \geq 3 \) and \( \text{deg}(v) \geq 3 \). We consider \( T_1 \) and \( T_2 \) are two components of \( T - u \) where \( u \in V(T_i) \) and \( v \in V(T_2) \). Clearly, \( T_i, (i = 1, 2) \) is a Star of order at least 3. Therefore \( \gamma_R(T_i) = 2 \) and \( \gamma_{SR}(T_i) = 3 \), \( (i = 1, 2) \). If \( \text{deg}(v) = 3 \), since \( \gamma_R(T_1 - u) \geq \gamma_R(T_1) \), then the tree \( T \) is obtained from \( T_1 \in \mathcal{F} \) by Operation \( \varphi_2 \).

Now, assume that \( \text{deg}(v) \geq 4 \).
and \( w \) is adjacent leaf to \( v \). We put \( T' = T - w \). In this case, it is obvious that \( \gamma (T') = 4 \) and \( \gamma_{SR} (T') = 6 \). So, \( \gamma (T') = \frac{2}{3} \gamma_{SR} (T') \). Therefore under the hypothesis, \( T' \in \mathcal{F} \). Thus \( T \) is obtained from \( T' \) by Operation \( \varphi_1 \).

Hence we can assume that \( \text{diam} (T) \geq 4 \). Let \( T \) has support vertex \( u \) with at least three adjacent leaves and \( w \) be adjacent leaf to \( u \). We put \( T'' = T - w \). We show that \( T' \in \mathcal{F} \). It is enough to prove that \( \gamma (T') = 2 \gamma (T') \) and \( \gamma_{SR} (T') = 3 \gamma (T') \). Assume that \( D \) is a \( \gamma (T') \)-set on tree \( T' \). Since \( u \) is a support vertex on tree \( T' \), we get \( u \in D \). Since in the tree \( T \) and the vertex \( u \) is adjacent to \( w \), we get \( D \) is a dominating set on tree \( T \). Thus \( \gamma (T) \leq |D| = \gamma (T') \). Always \( \gamma (T') \leq \gamma (T) \). Therefore \( \gamma (T) = \gamma (T') \).

Let \( f \) be a \( \gamma (T') \)-function. Since \( u \) is a Strong support vertex, we get \( f(u) = 2 \) and thus \( f \) can be extended to an RDF on tree \( T \) by assigning the weight 0 to the vertex \( w \). Therefore, based on Theorem 1 and Lemma 6 we have

\[
\gamma (T) = \gamma (T') \\
\leq 2 \gamma (T') \\
= 2 \gamma (T) \\
= 2 \gamma (T) \\
= \gamma (T)
\]

Hence \( \gamma (T) = 2 \gamma (T) \). Now, assume that \( g \) is a \( \gamma_{SR} (T') \)-function. Since \( u \) is a Strong support vertex, we get \( g(u) = 3 \). Thus \( g \) can be extended to an SRDF on tree \( T \) by assigning the weight 0 to the vertex \( w \). Therefore, from Theorem 1 and Lemma 6 we have

\[
\gamma_{SR} (T) \leq g(V) \\
= \gamma_{SR} (T') \\
\leq 3 \gamma (T') \\
= 3 \gamma (T) \\
= \gamma_{SR} (T)
\]

Then \( \gamma_{SR} (T') = 3 \gamma (T') \). So, \( T' \in \mathcal{F} \) and thus \( T \) is obtained from \( T' \) by Operation \( \varphi_1 \). Hence \( T \in \mathcal{F} \).

We say assume for the next, the following holds.

Fact: \( T \) has no Strong support vertex with three adjacent leaves to it.

Let \( x_0 x_1 \ldots x_d \) be a Diagonal path. We root the tree \( T \) at vertex \( x_0 \). Based on Fact, \( \deg(x_{d-1}) \leq 3 \). We consider the following cases:

**Case 1:** \( \deg(x_{d-1}) = 3 \). Let \( \deg(x_{d-1}) \geq 3 \). In this case, each child of \( x_{d-2} \) is a support vertex with a leaf.

We put \( T'' = T - T_{x_{d-1}} \). If \( D \) is a \( \gamma (T) \)-set, since \( x_{d-2} \) is either a support vertex or has a support vertex adjacent to it. Therefore, \( D - \{x_{d-1}\} \) is a dominating set on tree \( T'' \). Thus
Any $\gamma(T')$--set can be extended to a dominating set on tree $T$ by adding $x_{d-1}$ to it. Therefore $\gamma(T) \leq \gamma(T') + 1$. Hence $\gamma(T) = \gamma(T') + 1$. Let $f$ be a $\gamma_{SR}(T')$--function. In this case, the function $f$ can be extended to an SRDF on tree $T$ by assigning the weight 3 to $x_{d-1}$ and the weight 0 to all their neighbouring leaves. Therefore $\gamma_{SR}(T) \leq \gamma_{SR}(T') + 3$. Now, from Lemma 6 and Theorem 1 we have

$$\gamma_{SR}(T) \leq \gamma_{SR}(T') + 3$$

$$\leq 3\gamma(T') + 3$$

$$= 3\gamma(T)$$

$$= \gamma_{SR}(T).$$

Therefore $\gamma_{SR}(T') = 3\gamma(T')$. Also, if $g$ is a $\gamma_R(T')$--function, then $g$ can be extended to an RDF on tree $T$ by assigning the weight 2 to $x_{d-1}$ and the weight 0 to all their neighbouring leaves. Therefore $\gamma_R(T) \leq \gamma_R(T') + 2$. Now, from Theorem 1 and Lemma 6 we have

$$\gamma_R(T) \leq \gamma_R(T') + 2$$

$$\leq 2\gamma(T') + 2$$

$$= 2\gamma(T)$$

$$= \gamma_R(T).$$

It follows from above relation that $\gamma_R(T') = 2\gamma(T')$. Also, we showed that before $\gamma_{SR}(T') = 3\gamma(T')$. Therefore it follows from Lemma 6 that $\gamma_R(T') = \frac{2}{3} \gamma_{SR}(T')$. So, by the induction hypothesis $T' \in \mathcal{F}$. Now, suppose that $\gamma_R(T' - x_{d-2}) < \gamma_R(T')$. In this case, if $f'$ is a $\gamma_R(T' - x_{d-2})$--function, then the function $f'$ can be extended to an RDF on tree $T$ by assigning the weight 2 to $x_{d-1}$ and the weight 0 to all their neighbouring. Therefore $\gamma_R(T) \leq \gamma_R(T' - x_{d-2}) + 2$. So,

$$\gamma_R(T) \leq \gamma_R(T' - x_{d-2}) + 2$$

$$\leq \gamma_R(T') + 2$$

$$= 2\gamma(T') + 2$$

$$= 2\gamma(T)$$

$$= \gamma_R(T).$$

Which is a contradiction. Thus $\gamma_R(T' - x_{d-2}) \geq \gamma_R(T')$ and therefore $T$ is obtained from $T'$ by Operation $\varphi_2$. Hence $T \in \mathcal{F}$. Now, suppose that $\deg(x_{d-2}) = 2$. We put $T' = T - T_{x_{d-2}}$. Let $D$ be a $\gamma(T)$--set.
Clearly, we can assume \( x_{d-2} \notin D \) and \( x_{d-1} \in D \). In this case, \( D - \{ x_{d-1} \} \) is a dominating set on tree \( T' \). Therefore \( \gamma(T') \leq \gamma(T) - 1 \). Any \( \gamma(T') - \text{set} \) can be extended to a dominating set on tree \( T \) by adding \( x_{d-1} \) to it. Thus \( \gamma(T) \leq \gamma(T') + 1 \) and therefore \( \gamma(T) = \gamma(T') + 1 \). Now, let \( g' \) be a \( \gamma_R(T') - \text{function} \). In this case, the function \( g' \) can be extended to an RDF on tree \( T \) by assigning the weight \( 2 \) to \( x_{d-1} \) and the weight \( 0 \) to all its neighbours. Therefore \( \gamma_R(T) \leq \gamma_R(T') + 2 \). From Lemma 6 and Theorem 1 we have

\[
\gamma_R(T) \leq \gamma_R(T') + 2 \\
\leq 2\gamma(T') + 2 \\
= 2\gamma(T) \\
= \gamma_R(T).
\]

It is implied from above relation that \( \gamma_R(T') = 2\gamma_R(T') \). Also, if \( f' \) is a \( \gamma_{SR}(T') - \text{function} \), then the function \( f' \) can be extended to an SRDF on tree \( T \) by assigning the weight \( 3 \) to \( x_{d-1} \) and the weight \( 0 \) to all its neighbours. Therefore \( \gamma_{SR}(T) \leq \gamma_{SR}(T') + 3 \). From Lemma 6 and Theorem 1 we have

\[
\gamma_{SR}(T) \leq \gamma_{SR}(T') + 2 \\
\leq 3\gamma(T') + 3 \\
= 3\gamma(T) \\
= \gamma_{SR}(T).
\]

Above relation implies that \( \gamma_{SR}(T') = 3\gamma(T') \). Considering the Lemma 6 and what we already showed \( \gamma_R(T') = 2\gamma(T') \), we conclude \( \gamma_R(T') = \frac{2}{3}\gamma_{SR}(T') \). So, with the inductive hypothesis we have \( T' \in \mathcal{F} \). Thus \( T \) is obtained from \( T' \) by Operation \( \phi_2 \). Hence \( T \in \mathcal{F} \).

**Case 2:** \( \deg(x_{d-2}) = 2 \). If \( \deg(x_{d-2}) \geq 3 \), then each child of \( x_{d-2} \) is either a leaf or a support vertex of degree 2. Since each support child of \( x_{d-2} \) other than \( x_{d-1} \) plays the same role of \( x_{d-1} \), we can assume that have degree 2. Let \( f = (V_0, V_1, V_2, V_3) \), where \( V_1 = \phi \) be a \( \gamma_{SR}(T') - \text{function} \). Since \( \gamma_R(T) = \frac{2}{3}\gamma_{SR}(T) \), based on Lemma 7, we get \( V_2 = \phi \). Therefore if \( f(x_{d-2}) \neq 0 \), then \( f(x_{d-2}) = 3 \) holds. In this case, we can assume that \( f(x_{d-1}) = 0 \) and \( f(x_{d-2}) = 2 \) which is a contradiction with \( V_2 = \phi \). Therefore \( f(x_{d-2}) = 0 \). In this case, \( x_{d-2} \) does not have adjacent leaf. Otherwise, the adjacent leaf should be weighted 2 which is a contradiction with \( V_2 = \phi \). We show that in this case \( x_{d-2} \) has at most an adjacent support other than \( x_{d-1} \). On contrary suppose that \( u_1 \) and \( u_2 \) are two support children of \( x_{d-2} \) other
than $x_{d-1}$. Let $v_1$ and $v_2$ be adjacent leaves to $u_1$ and $u_2$, respectively. Since $V_2 = \phi$, we can assume $f(u_1) = f(u_2) = 3$ and $f(v_1) = f(v_2) = v$. In this case, we define the function $g$ as follows: Let

$$
g(x_d) = g(v_1) = g(v_2) = 2,
g(x_{d-1}) = g(u_1) = g(u_2) = 0,
g(x_{d-2}) = 3,
$$

and other vertices take the same weight of the function. In this case, $g$ is a $\gamma_{SR}(T)$-function, where the set of vertices have weight 2 are non-empty is contradiction with Lemma 7. Therefore $x_{d-2}$ has at most a support child other than $x_{d-1}$. Let $u$ be a support child of $x_{d-2}$ other than $x_{d-1}$ and $v$ be an adjacent leaf to $x_{d-2}$. We put $T' = T - T_{x_{d-2}}$. If $D'$ is a $\gamma(T')$-set, then $D' \cup \{x_{d-1}, u\}$ is a dominating set on tree $T$. Therefore $\gamma(T') \leq \gamma(T') + 2$. On the other hand, if $\{x_{d-1}, u\} \subseteq D$ and $x_{d-2} \notin D$, then $D - \{x_{d-1}, u\}$ is a dominating set on tree $T'$. Thus $\gamma(T') \leq \gamma(T) - 2$. Consequently, we conclude that $\gamma(T) = \gamma(T') + 2$. Now, let $f'$ be a $\gamma_{SR}(T')$-function, then the function $f'$ can be extended to an SRDF on tree $T$ by assigning the weight 3 to $x_{d-1}$ and $u$ and the weight 0 to all their neighbouring. Therefore $\gamma_{SR}(T) \leq \gamma_{SR}(T') + 6$. Now, from Lemma 6 and Theorem 1, we conclude

$$
\gamma_{SR}(T) \leq \gamma_{SR}(T') + 6 \\
\leq 3\gamma(T') + 6 \\
= 3\gamma(T) \\
= \gamma_{SR}(T).
$$

Above relation implies that $\gamma_{SR}(T') = 3\gamma(T')$. Also, any $\gamma_{SR}(T')$-function can be extended to an RDF on tree $T$ by assigning the weight 2 to $x_{d-1}$ and $u$ and the weight 0 to all their neighbouring. Therefore $\gamma_{R}(T) \leq \gamma_{R}(T') + 4$. From Lemma 6 and Theorem 1, we have

$$
\gamma_{R}(T) \leq \gamma_{SR}(T') + 4 \\
\leq 2\gamma(T') + 4 \\
= 2\gamma(T) \\
= \gamma_{R}(T).
$$

Therefore $\gamma_{R}(T') = 2\gamma_{R}(T')$. And so consider that $\gamma_{SR}(T') = 3\gamma(T')$, from Lemma 6 it implies that $\gamma_{R}(T') = \frac{2}{3}\gamma_{SR}(T')$. Hence assuming the induction $T' \in \mathcal{J}$. Now, suppose that $\gamma_{R}(T' - x_{d-3}) < \gamma_{R}(T')$. In this case, any $\gamma_{R}(T' - x_{d-3})$-function can be extended to an RDF on tree $T$ by assigning the weight 2
to \(x_{d-2}\), the weight 0 to \(u\) and \(x_{d-1}\) and the weight 1 to \(v\) and \(x_d\). Thus \(\gamma_R(T) \leq \gamma_R(T' - x_{d-3}) + 4\). And therefore

\[
\gamma_R(T) \leq \gamma_R(T' - x_{d-3}) + 4 \\
\leq \gamma_R(T') + 4 \\
\leq 2\gamma(T') + 4 \\
= 2\gamma(T) \\
= \gamma_R(T),
\]

which is a contradiction. So, \(\gamma_R(T' - x_{d-3}) \geq \gamma_R(T')\) and thus \(T\) is obtained from \(T'\) with Operation \(\phi_3\). Hence \(T \in \mathcal{F}\). Now, suppose that \(\deg(x_{d-2}) = 2\). We put \(T = T - x_{d-2}\). If \(D\) is a \(\gamma(T)\)-set, then we can assume that \(x_{d-1} \in D\) and \(x_{d-2} \notin D\). Therefore \(D - \{x_{d-1}\}\) is a dominating set on tree \(T'\). Thus \(\gamma(T') \leq \gamma(T) - 1\). Since \(\gamma(T')\) - set can be extended to a dominating set on tree \(T\) by adding \(x_{d-1}\) to it, we have \(\gamma(T) \leq \gamma(T') + 1\) and thus \(\gamma(T) = \gamma(T') + 1\). Now, suppose that \(f\) is a \(\gamma_{SR}(T)\)-function. Based on Lemma 7 there is no vertex which has weight 2. Therefore we can assume that \(f(x_{d}) = f(x_{d-2}) = 0\) and \(f(x_{d-1}) = 3\). In this case, \(f\mid_T\) is a \(\gamma_{SR}(T')\)-function. Thus \(\gamma_{SR}(T') \leq \gamma_{SR}(T) + 3\). Any \(\gamma_{SR}(T')\) - function can be extended to an SRDF on tree \(T\) by assigning the weight 3 to \(x_{d-1}\) and the weight 0 to all its neighbouring. Therefore, \(\gamma_{SR}(T) \leq \gamma_{SR}(T') + 3\). Consequently, we conclude that \(\gamma_{SR}(T) = \gamma_{SR}(T') + 3\) and in this case, based on Lemma 6, Theorem 1 and previous relation we have

\[
\gamma_{SR}(T') \leq \gamma_{SR}(T) - 3 \\
\leq 3\gamma(T) - 3 \\
= 3(\gamma(T) - 1) \\
= 3\gamma(T').
\]

Now, let \(f\) be a \(\gamma_R(T)\)-function. If \(f(x_{d-2}) = 2\) and \(f(x_{d-3}) = 0\), then \(f(x_{d-1}) = 0\) and \(f(x_{d}) = 1\). By changing the weight of \(x_{d-2}\) and \(x_{d}\) to 0, the weight of \(x_{d-1}\) to 2 and the weight of \(x_{d-3}\) to 1 a new \(\gamma_R(T)\)-function is obtained such that \(f(x_{d-2}) = 0\). Also, if \(f(x_{d-2}) = 1\), then \(f(x_{d}) + f(x_{d-1}) = 2\). By changing the weight of \(x_{d-2}\) and \(x_{d}\) to 0 and the weight of \(x_{d-1}\) to 2 an RDF with weight less than \(\gamma_R(T)\) is obtained which is a contradiction. Therefore we can assume that \(f(x_{d}) + f(x_{d-2}) = 0\) and \(f(x_{d-1}) = 2\). Thus \(f\mid_T\) is an RDF on tree \(T'\). Hence \(\gamma_R(T') \leq \gamma_R(T) - 2\). Any \(\gamma_R(T')\) - function can be extended to an RDF on tree \(T\) by assigning the weight 2 to \(x_{d-1}\) and the weight 0 to \(x_{d}\) and \(x_{d-2}\). Thus \(\gamma_R(T) \leq \gamma_R(T') + 2\). It follows from above
inequality that $\gamma_R(T) = \gamma_R(T') + 2$. From Lemma 6, Theorem 1 and $\gamma_R(T) = \gamma_R(T') + 1$ we conclude that

$$\gamma_R(T') \leq \gamma_R(T) - 2 = 2\gamma(T) - 2 = 2\gamma(T').$$

We have already show that $\gamma_{SR}(T') = 3\gamma(T')$ and the previous relation conclude that $T' \in \mathcal{F}$ and $T$ is obtained from $T'$ with Operation $\varphi_3$. Therefore $T \in \mathcal{F}$.

Hence in each case $T \in \mathcal{F}$.

Conversely, let $T \in \mathcal{F}$. In this case, by induction on the number of Operations that constructed the tree $T$ and by using Lemma 8, 9, 10, 11 and 12 we can show that $\gamma_R(T) = \frac{2}{3}\gamma_{SR}(T)$. □

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