

Roman and Strong Roman Domination Numbers in Graphs

Narges Mohsenitonekaboni¹, G. Suresh Singh²

Research Scholar, Department of Mathematics, University of Kerala, Kariavattom, Thiruvananthapuram, India¹

Professor and Head, Department of Mathematics, University of Kerala, Kariavattom, Thiruvananthapuram, India²

ABSTRACT: Let $f : V \rightarrow \{0, 1, 2\}$ be a function having the property that for every vertex $v \in V$ with $f(v) = 0$, there exists a neighbourhood $u \in N(v)$ with $f(u) = 2$. Such a function is called a *Roman dominating function* or just an RDF. The weight of an RDF is the value $f(V) = \sum_{u \in V} f(u)$. The minimum weight of an RDF on G is called the *Roman domination number* of G and is denoted by $\gamma_R(G)$. The function $f : V \rightarrow \{0, 1, 2, 3\}$ is named a *Strong Roman dominating function* on a graph G whenever for each vertex u with $f(u) = 0$ there exists an adjacent vertex v to u such that $f(v) = 3$. Also for each vertex u with $f(u) = 1$ there should be at least an adjacent vertex v to u weighted 2. The weight of an SRDF is the value $f(V) = \sum_{u \in V} f(u)$. *Strong Roman domination number* of graph G which we denoted by $\gamma_{SR}(G)$ is minimum weight of an SRDF on G . In this paper, we show that

$$\frac{1}{2}\gamma_{SR}(G) \leq \gamma_R(G) \leq \frac{2}{3}\gamma_{SR}(G).$$

And we study the Roman and Strong Roman domination of the trees and we show that $\gamma_R(T) = \frac{2}{3}\gamma_{SR}(T)$.

KEYWORDS: Roman domination, Strong Roman domination, Tree.

I. INTRODUCTION

For notations and graph theoretic terminologies not defined herein, we refer [3, 4, 9]. Let $G = (V, E)$ be a finite simple graph with the vertex set $V = V(G)$ and the edge set $E = E(G)$. Let $|V(G)| = n(G) = n$ be the order of graph G . For any vertex $v \in V$, the open neighbourhood of v is the set $N(v) = \{u \in V \mid uv \in E\}$ and the closed neighbourhood of v is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighbourhood of S is the set $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighbourhood of S is the set $N[S] = N(S) \cup S$. The degree of a vertex $v \in V$ is equals to $|N(v)|$ and denoted by $\deg(v)$. The maximum and minimum degree of a graph G are

International Journal of Innovative Research in Science, Engineering and Technology

(A High Impact Factor, Monthly, Peer Reviewed Journal)

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denoted by $\Delta(G)$ and $\delta(G)$, respectively. Let $S \subset V$ be any subset of vertices of G . Then the *induced sub-graph* $G[S]$ is the graph whose set is S and whose edge set consists of all of the edges in E that have both endpoints in S . The induced sub-graph $G[S]$ may also be called the *sub-graph induced in G by S* . A *walk* in a graph G is a finite sequence $W = v_0 e_1 v_1 e_2 v_2 \dots e_n v_n$ whose terms are alternatively vertices and edges such that for $1 \leq i \leq n$ the ends of e_i are v_{i-1} and v_i . The *length* of a walk is the number of edges occur in the walk. The *distance* between two vertices u and v in graph G is the minimum length of a path joining them. The *diameter* of a graph G is the maximum distance between vertices of G , denoted $diam(G)$. The minimum diameter of graph G is called a *diagonal* in G . A *non-trivial* graph, is a graph G of order at least 2. A walk consisting of $k + 1$ distinct vertices $v_0 v_1 \dots v_k$ is a *path*, and if $v_0 = v_k$ then these vertices form a *cycle*. A graph G is *connected* if for every pair of vertices v and x in $V(G)$, there is a $v-x$ path. Otherwise, G is *disconnected*. A tree T is a connected graph with no cycles. Every tree T with n vertices has $m = n - 1$ edges. A Star $K_{1,n}$ is a *tree* with a vertex of degree n and n vertices of degree 1. Observe that a star is a type of tree. A *leaf* is a vertex with degree 1 and a *Support vertex* is an adjacent vertex to a leaf. A *Strong support vertex* is a vertex which adjacent to at least two leaves. In this paper, all support vertices and all leaves of tree T are denoted by $S(T)$ and $L(T)$, respectively. A rooted tree T distinguishes one vertex r called the *root*. For each vertex $v \neq r$ of T the *parent* of v is the neighbourhood of v on the unique (r, v) -*path*, while a *child* of v is any other neighbourhood of v . A simple graph G , is said to be *complete* if each pair of distinct vertices is joined by an edge. Generally K_n denotes a complete graph on n vertices. The *complement* \bar{G} of a graph G has $V(\bar{G}) = V(G)$ and $\{u, v\} \in E(\bar{G})$ if and only if $\{u, v\} \notin E(G)$. Thus, the complement of a complete graph is the empty graph.

A set $S \subseteq V$ is a *dominating set* if $N[S] = V$. The *domination number* is the minimum cardinality of a dominating set in G and denoted by $\gamma(G)$, see [2].

A *Roman dominating function* on a graph $G = (V, E)$ is a function $f : V \rightarrow \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a Roman dominating function is the value $f(V) = \sum_{u \in V} f(u)$. The minimum weight of a Roman

dominating function on a graph G is called the *Roman domination number* of G which is denoted by $\gamma_R(G)$, see [1, 10, 11].

Other notations and terminologies not explicitly given here could be find in [3, 4, 5, 6, 12]. K. Selvakumar et al. in [7] introduced Strong Roman domination in 2016. A *Strong Roman dominating function* on a graph $G = (V, E)$ is a function $f : V \rightarrow \{0, 1, 2, 3\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 3$ and also every vertex u for which $f(u) = 1$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a Strong Roman dominating function is the value $f(V) = \sum_{u \in V} f(u)$. The minimum weight of a Strong Roman dominating function on a graph G is called

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(A High Impact Factor, Monthly, Peer Reviewed Journal)

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the Strong Roman domination number of G which is denoted by $\gamma_{SR}(G)$. A Strong Roman dominating function can be displayed with $f = (V_0, V_1, V_2, V_3)$, or $f = (V_0^f, V_1^f, V_2^f, V_3^f)$ to refer to f , where $V_i = \{v \in V(G) \mid f(v) = i\}$, ($i = 0, 1, 2, 3$). Clearly in a Strong Roman dominating function of weight $\gamma_{SR}(G)$, no vertex needs to be assigned the value 1. Therefore, we can assume that for each Strong Roman domination function, $V_1 = \emptyset$. In this paper, we show that

$$\frac{1}{2}\gamma_{SR}(G) \leq \gamma_R(G) \leq \frac{2}{3}\gamma_{SR}(G)$$

and characterize the Lower bound. Also, we characterize all trees T with $\gamma_R(T) = \frac{2}{3}\gamma_{SR}(T)$.

In this paper we quote the following Theorems from [7, 8].

Theorem 1 [8]: For any graph G ,

$$2\gamma(G) \leq \gamma_{SR}(G) \leq 3\gamma(G).$$

Theorem 2 [7]: For any graph G ,

$$\gamma_R(G) \leq \gamma_{SR}(G).$$

Theorem 3 [7]: Let $f = (V_0, V_1, V_2, V_3)$ be any $\gamma_{SR}(G)$ -function. Then

- $G[V_2]$ has maximum degree 0.
- No edge of G joins V_2 and V_3 .
- No edge of G joins V_1 and V_3 .

II. MAIN RESULTS

First, we present the Lower and Upper bound on Roman domination number on Strong Roman domination number.

Theorem 4: For any graph G ,

$$\frac{1}{2}\gamma_{SR}(G) \leq \gamma_R(G) \leq \frac{2}{3}\gamma_{SR}(G).$$

Proof: Let $f = (V_0, \emptyset, V_2, V_3)$ be a $\gamma_{SR}(G)$ -function. In this case, the function $f' = (V'_0, V'_1, V'_2)$, where $V'_0 = V_0$, $V'_1 = V_2$ and $V'_2 = V_3$ is an RDF on graph G . Therefore

$$\begin{aligned} \gamma_R(G) &\leq f'(V) \\ &= |V'_1| + 2|V'_2| \\ &= |V_2| + 2|V_3| \\ &= 2|V_2| + 3|V_3| - (|V_2| + |V_3|) \\ &= \gamma_{SR}(G) - (|V_2| + |V_3|). \end{aligned} \tag{1}$$

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Since $V_2 \cup V_3$ is dominating set on graph G , we have $\gamma(G) \leq |V_2| + |V_3|$ and thus from (1) we have

$$\begin{aligned} \gamma_R(G) &\leq \gamma_{SR}(G) - (|V_2| + |V_3|) \\ &\leq \gamma_{SR}(G) - \gamma(G). \end{aligned} \tag{2}$$

Using Theorem 1, $\gamma_{SR}(G) \leq 3\gamma(G)$ implies $\frac{1}{3}\gamma_{SR}(G) \leq \gamma(G)$. Therefore from (2) we have

$$\begin{aligned} \gamma_R(G) &\leq \gamma_{SR}(G) - \gamma(G) \\ &\leq \gamma_{SR}(G) - \frac{1}{3}\gamma_{SR}(G) \\ &= \frac{2}{3}\gamma_{SR}(G). \end{aligned}$$

Hence the Upper bound exists. Now, to prove the existence of Lower bound, assume that $g = (V_0, V_1, V_2)$ is a $\gamma_R(G) - function$. In this case, the function $g' = (V'_0, \phi, V'_2, V'_3)$, where $V'_0 = V_0$, $V'_2 = V_1$ and $V'_3 = V_2$ is an SRDF on graph G . Therefore

$$\begin{aligned} \gamma_R(G) &\leq g'(V) \\ &= 2|V'_2| + 3|V'_3| \\ &= 2|V_1| + 3|V_2| \\ &\leq 2|V_1| + 4|V_2| \\ &= 2\gamma_{SR}(G). \end{aligned}$$

So, $\gamma_{SR}(G) \leq 2\gamma_R(G)$ and thus $\frac{1}{2}\gamma_{SR}(G) \leq \gamma_R(G)$.

Hence the proof follows. \square

Note that, the Upper bound in Theorem 4 can be viewed as an extension of the Upper bound in Theorem 2 for all graphs. Also, the boundaries presented in Theorem 4 are sharp. For sharpness of Lower and Upper bound we consider the trivial graph K_1 and path P_5 , respectively. In next theorem we characterize the boundaries with

$$\frac{1}{2}\gamma_{SR}(G) = \gamma_R(G).$$

Theorem 5: Let G be a graph of order n . Then $\frac{1}{2}\gamma_{SR}(G) = \gamma_R(G)$ if and only if $G = \bar{K}_n$.

Proof: Suppose that $f = (V_0, V_1, V_2)$ is a $\gamma_R(G) - function$, such that V_2 is maximum. In this case, the function $f' = (V'_0, \phi, V'_2, V'_3)$, where $V'_0 = V_0$, $V'_2 = V_1$ and $V'_3 = V_2$ is an SRDF on graph G . Therefore

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$$\begin{aligned} 2\gamma_R(G) &= \gamma_{SR}(G) \\ &\leq f'(V) \\ &= 2|V_2'| + 3|V_3'| \\ &= 2|V_1| + 3|V_2| \\ &\leq 2|V_1| + 4|V_2| \\ &= 2\gamma_R(G). \end{aligned}$$

From this above relation it follows that

$$2|V_1| + 3|V_2| = 2|V_1| + 4|V_2|,$$

therefore $|V_2| = 0$. Since we chose f such that V_2 is maximum, we have V_1 is an independent set and thus $G = \bar{K}_n$.

The converse part is obvious. \square

III. CHARACTERIZATION OF THE TREES WITH $\gamma_R(T) = \frac{2}{3}\gamma_{SR}(T)$

First, we begin with following supporting Lemmas.

Lemma 6: For any graph G ,

$$\gamma_R(G) = \frac{2}{3}\gamma_{SR}(G),$$

if and only if $\gamma_{SR}(G) = 3\gamma(G)$ and $\gamma_R(G) = 2\gamma(G)$.

Proof: Suppose that $\gamma_R(G) = \frac{2}{3}\gamma_{SR}(G)$ and $f = (V_0, \phi, V_2, V_3)$ is a $\gamma_{SR}(G)$ -function. In this case, the function $g = (V_0', V_1', V_2')$, where $V_0' = V_0$, $V_1' = V_2$ and $V_2' = V_3$ is an SRDF on graph G . Therefore

$$\begin{aligned} \frac{2}{3}\gamma_{SR}(G) &= \gamma_R(G) \\ &\leq g(V) \\ &= |V_1'| + 2|V_2'| \\ &= |V_2| + 2|V_3| \\ &\leq \gamma_{SR}(G) - |V_2| - |V_3| \\ &\leq \gamma_{SR}(G) - \gamma(G). \end{aligned}$$

So, $\frac{2}{3}\gamma_{SR}(G) \leq \gamma_{SR}(G) - \gamma(G)$ and thus $3\gamma(G) \leq \gamma_{SR}(G)$. From Theorem 1, we have $\gamma_{SR}(G) \leq 3\gamma(G)$. Hence $\gamma_{SR}(G) = 3\gamma(G)$, also

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$$\begin{aligned}\gamma_R(G) &= \frac{2}{3}\gamma_{SR}(G) \\ &= \frac{2}{3}(3\gamma(G)) \\ &= 2\gamma(G).\end{aligned}$$

The converse part is obvious. \square

Lemma 7: Let for any graph G with $\gamma_R(G) = \frac{2}{3}\gamma_{SR}(G)$, the function $f = (V_0, \phi, V_2, V_3)$ be a $\gamma_{SR}(G)$ -function. Then $V_2 = \phi$.

Proof: Let $f' = (V'_0, V'_1, V'_2)$, where $V'_0 = V_0$, $V'_1 = V_2$ and $V'_2 = V_3$ be an RDF on graph G . Therefore

$$\begin{aligned}\frac{2}{3}\gamma_{SR}(G) &= \gamma_R(G) \\ &\leq f'(V) \\ &= |V'_1| + 2|V'_2| \\ &= |V_2| + 2|V_3| \\ &\leq \gamma_{SR}(G) - |V_2| - |V_3|.\end{aligned}$$

Thus

$$\begin{aligned}|V_2| + 2|V_3| &\leq \frac{1}{3}\gamma_{SR}(G) \\ &= \frac{2|V_2| + 3|V_3|}{3} \\ &= \frac{2}{3}|V_2| + |V_3|.\end{aligned}$$

So, $|V_2| \leq \frac{2}{3}|V_2|$ and hence $|V_2| = 0$. Then $V_2 = \phi$. \square

Our aim is to characterize all trees with Upper bound where equality holds in Theorem 4. Let \mathcal{F} be a family of trees T which is obtained from a sequence of trees T_1, T_2, \dots, T_j , ($j \geq 1$) such that T_1 is a Star $K_{1,r}$, ($r \geq 1$) and if $j \geq 2$, then T_j can be obtained from T_{j-1} with one of the five Operations $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ and φ_5 as shown in figures 1, 2, 3, 4 and 5, respectively.

Operation φ_1 : Let u be a Strong support vertex of T_i . In this case, T_{i+1} is obtained from T_i by adding a new vertex v and adding the edge uv . (see Figure 1)

Operation φ_2 : Let $u \in V(T_i)$ and $\gamma_R(T_i - u) \geq \gamma_R(T_i)$. In this case, T_{i+1} is obtained from T_i by adding a Star $K_{1,2}$ with central vertex v and adding the edge uv . (see Figure 2)

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Operation φ_2 : Let $u \in V(T_i)$. In this case, T_{i+1} is obtained from T_i by adding a Star $K_{1,2}$ with central vertex v and leaf w and adding the edge uw . (see Figure 3)

Operation φ_4 : Let $u \in V(T_i)$. In this case, T_{i+1} is obtained from T_i by adding a Star $K_{1,3}$ with central vertex v and leaf w and adding the edge uw . (see Figure 4)

Operation φ_5 : Let $u \in V(T_i)$ and $\gamma_R(T_i - u) \geq \gamma_R(T_i)$. In this case, T_{i+1} is obtained from T_i by adding a Path P_5 with central vertex v and adding the edge uv . (see Figure 5)

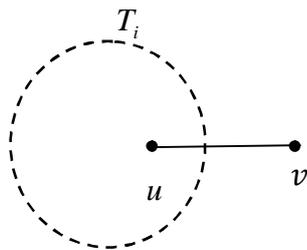


Figure 1. Operation φ_1 .

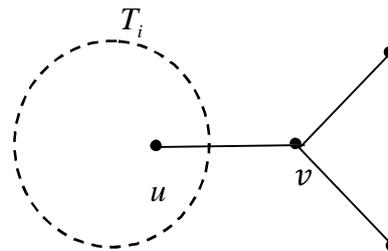


Figure 2. Operation φ_2 .

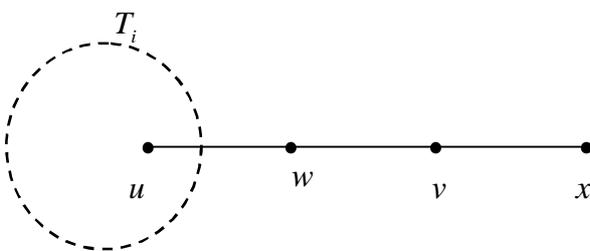


Figure 3. Operation φ_3 .

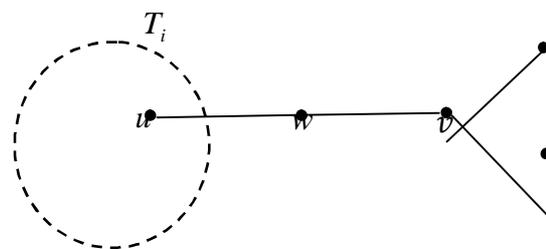


Figure 4. Operation φ_4 .

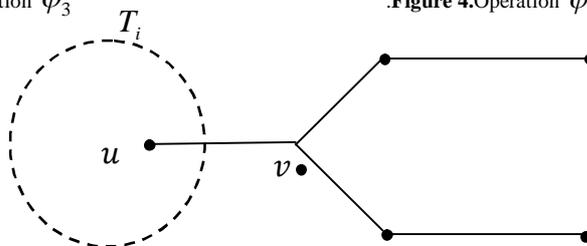


Figure 5. Operation φ_5 .

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In the following, we show that each tree T of family F satisfy the condition $\gamma_R(T_i) = \frac{2}{3}\gamma_{SR}(T_i)$.

Lemma 8: If $\gamma_R(T_i) = \frac{2}{3}\gamma_{SR}(T_i)$ and T_{i+1} is obtained from T_i with Operation φ_1 , then

$$\gamma_R(T_{i+1}) = \frac{2}{3}\gamma_{SR}(T_{i+1}).$$

Proof: Suppose that f is a $\gamma_{SR}(T_i)$ -function. Then, since u is a strong support vertex, we have $f(u) = 3$. We can extend the function f to an SRDF on tree T_{i+1} by assigning the weight 0 to the vertex v . Therefore

$$\begin{aligned} \gamma_{SR}(T_{i+1}) &\leq f(V) \\ &= \gamma_{SR}(T_i). \end{aligned}$$

Always $\gamma_R(T_i) \leq \gamma_{SR}(T_{i+1})$. Hence $\gamma_{SR}(T_{i+1}) = \gamma_{SR}(T_i)$. Assume the same way, g is a $\gamma_R(T_i)$ -function. Since u is a strong support vertex, we get $f(u) = 2$ and thus we can extend the function g to an RDF on tree T_{i+1} by assigning the weight 0 to the vertex v . Therefore

$$\begin{aligned} \gamma_{SR}(T_{i+1}) &\leq g(V) \\ &= \gamma_R(T_i). \end{aligned}$$

Since always $\gamma_R(T_i) \leq \gamma_{SR}(T_{i+1})$, we have $\gamma_{SR}(T_i) = \gamma_{SR}(T_{i+1})$.

So,

$$\begin{aligned} \gamma_R(T_{i+1}) &= \gamma_R(T_i) \\ &= \frac{2}{3}\gamma_{SR}(T_i) \\ &= \frac{2}{3}\gamma_{SR}(T_{i+1}). \end{aligned}$$

Hence $\gamma_R(T_{i+1}) = \frac{2}{3}\gamma_{SR}(T_{i+1})$. \square

Lemma 9: If $\gamma_R(T_i) = \frac{2}{3}\gamma_{SR}(T_{i+1})$ and T_{i+1} is obtained from T_i with Operation φ_2 , then

$$\gamma_R(T_{i+1}) = \frac{2}{3}\gamma_{SR}(T_{i+1}).$$

Proof: Suppose that $u \in V(T_i)$ and $\gamma_R(T_i - u) \geq \gamma_R(T_i)$. Let v be a vertex of Star $K_{1,2}$ relative to Operation φ_2 . If D is a dominating set for tree T_i , then $D \cup \{v\}$ is a dominating set for tree T_{i+1} . Therefore

$$\gamma(T_{i+1}) < \gamma(T_i) + 1. \tag{1}$$

Now, suppose that f is a $\gamma_{SR}(T_{i+1})$ -function. Since $f|_{T_i}$ is an SRDF on tree T_i , we have

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$$\begin{aligned}\gamma_{SR}(T_i) &\leq g(V |_{T_i}) \\ &= f(V) - 3 \\ &= \gamma_{SR}(T_{i+1}) - 3.\end{aligned}$$

Otherwise, $f|_{T_i-u}$ is an SRDF on tree $T_i - u$. Thus

$$\begin{aligned}\gamma_{SR}(T_i) &= \frac{3}{2}\gamma_R(T_i) \\ &\leq \frac{3}{2}\gamma_R(T_i - u) \\ &\leq \gamma_{SR}(T_i - u) \\ &\leq f(V |_{T_i-u}) \\ &\leq f(V) - 3 \\ &= \gamma_{SR}(T_{i+1}) - 3.\end{aligned}$$

Therefore, in this case, $\gamma_{SR}(T_i) \leq \gamma_{SR}(T_{i+1}) - 3$.

Any $\gamma_{SR}(T_i)$ -function can be extended to an SRDF on tree T_{i+1} by assigning the weight 3 to the vertex v and the weight 0 to all its neighbouring vertices. Therefore $\gamma_{SR}(T_{i+1}) \leq \gamma_{SR}(T_i) + 3$ which implies that $\gamma_{SR}(T_{i+1}) = \gamma_{SR}(T_i) + 3$. Now, from (1) and Lemma 6 we have

$$\begin{aligned}\gamma_{SR}(T_{i+1}) &= \gamma_{SR}(T_i) + 3 \\ &= 3\gamma(T_i) + 3 \\ &= 3\gamma(T_{i+1}).\end{aligned}$$

Any $\gamma_R(T_i)$ -function can be extended to an RDF on tree T_{i+1} by assigning the weight 2 to the vertex v and 0 to all its neighbouring vertices. Therefore $\gamma_R(T_{i+1}) \leq \gamma_R(T_i) + 2$. Also, if g is a $\gamma_R(T_{i+1})$ -function, then since v is a Strong support vertex, we have $g(v) = 2$. If $f(u) \neq 0$, then $g|_{T_i}$ is an RDF on tree T_i . Thus

$$\begin{aligned}\gamma_R(T_i) &\leq g(V |_{T_i}) \\ &= g(V) - 2 \\ &= \gamma_R(T_{i+1}) - 2.\end{aligned}$$

And in case $g(u) = 0$, then $g|_{T_i-u}$ is an RDF on tree $T_i - u$. Therefore

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$$\begin{aligned}\gamma_R(T_i) &\leq \gamma_R(T_i - u) \\ &\leq g(V |_{T_i - u}) \\ &= g(V) - 2 \\ &= \gamma_R(T_{i+1}) - 2.\end{aligned}$$

Thus in each case, $\gamma_R(T_i) \leq \gamma_R(T_{i+1}) - 2$ and since we showed that $\gamma_R(T_{i+1}) \leq \gamma_R(T_i) + 2$. So,

$$\gamma_R(T_{i+1}) = \gamma_R(T_i) + 2.$$

Now, from (1) and Lemma 6 we have

$$\begin{aligned}\gamma_R(T_{i+1}) &= \gamma_R(T_i) + 3 \\ &= 2\gamma(T_i) + 2 \\ &= 2\gamma(T_{i+1}).\end{aligned}$$

Hence $\gamma_{SR}(T_{i+1}) = 3\gamma(T_{i+1})$ and $\gamma_R(T_{i+1}) = 2\gamma(T_{i+1})$. Thus Lemma 6 implying that

$$\gamma_R(T_{i+1}) = \frac{2}{3}\gamma(T_{i+1}). \quad \square$$

Lemma 10: If $\gamma_R(T_i) = \frac{2}{3}\gamma(T_i)$ and T_{i+1} is obtained from T_i with Operation ϕ_3 , then

$$\gamma_R(T_{i+1}) = \frac{2}{3}\gamma(T_{i+1}).$$

Proof: Suppose that $u \in V(T_i)$ and v is the central vertex and w and x are the leaves of Star $K_{1,2}$, respectively. If D is a $\gamma(T_i)$ -set, then $D \cup \{v\}$ is a dominating set on tree T_{i+1} . Therefore

$$\gamma(T_{i+1}) \leq \gamma(T_i) + 1. \quad (1)$$

Suppose that f is a $\gamma_{SR}(T_{i+1})$ -function. Clearly, we can assume that $f(v) = 3$ and $f(x) = f(w) = 0$.

Therefore $f|_{T_i}$ is an SRDF on tree T_i and thus

$$\begin{aligned}\gamma_{SR}(T_i) &\leq f(V |_{T_i}) \\ &= f(V) - 3 \\ &= \gamma_{SR}(T_{i+1}) - 3.\end{aligned}$$

Any $\gamma_{SR}(T_i)$ -function can be extended to an SRDF on tree T_{i+1} by assigning the weight 3 to the vertex v and the weight 0 to all its neighbouring vertices. Therefore

$$\gamma_{SR}(T_{i+1}) \leq \gamma_{SR}(T_i) + 3.$$

And thus based on previous

$$\begin{aligned}\gamma_{SR}(T_{i+1}) &= \gamma_{SR}(T_i) + 3 \\ &= \gamma_{SR}(T_{i+1}).\end{aligned}$$

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Now, from (1) and Lemma 6 we have

$$\begin{aligned}\gamma_{SR}(T_{i+1}) &= 3\gamma(T_{i+1}) \\ &\leq 3\gamma(T_i) + 3 \\ &\leq \gamma_{SR}(T_i) + 3 \\ &= \gamma_{SR}(T_{i+1}).\end{aligned}$$

Therefore $\gamma_{SR}(T_{i+1}) = 3\gamma(T_{i+1})$. Now, to prove the problem based on Lemma 6, we have its enough to show that $\gamma_R(T_{i+1}) = 2\gamma(T_{i+1})$. Let g be a $\gamma_R(T_{i+1})$ -function. In this case, we can assume that $g(v) = 2$ and $g(x) = g(w) = 0$. Therefore $g|_{T_i}$ is an RDF on tree T_i . And thus

$$\begin{aligned}\gamma_R(T_i) &\leq g(V|_{T_i}) \\ &= g(V) - 2 \\ &= \gamma_R(T_{i+1}) - 2.\end{aligned}$$

Any $\gamma_R(T_i)$ -function can be extended to an RDF on tree T_{i+1} by assigning the weight 2 to the vertex v and the weight 0 to all its neighbouring vertices. Therefore $\gamma_R(T_{i+1}) \leq \gamma_R(T_i) + 2$.

Hence $\gamma_R(T_{i+1}) = \gamma_R(T_i) + 2$. Now, from (1) and Lemma 6 we have

$$\begin{aligned}\gamma_R(T_{i+1}) &\leq 2\gamma(T_{i+1}) \\ &\leq 2\gamma(T_i) + 2 \\ &= \gamma_R(T_i) + 2 \\ &= \gamma_R(T_{i+1}).\end{aligned}$$

So,

$$\gamma_R(T_{i+1}) = 2\gamma(T_{i+1}).$$

Hence the proof. \square

Lemma 11: If $\gamma_R(T_i) = \frac{2}{3}\gamma_{SR}(T_i)$ and T_{i+1} is obtained from T_i with Operation φ_4 , then

$$\gamma_R(T_{i+1}) = \frac{2}{3}\gamma_{SR}(T_{i+1}).$$

Proof: Suppose that u is a vertex of tree T_i and w is a leaf of Star $K_{1,2}$ with central vertex v relative to Operation φ_4 . If D is a dominating set on tree T_i , then $D \cup \{v\}$ is a dominating set on tree T_{i+1} . Therefore

$$\gamma(T_{i+1}) \leq \gamma(T_i) + 1. \tag{1}$$

Any $\gamma_{SR}(T_i)$ -function can be extended to an SRDF on tree T_{i+1} by assigning the weight 3 to the vertex v and the weight 0 to all its neighbouring vertices. Therefore

$$\gamma_{SR}(T_{i+1}) \leq \gamma_{SR}(T_i) + 3.$$

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Suppose that f is a $\gamma_{SR}(T_{i+1})$ -function. Since v is a Strong support vertex, we have $f(v) = 3$. Also, we can assume that $f(w) = 0$. In this case, $f|_{T_i}$ is an SRDF on tree T_i . Thus

$$\begin{aligned}\gamma_{SR}(T_i) &\leq f(V|_{T_i}) \\ &= f(V) - 3 \\ &= \gamma_{SR}(T_{i+1}) - 3.\end{aligned}$$

Therefore $\gamma_{SR}(T_{i+1}) = \gamma_{SR}(T_i) + 3$. Now, from (1) and Lemma 6 we have

$$\begin{aligned}\gamma_{SR}(T_{i+1}) &= 3\gamma(T_{i+1}) \\ &\leq 3\gamma(T_i) + 3 \\ &\leq \gamma_{SR}(T_i) + 3 \\ &= \gamma_{SR}(T_{i+1}).\end{aligned}$$

Hence $\gamma_{SR}(T_{i+1}) = 3\gamma(T_{i+1})$. Now, suppose that f is a $\gamma_R(T_{i+1})$ -function. Since v is a Strong support vertex, we can suppose that $f(v) = 3$ and each of its neighbouring vertices has the weight 0. Therefore $f|_{T_i}$ is an RDF on tree T_i . Thus $\gamma_R(T_i) \leq \gamma_R(T_{i+1}) - 2$. Any $\gamma_R(T_i)$ -function can be extended to an RDF by assigning the weight 2 to the vertex v and the weight 0 to all its neighbouring vertices. Therefore $\gamma_R(T_{i+1}) \leq \gamma_R(T_i) + 2$. So,

$$\gamma_R(T_{i+1}) = \gamma_R(T_i) + 2.$$

Hence based on Lemma 6 we have $\gamma_R(T_{i+1}) = \frac{2}{3}\gamma_{SR}(T_{i+1})$. \square

Lemma 12: If $\gamma_R(T_i) = \frac{2}{3}\gamma_{SR}(T_i)$ and T_{i+1} is obtained from T_i with Operation ϕ_5 , then

$$\gamma_R(T_{i+1}) = \frac{2}{3}\gamma_{SR}(T_{i+1}).$$

Proof: Let $u \in V(T_i)$ and $P_5 = xyvwz$ be a Path with five vertices relative to Operation ϕ_5 . If D is a $\gamma(T_i)$ -set, then $D \cup \{y, w\}$ is a dominating set on tree T_{i+1} . Therefore

$$\gamma(T_{i+1}) \leq \gamma(T_i) + 2. \tag{1}$$

Assume that f is a $\gamma_R(T_{i+1})$ -function. If $f(v) \neq 2$, then $f|_{T_i}$ is an RDF on tree T_i .

Since $f(V(P_5)) \geq 4$, we have

$$\begin{aligned}\gamma_R(T_i) &\leq f(V|_{T_i}) \\ &= f(V) - 4 \\ &= \gamma_R(T_{i+1}) - 4.\end{aligned}$$

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Now, suppose that $f(v) = 2$. In this case, we can assume that $f(u) = 0$ and therefore $f|_{T_i - u}$ is an RDF on tree $T_i - u$. Based on assumption we have

$$\begin{aligned} \gamma_R(T_i) &\leq \gamma_R(T_i - u) \\ &\leq f(V|_{T_i - u}) \\ &= f(V) - 4 \\ &= \gamma_R(T_{i+1}) - 4. \end{aligned}$$

Thus in each case, $\gamma_R(T_i) \leq \gamma_R(T_{i+1}) - 4$. Any $\gamma_R(T_i)$ -function can be extended to an RDF on tree T_{i+1} by assigning the weight 2 to y and w and the weight 0 to x , v and z . Therefore $\gamma_R(T_{i+1}) \leq \gamma_R(T_i) + 4$. Then same as before $\gamma_R(T_{i+1}) = \gamma_R(T_i) + 4$. Now, from (1) and Lemma 6 we have

$$\begin{aligned} \gamma_R(T_{i+1}) &\leq 2\gamma(T_{i+1}) \\ &\leq 2\gamma(T_i) + 4 \\ &= \gamma_R(T_i) + 4 \\ &= \gamma_R(T_{i+1}). \end{aligned}$$

So, we conclude that $\gamma_R(T_{i+1}) = 2\gamma(T_{i+1})$. Based on Lemma 6, it is enough to show that $\gamma_{SR}(T_{i+1}) = 3\gamma(T_{i+1})$. If g is a $\gamma_{SR}(T_i)$ -function, then the function g can be extended to an SRDF on tree T_{i+1} by assigning the weight 3 to y and w and the weight 0 to x , v and z . Therefore $\gamma_{SR}(T_{i+1}) \leq \gamma_{SR}(T_i) + 6$. Now, suppose that f is a $\gamma_{SR}(T_{i+1})$ -function. If $f(u) \neq 0$, then $f|_{T_i}$ is an SRDF on tree T_i and also $f(V(P_5)) \geq 6$. Therefore

$$\begin{aligned} \gamma_{SR}(T_i) &\leq g(V|_{T_i}) \\ &= g(V) - 6 \\ &= \gamma_{SR}(T_{i+1}) - 6. \end{aligned}$$

Otherwise, $g(u) = 0$. Then $g|_{T_i - u}$ is an SRDF on tree $T_i - u$. Based on assumption $\gamma_R(T_{i-u}) \geq \gamma_R(T_i)$. So,

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$$\begin{aligned} \gamma_{SR}(T_i) &= \frac{3}{2} \gamma_R(T_i) \\ &\leq \frac{3}{2} \gamma_R(T_i - u) \\ &\leq \gamma_{SR}(T_i - u) \\ &\leq g(V |_{T_i - u}) \\ &\leq g(V) - 6 \\ &= \gamma_{SR}(T_{i+1}) - 6. \end{aligned}$$

Therefore in each case, $\gamma_{SR}(T_i) \leq \gamma_{SR}(T_{i+1}) - 6$ and since we showed that earlier $\gamma_{SR}(T_{i+1}) \leq \gamma_{SR}(T_i) + 6$. Hence $\gamma_{SR}(T_{i+1}) = \gamma_{SR}(T_i) + 6$. Now, from (1), Lemma 6 and Theorem 1 we have

$$\begin{aligned} \gamma_{SR}(T_{i+1}) &\leq 3\gamma(T_{i+1}) \\ &\leq 3\gamma(T_i) + 6 \\ &\leq \gamma_{SR}(T_i) + 6 \\ &= \gamma_{SR}(T_{i+1}). \end{aligned}$$

Therefore $\gamma_{SR}(T_{i+1}) = 3\gamma(T_{i+1})$ and thus the problem is solved. \square

In this section we provide a constructive characterization of trees with $\gamma_R(T) = \frac{2}{3} \gamma_{SR}(T)$.

Theorem 13: The tree T satisfies the condition $\gamma_R(T) = \frac{2}{3} \gamma_{SR}(T)$ if and only if $T \in \mathcal{F}$.

Proof: First we use induction on the order $n \geq 3$ of the tree T in which $\gamma_R(T) = \frac{2}{3} \gamma_{SR}(T)$. If $\text{diam}(T) = 2$, then T is a Star. Therefore $T \in \mathcal{F}$. Now, assume that $\text{diam}(T) = 3$. Thus T is a Double Star $(S_{r,s})$ on $1 \leq s \leq r$. Let $s = 1$. In this case, T includes a vertex v of degree 2 which has exactly an adjacent leaf. The function in which the adjacent leaves to v have the weight 3 and all the other vertices have the weight 0 is a $\gamma_{SR}(T)$ -function. Therefore $\gamma_{SR}(T) = 5$. Also, the function in which the adjacent leaves to v have the weight 2, the non-adjacent leaves to v have the weight 3 and all the other vertices have the weight 0 is a $\gamma_R(T)$ -function. Therefore $\gamma_R(T) = 3$. Hence $\gamma_R(T) \neq \frac{2}{3} \gamma_{SR}(T)$ which is a contradiction. Thus $s \geq 2$. Let u be a non-adjacent leaf to v . Then T is a Double Star with central vertices u and v where $\deg(u) \geq 3$ and $\deg(v) \geq 3$. We consider T_1 and T_2 are two components of $T - uv$ where $u \in V(T_1)$ and $v \in V(T_2)$. Clearly, T_i , ($i = 1, 2$) is a Star of order at least 3. Therefore $\gamma_R(T_i) = 2$ and $\gamma_{SR}(T_i) = 3$, ($i = 1, 2$). If $\deg(v) = 3$, since $\gamma_R(T_1 - u) \geq \gamma_R(T_1)$, then the tree T is obtained from $T_1 \in \mathcal{F}$ by Operation φ_2 . Now, assume that $\deg(v) \geq 4$

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and w is adjacent leaf to v . We put $T' = T - w$. In this case, it is obvious that $\gamma_R(T') = 4$ and $\gamma_{SR}(T') = 6$. So, $\gamma_R(T') = \frac{2}{3}\gamma_{SR}(T')$. Therefore under the hypothesis, $T' \in \mathcal{F}$. Thus T is obtained from T' by Operation ϕ_1 . Hence we can assume that $diam(T) \geq 4$. Let T has support vertex u with at least three adjacent leaves and w be adjacent leaf to u . We put $T' = T - w$. We show that $T' \in \mathcal{F}$. It is enough to prove that $\gamma_R(T') = 2\gamma(T')$ and $\gamma_{SR}(T') = 3\gamma(T')$. Assume that D is a $\gamma(T')$ -set on tree T' . Since u is a support vertex on tree T' , we get $u \in D$. Since in the tree T and the vertex u is adjacent to w , we get D is a dominating set on tree T . Thus $\gamma(T) \leq |D| = \gamma(T')$. Always $\gamma(T') \leq \gamma(T)$. Therefore $\gamma(T) = \gamma(T')$.

Let f be a $\gamma_R(T')$ -function. Since u is a Strong support vertex, we get $f(u) = 2$ and thus f can be extended to an RDF on tree T by assigning the weight 0 to the vertex w . Therefore, based on Theorem 1 and Lemma 6 we have

$$\begin{aligned} \gamma_R(T) &\leq f(V) \\ &= \gamma_R(T') \\ &\leq 2\gamma(T') \\ &= 2\gamma(T) \\ &= 2\gamma_R(T). \end{aligned}$$

Hence $\gamma_R(T) = 2\gamma(T)$. Now, assume that g is a $\gamma_{SR}(T')$ -function. Since u is a Strong support vertex, we get $g(u) = 3$. Thus g can be extended to an SRDF on tree T by assigning the weight 0 to the vertex w . Therefore, from Theorem 1 and Lemma 6 we have

$$\begin{aligned} \gamma_{SR}(T) &\leq g(V) \\ &= \gamma_{SR}(T') \\ &\leq 3\gamma(T') \\ &= 3\gamma(T) \\ &= \gamma_{SR}(T). \end{aligned}$$

Then $\gamma_{SR}(T') = 3\gamma(T')$. So, $T' \in \mathcal{F}$ and thus T is obtained from T' by Operation ϕ_1 . Hence $T \in \mathcal{F}$.

We say assume for the next, the following holds.

Fact: T has no Strong support vertex with three adjacent leaves to it.

Let $x_0 x_1 \dots x_d$ be a Diagonal path. We root the tree T at vertex x_0 . Based on Fact, $\deg(x_{d-1}) \leq 3$. We consider the following cases:

Case 1: $\deg(x_{d-1}) = 3$. Let $\deg(x_{d-1}) \geq 3$. In this case, each child of x_{d-2} is a support vertex with a leaf.

We put $T' = T - T_{x_{d-1}}$. If D is a $\gamma(T)$ -set, since x_{d-2} is either a support vertex or has a support vertex adjacent to it. Therefore, $D - \{x_{d-1}\}$ is a dominating set on tree T' . Thus

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$$\begin{aligned}\gamma(T') &\leq |D| - 1 \\ &= \gamma(T) - 1.\end{aligned}$$

Any $\gamma(T')$ -set can be extended to a dominating set on tree T by adding x_{d-1} to it. Therefore $\gamma(T) \leq \gamma(T') + 1$. Hence $\gamma(T) = \gamma(T') + 1$. Let f be a $\gamma_{SR}(T')$ -function. In this case, the function f can be extended to an SRDF on tree T by assigning the weight 3 to x_{d-1} and the weight 0 to all their neighbouring leaves. Therefore $\gamma_{SR}(T) \leq \gamma_{SR}(T') + 3$. Now, from Lemma 6 and Theorem 1 we have

$$\begin{aligned}\gamma_{SR}(T) &\leq \gamma_{SR}(T') + 3 \\ &\leq 3\gamma(T') + 3 \\ &= 3\gamma(T) \\ &= \gamma_{SR}(T).\end{aligned}$$

Therefore $\gamma_{SR}(T') = 3\gamma(T')$. Also, if g is a $\gamma_R(T')$ -function, then g can be extended to an RDF on tree T by assigning the weight 2 to x_{d-1} and the weight 0 to all their neighbouring leaves. Therefore $\gamma_R(T) \leq \gamma_R(T') + 2$. Now, from Theorem 1 and Lemma 6 we have

$$\begin{aligned}\gamma_R(T) &\leq \gamma_R(T') + 2 \\ &\leq 2\gamma(T') + 2 \\ &= 2\gamma(T) \\ &= \gamma_R(T).\end{aligned}$$

It follows from above relation that $\gamma_R(T') = 2\gamma(T')$. Also, we showed that before $\gamma_{SR}(T') = 3\gamma(T')$.

Therefore it follows from Lemma 6 that $\gamma_R(T') = \frac{2}{3}\gamma_{SR}(T')$. So, by the induction hypothesis $T' \in \mathcal{F}$. Now, suppose that $\gamma_R(T' - x_{d-2}) < \gamma_R(T')$. In this case, if f' is a $\gamma_R(T' - x_{d-2})$ -function, then the function f' can be extended to an RDF on tree T by assigning the weight 2 to x_{d-1} and the weight 0 to all their neighbouring. Therefore $\gamma_R(T) \leq \gamma_R(T' - x_{d-2}) + 2$. So,

$$\begin{aligned}\gamma_R(T) &\leq \gamma_R(T' - x_{d-2}) + 2 \\ &\leq \gamma_R(T') + 2 \\ &= 2\gamma(T') + 2 \\ &= 2\gamma(T) \\ &= \gamma_R(T),\end{aligned}$$

Which is a contradiction. Thus $\gamma_R(T' - x_{d-2}) \geq \gamma_R(T')$ and therefore T is obtained from T' by Operation φ_2 . Hence $T \in \mathcal{F}$. Now, suppose that $\deg(x_{d-2}) = 2$. We put $T' = T - T_{x_{d-2}}$. Let D be a $\gamma(T)$ -set.

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Clearly, we can assume $x_{d-2} \notin D$ and $x_{d-1} \in D$. In this case, $D - \{x_{d-1}\}$ is a dominating set on tree T' . Therefore $\gamma(T') \leq \gamma(T) - 1$. Any $\gamma(T')$ -set can be extended to a dominating set on tree T by adding x_{d-1} to it. Thus $\gamma(T) \leq \gamma(T') + 1$ and therefore $\gamma(T) = \gamma(T') + 1$. Now, let g' be a $\gamma_R(T')$ -function. In this case, the function g' can be extended to an RDF on tree T by assigning the weight 2 to x_{d-1} and the weight 0 to all its neighbouring. Therefore $\gamma_R(T) \leq \gamma_R(T') + 2$. From Lemma 6 Theorem 1 we have

$$\begin{aligned} \gamma_R(T) &\leq \gamma_R(T') + 2 \\ &\leq 2\gamma(T') + 2 \\ &= 2\gamma(T) \\ &= \gamma_R(T). \end{aligned}$$

It is implied from above relation that $\gamma_R(T) = 2\gamma(T)$. Also, if f' is a $\gamma_{SR}(T')$ -function, then the function f' can be extended to an SRDF on tree T by assigning the weight 3 to x_{d-1} and the weight 0 to all its neighbouring. Therefore $\gamma_{SR}(T) \leq \gamma_{SR}(T') + 3$. From Lemma 6 and Theorem 1 we have

$$\begin{aligned} \gamma_{SR}(T) &\leq \gamma_{SR}(T') + 3 \\ &\leq 3\gamma(T') + 3 \\ &= 3\gamma(T) \\ &= \gamma_{SR}(T). \end{aligned}$$

Above relation implies that $\gamma_{SR}(T) = 3\gamma(T)$. Considering the Lemma 6 and what we already showed $\gamma_R(T) = 2\gamma(T)$, we conclude $\gamma_R(T) = \frac{2}{3}\gamma_{SR}(T)$. So, with the inductive hypothesis we have $T' \in \mathcal{F}$. Thus T is obtained from T' by Operation φ_4 . Hence $T \in \mathcal{F}$.

Case 2: $\deg(x_{d-2}) = 2$. If $\deg(x_{d-2}) \geq 3$, then each child of x_{d-2} is either a leaf or a support vertex of degree 2. Since each support child of x_{d-2} other than x_{d-1} plays the same role of x_{d-1} , we can assume that have degree 2. Let $f = (V_0, V_1, V_2, V_3)$, where $V_1 = \emptyset$ be a $\gamma_{SR}(T)$ -function. Since $\gamma_R(T) = \frac{2}{3}\gamma_{SR}(T)$, based on Lemma 7, we get $V_2 = \emptyset$. Therefore if $f(x_{d-2}) \neq 0$, then $f(x_{d-2}) = 3$ holds. In this case, we can assume that $f(x_{d-1}) = 0$ and $f(x_{d-2}) = 2$ which is a contradiction with $V_2 = \emptyset$. Therefore $f(x_{d-2}) = 0$. In this case, x_{d-2} does not have adjacent leaf. Otherwise, the adjacent leaf should be weighted 2 which is a contradiction with $V_2 = \emptyset$. We show that in this case x_{d-2} has at most an adjacent support other than x_{d-1} . On contrary suppose that u_1 and u_2 are two support children of x_{d-2} other

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than x_{d-1} . Let v_1 and v_2 be adjacent leaves to u_1 and u_2 , respectively. Since $V_2 = \emptyset$, we can assume $f(u_1) = f(u_2) = 3$ and $f(v_1) = f(v_2) = v$. In this case, we define the function g as follows: Let us

$$\begin{aligned} g(x_d) &= g(v_1) = g(v_2) = 2, \\ g(x_{d-1}) &= g(u_1) = g(u_2) = 0, \\ g(x_{d-2}) &= 3, \end{aligned}$$

and other vertices take the same weight of the function. In this case, g is a $\gamma_{SR}(T)$ -function, where the set of vertices have weight 2 are non-empty is contradiction with Lemma 7. Therefore x_{d-2} has at most a support child other than x_{d-1} . Let u be a support child of x_{d-2} other than x_{d-1} and v be an adjacent leaf to x_{d-2} . We put $T' = T - T_{x_{d-2}}$. If D' is a $\gamma(T')$ -set, then $D' \cup \{x_{d-1}, u\}$ is a dominating set on tree T . Therefore $\gamma(T) \leq \gamma(T') + 2$. On the other hand, if $\{x_{d-1}, u\} \subseteq D$ and $x_{d-2} \notin D$, then $D - \{x_{d-1}, u\}$ is a dominating set on tree T' . Thus $\gamma(T') \leq \gamma(T) - 2$. Consequently, we conclude that $\gamma(T) = \gamma(T') + 2$. Now, let f' be a $\gamma_{SR}(T')$ -function, then the function f' can be extended to an SRDF on tree T by assigning the weight 3 to x_{d-1} and u and the weight 0 to all their neighbouring. Therefore $\gamma_{SR}(T) \leq \gamma_{SR}(T') + 6$. Now, from Lemma 6 and Theorem 1, we conclude

$$\begin{aligned} \gamma_{SR}(T) &\leq \gamma_{SR}(T') + 6 \\ &\leq 3\gamma(T') + 6 \\ &= 3\gamma(T) \\ &= \gamma_{SR}(T). \end{aligned}$$

Above relation implies that $\gamma_{SR}(T') = 3\gamma(T')$. Also, any $\gamma_R(T')$ -function can be extended to an RDF on tree T by assigning the weight 2 to x_{d-1} and u and the weight 0 to all their neighbouring. Therefore $\gamma_R(T) \leq \gamma_R(T') + 4$. From Lemma 6 and Theorem 1, we have

$$\begin{aligned} \gamma_R(T) &\leq \gamma_{SR}(T') + 4 \\ &\leq 2\gamma(T') + 4 \\ &= 2\gamma(T) \\ &= \gamma_R(T). \end{aligned}$$

Therefore $\gamma_R(T') = 2\gamma_R(T')$. And so consider that $\gamma_{SR}(T') = 3\gamma(T')$, from Lemma 6 it implies that $\gamma_R(T') = \frac{2}{3}\gamma_{SR}(T')$. Hence assuming the induction $T' \in \mathcal{F}$. Now, suppose that $\gamma_R(T' - x_{d-3}) < \gamma_R(T')$. In this case, any $\gamma_R(T' - x_{d-3})$ -function can be extended to an RDF on tree T by assigning the weight 2

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to x_{d-2} , the weight 0 to u and x_{d-1} and the weight 1 to v and x_d . Thus $\gamma_R(T) \leq \gamma_R(T' - x_{d-3}) + 4$. And therefore

$$\begin{aligned} \gamma_R(T) &\leq \gamma_R(T' - x_{d-3}) + 4 \\ &\leq \gamma_R(T') + 4 \\ &\leq 2\gamma(T') + 4 \\ &= 2\gamma(T) \\ &= \gamma_R(T), \end{aligned}$$

which is a contradiction. So, $\gamma_R(T' - x_{d-3}) \geq \gamma_R(T')$ and thus T is obtained from T' with Operation φ_5 . Hence $T \in \mathcal{F}$. Now, suppose that $\deg(x_{d-2}) = 2$. We put $T' = T - T_{x_{d-2}}$. If D is a $\gamma(T)$ -set, then we can assume that $x_{d-1} \in D$ and $x_{d-2} \notin D$. Therefore $D - \{x_{d-1}\}$ is a dominating set on tree T' . Thus $\gamma(T') \leq \gamma(T) - 1$. Since $\gamma(T')$ -set can be extended to a dominating set on tree T by adding x_{d-1} to it, we have $\gamma(T) \leq \gamma(T') + 1$ and thus $\gamma(T) = \gamma(T') + 1$. Now, suppose that f is a $\gamma_{SR}(T)$ -function. Based on Lemma 7 there is no vertex which has weight 2. Therefore we can assume that $f(x_d) = f(x_{d-2}) = 0$ and $f(x_{d-1}) = 3$. In this case, $f|_{T'}$ is a $\gamma_{SR}(T')$ -function. Thus $\gamma_{SR}(T') \leq \gamma_{SR}(T) + 3$. Any $\gamma_{SR}(T')$ -function can be extended to an SRDF on tree T by assigning the weight 3 to x_{d-1} and the weight 0 to all its neighbouring. Therefore, $\gamma_{SR}(T) \leq \gamma_{SR}(T') + 3$. Consequently, we conclude that $\gamma_{SR}(T) = \gamma_{SR}(T') + 3$ and in this case, based on Lemma 6, Theorem 1 and previous relation we have

$$\begin{aligned} \gamma_{SR}(T') &\leq \gamma_{SR}(T) - 3 \\ &\leq 3\gamma(T) - 3 \\ &= 3(\gamma(T) - 1) \\ &= 3\gamma(T'). \end{aligned}$$

Now, let f be a $\gamma_R(T)$ -function. If $f(x_{d-2}) = 2$ and $f(x_{d-3}) = 0$, then $f(x_{d-1}) = 0$ and $f(x_d) = 1$. By changing the weight of x_{d-2} and x_d to 0, the weight of x_{d-1} to 2 and the weight of x_{d-3} to 1 a new $\gamma_R(T)$ -function is obtained such that $f(x_{d-2}) = 0$. Also, if $f(x_{d-2}) = 1$, then $f(x_d) + f(x_{d-1}) = 2$. By changing the weight of x_{d-2} and x_d to 0 and the weight of x_{d-1} to 2 an RDF with weight less than $\gamma_R(T)$ is obtained which is a contradiction. Therefore we can assume that $f(x_d) + f(x_{d-2}) = 0$ and $f(x_{d-1}) = 2$. Thus $f|_{T'}$ is an RDF on tree T' . Hence $\gamma_R(T') \leq \gamma_R(T) - 2$. Any $\gamma_R(T')$ -function can be extended to an RDF on tree T by assigning the weight 2 to x_{d-1} and the weight 0 to x_d and x_{d-2} . Thus $\gamma_R(T) \leq \gamma_R(T') + 2$. It follows from above

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inequality that $\gamma_R(T) = \gamma_R(T') + 2$. From Lemma 6, Theorem 1 and $\gamma_R(T) = \gamma_R(T') + 1$ we conclude that

$$\begin{aligned}\gamma_R(T') &\leq \gamma_R(T) - 2 \\ &= 2\gamma(T) - 2 \\ &= 2\gamma(T').\end{aligned}$$

We have already show that $\gamma_{SR}(T') = 3\gamma(T')$ and the previous relation conclude that $T' \in \mathcal{F}$ and T is obtained from T' with Operation ϕ_3 . Therefore $T \in \mathcal{F}$.

Hence in each case $T \in \mathcal{F}$.

Conversely, let $T \in \mathcal{F}$. In this case, by induction on the number of Operations that constructed the tree T and by using Lemma 8, 9, 10, 11 and 12 we can show that $\gamma_R(T) = \frac{2}{3}\gamma_{SR}(T)$. \square

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