

Connected Domination Number of a Graph and Its Complement

P.Bama¹, T.Rama Devi², A.Thiru murugan³

Department of Mathematics, Parvathy, S Arts And Science College, Dindigul, India

ABSTRACT: In this chapter, we discuss that when G and G^c are both connected, $\gamma_c(G) + \gamma_c(G^c) \leq \delta^*(G) + 4 - (\gamma_c(G) - 3)(\gamma_c(G^c) - 3)$. We also prove that $\gamma_c(G) + \gamma_c(G^c) \leq \delta^*(G) + 2$ when $\gamma_c(G), \gamma_c(G^c) \geq 4$. This bound is sharp when $\delta^*(G) = 6$ and equality can only when $\delta^*(G) = 6$.

KEYWORDS: Graph, Complement, Connected, Domination, Connected Domination, Connected Domination Number.

I. INTRODUCTION

Many problems in external graph theory seek the extreme values of graph parameters on families of graphs. Results of Nordhaus_Gaddum type study the extreme values of the sum (or product) of a parameter on a graph and its complement, following the classic paper of Nordhaus and Gaddum [8] solving these problems for the chromatic number on n -vertex graphs. In this paper, we study such problems for the connected domination number. For domination problems, multiple edges and loops are irrelevant, so we forbid them. We use $V(G)$ and $E(G)$ for the set and edge set of a graph G . For a vertex $v \in V(G)$, the open neighborhood $N(v)$ is the set $\{u \in V(G) : uv \in E(G)\}$ and the closed neighborhood $N[v]$ is the set $N(v) \cup \{v\}$. The open neighborhood $N[S]$ of a set $S \subseteq V$ is the set $\bigcup_{v \in S} N(v)$, and the closed neighborhood $N[S]$ of S is the set $N(S) \cup S$. The minimum and maximum vertex degrees in G are denoted $\delta(G)$ and $\Delta(G)$, respectively. Given graphs G and H , the Cartesian product $G \square H$ is the graph with vertex set $V(G) \times V(H)$ and edge set defined by making (u, v) and (u', v') adjacent if and only if either (1) $u = u'$ and $vv' \in E(H)$ or (2) $v = v'$ and $uu' \in E(G)$. For a graph G , a set $S \subseteq V(G)$ is a dominating set if $N[S] = V(G)$, and S is a connected dominating set if also the subgraph induced by S , denoted $G[S]$, is connected. The minimum size of a dominating set and a connected dominating set are the domination number $\gamma(G)$ and the connected domination number $\gamma_c(G)$, respectively. Inequalities of Nordhaus_Gaddum type have been proved for many variations of domination parameters. Some of them can be improved when constraints on G and G^c are imposed.

- 1) $\gamma(G) + \gamma(G^c) \leq n + 1$ for every graph G [6];
- 2) $\gamma(G) + \gamma(G^c) \leq n/2 + 2$ if $\delta(G), \delta(G^c) \geq 1$ [5];
- 3) $\gamma(G) + \gamma(G^c) \leq 2n/5 + 3$ if $\delta(G), \delta(G^c) \geq 2$, with some small exceptions [2];
- 4) $\gamma(G) + \gamma(G^c) \leq 3n/8 + 2$ if $\delta(G), \delta(G^c) \geq 3$ with some small exceptions [2].
- 5) Throughout this paper we impose the following condition: G is a connected n -vertex graph whose complement G^c is also connected. Note that this requires $n \geq 4$. For such G , we establish sharp upper bounds for $\gamma_c(G) + \gamma_c(G^c)$ and $\gamma_c(G) \cdot \gamma_c(G^c)$ in terms of n and the minimum degrees of G and G^c . In the list below of our results, (1) is our main result, and most of the others follow from closer examination of its proof.

Let $\delta^*(G) = \min\{\delta(G), \delta(G^c)\}$.

1. $\gamma_c(G) + \gamma_c(G^c) \leq \delta^*(G) + 4 - (\gamma_c(G) - 3)(\gamma_c(G^c) - 3)$; sharp for $\delta(G) \geq 2$
2. $\gamma_c(G) + \gamma_c(G^c) \leq 3n/4$ when $\delta^*(G) \geq 3$ and $n \geq 14$; sharp when 4 divides n .
3. $\gamma_c(G) + \gamma_c(G^c) \leq \delta^*(G) + 2$ when $\gamma_c(G), \gamma_c(G^c) \geq 4$, with equality possible if and only if $\delta^*(G) = 6$.
4. $\gamma_c(G) \gamma_c(G^c) \leq 2n - 4$ when $n \geq 7$, with equality only when G or G^c is a path or cycle.
5. If G or G^c has a dominating vertex, then its complement is disconnected, so we may assume that $\gamma_c(G), \gamma_c(G^c) \geq 2$. When $\gamma_c(G^c) = 2$, our first inequality reduces to $2 \leq \delta^*(G) + 1$, which holds since both graphs are connected. Hence we may assume that $\gamma_c(G), \gamma_c(G^c) \geq 3$. This condition is equivalent to $\text{diam} G = \text{diam} G^c = 2$, since it is immediate from the definitions that $\text{diam} G \geq 3$ if and only if $\gamma_c(G^c) \leq 2$.

We note first that when $\gamma_c(G^c) = 3$, the best bound on the sum is $\gamma_c(G) + \gamma_c(G^c) \leq n$. The existence of a spanning tree (with at least two leaves) in a connected graph yields $\gamma_c(G) \leq n-2$; equality holds for paths and cycles. (The fact that $\gamma_c(G) = n - l(G)$, where $l(G)$ is the maximum number of leaves in a spanning tree, was first noted by Hedetniemi and Laskar in [4]). Restricting the problem in the case $\gamma_c(G^c) = 2$

By requiring also $\delta(G) \geq 3$ leads to $\gamma_c(G) + \gamma_c(G^c) \leq 3n/4$.

1.1 History of Graph Theory

Bridges in Königsberg is about the Swiss mathematician Leonhard Euler (1707-1783). While he lived in the Prussian town of Königsberg at the Baltic sea, he was thinking about the following problem. Königsberg is divided into four parts by the river Pregel, and connected by seven bridges. Is it possible to tour Königsberg along a path that crosses every bridge once, and at most once? You can start and finish wherever you want, not necessarily in the same place.

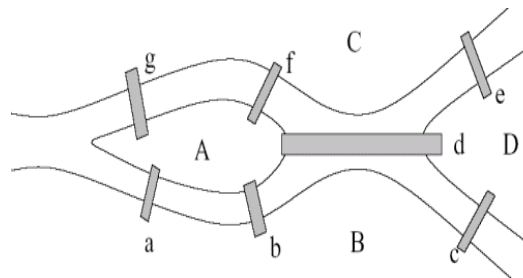


Figure 1 : The Königsberg Bridge

Graph Theory branch of mathematics concerned with networks of points connected by lines. The subject of graph theory had its beginnings in recreational math problems, but it has grown into a significant area of mathematical research with applications in chemistry, operations research, social sciences and computer science. The history of graph theory may be specifically traced to 1735, when the Swiss mathematician Leonhard Euler solved the Königsberg bridge problem. The Königsberg bridge problem was an old puzzle concerning the possibility of finding a path over every one of seven bridges that span a forked river flowing past an island but without crossing any bridge twice. Euler argued that no such path exists. In 1750 Euler discovered the formula $V-E+F=2$ relating the number of vertices (V), edges (E) and faces (F) of a polyhedron. The vertices and edges of a polyhedron form a graph on its surface.

1.2 Application of Graph Theory

- i. Everything in our world is linked: cities are linked by street, rail and flight networks, pages on the internet are linked by hyperlinks.
- ii. The different components of an electric circuit or computer chip are connected and the paths of disease outbreaks form a network; scientists, engineers, and many others want to analyse, understand and optimize these networks. And this can be done using graph theory.
- iii. Graph theory can apply to road networks trying to find a way to reduce traffic congestion. An idea which if successful, could save millions every year which are lost due to time spent on the road, as well as mitigating the enormous environmental impact. It could also make life safer by allowing emergency services to travel faster and avoid car accidents in the first place. These Intelligent Transportation Systems could work by collection location data from smart phones of motorists and telling them where and how fast to drive in order to reduce overall congestion.
- iv. Graph theory is already utilized in flight networks. Airlines want to connect countless cities in the most efficient way moving the most passengers with the fewest possible trips: a problem very similar to the Travelling Salesman.
- v. At the same time, air traffic controllers need to make sure hundreds of planes are at the right place at the right time and don't crash: an enormous task that would be almost impossible without computers and graph theory.

1.3 VERTICES AND EDGES

A graph G consists of a pair $(V(G), E(G))$ where $V(G)$ is a nonempty finite set whose elements are called points or vertices and $E(G)$ is a set of unordered pairs of distinct elements of $V(G)$. The elements of $E(G)$ are called lines or edges of the graph G .

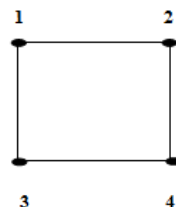
If $X = \{u, v\} \in E(G)$, the line X is said to join u and v .

We write $X = uv$ and we say that the points u and v are adjacent.

We also say that the point u and the line X are incident with each other.

If two distinct lines X and Y are incident with a common point then they are called *adjacent line*.

A graph with p points and q lines is called (p, q) graph. We write $V(G)=V$ and $E(G) = E$.

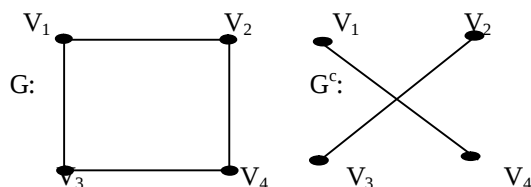


In the above example the graph G contains $(V(G), E(G))$.

$V(G) = \{1,2,3,4\}$ and $E(G) = \{e_1, e_2, e_3, e_4\}$.

1.4 COMPLEMENT OF A GRAPH

Let $G=(V, E)$ be a graph. The complement G^c of G is defined to be the graph which has V as its set of points and two points are adjacent in G^c if and only if they are not adjacent in G .



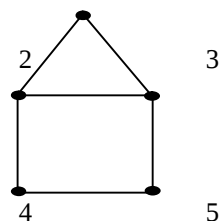
From this graph $(v_1, v_2), (v_1, v_3), (v_2, v_4), (v_3, v_4)$ are adjacent in G , but $(v_1, v_2), (v_1, v_3), (v_2, v_4), (v_3, v_4)$ are not adjacent in G^c .

1.5 CONNECTEDNESS

Two points u and v of a graph G are said to be connected if there exists a uv path in G .

A graph G is said to be connected if every pair of its points are connected.

A graph which is not connected is said to be disconnected. 1



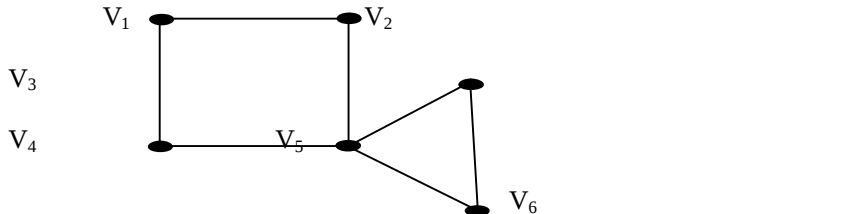
1.6 DOMINATING SET

A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of $V(G) - D$ has a neighbour in D .

That is A set $D \subseteq V(G)$ is a dominating set if $v \in V(G)$ is an element of D or adjacent to anyone element of D .

The minimum cardinality of a dominating set of G is called domination number and it is denoted by $\gamma(G)$.

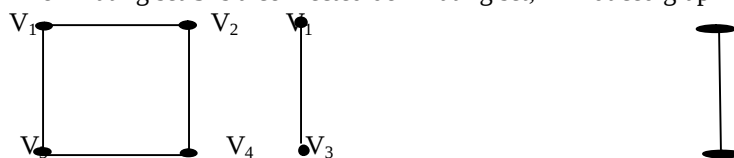
The dominating set of G of minimum cardinality is called a $\gamma(G)$ set .



From the above graph (v_1, v_5) , (v_2, v_4, v_3) are dominating sets and $\gamma(G)$ set is (v_1, v_5) . The domination number is $\gamma(G) = 2$.

1.7 CONNECTED DOMINATING SET

A Dominating set S is a connected dominating set, if induced graph $\langle S \rangle$ is connected



From the above graph the induced graph (v_1, v_2) , (v_1, v_3) , (v_2, v_4) are connected .Therefore they are connected dominating set.

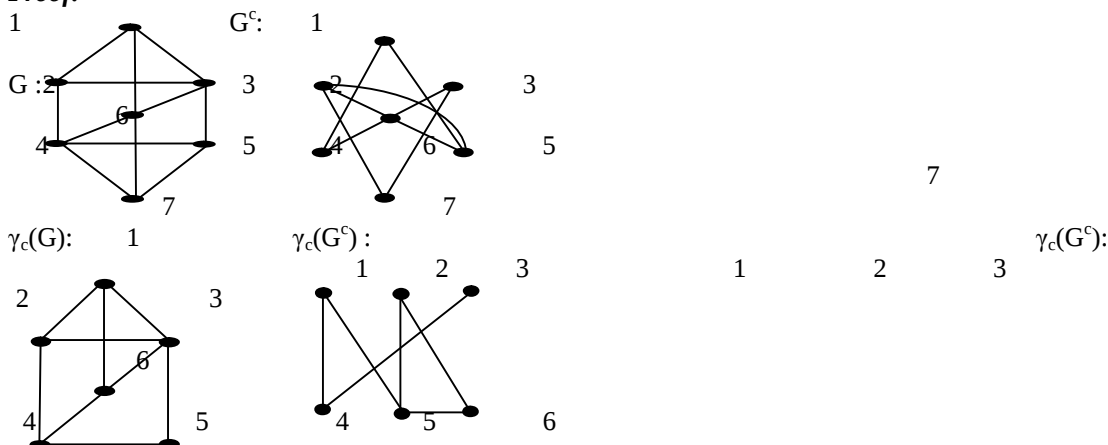
II. RESULT AND DISCUSSION

We begin with the following straightforward observations.

2.1 Theorem

If G and G^c are connected, then $\gamma_c(G) + \gamma_c(G^c) \leq \delta^*(G) + 4 - (\gamma_c(G) - 3)(\gamma_c(G^c) - 3)$.

Proof:



From the above graph,

$\delta(G)=3$, $\delta(G^c)=2$; $\delta^*(G)=\min(\delta(G),\delta(G^c))$; $\delta^*(G) = \min(3,2)$; $\gamma_c(G)=\{1,4\}$ and $\gamma_c(G^c)=\{3,5\}$; $\delta^*(G) = 2$; $\gamma_c(G) + \gamma_c(G^c) < \delta^*(G) + 4 - (\gamma_c(G) - 3)(\gamma_c(G^c) - 3)$

We may assume that $\gamma_c(G), \gamma_c(G^c) \geq 3$ and $\text{diam} G = \text{diam} G^c = 2$. Let x be a vertex of degree $\delta(G)$ and let $X = V(G) - N[x]$. Since $\gamma_c(G) \geq 3$, We have $X \neq \emptyset$. Also, $N(x)$ dominates X , since $\text{diam}(G) = 2$. We successively select disjoint sets $S_1, S_2, \dots, S_k$ in $N(x)$ that almost dominate X . Let $T_0 = N(x)$, and let S_0 be a largest subset of $N(x)$ that does not dominate X . Let $T_1 = T_0 - S_0$. By the maximality of S_0 , every vertex t of T_1 dominates X -

$N(S_0)$, but T_1 may or may not dominate X . We continue, constructing sets $T_1, T_2, \dots, T_k$ with $T_0 \supset T_1 \supset \dots \supset T_k$ (where $k \geq 1$) and sets $S_0, S_1, \dots, S_{k-1}$ such that,

- (a) For $I < k$, the set T_i dominates x .
- (b) For $I < k$, the set S_i is a largest subset of T_i not dominating X , and $T_{i+1} = T_i - S_i$.
- (c) T_k does not dominate X .

Since T_i dominates X but S_i does not (when $I < k$), all of $T_0, T_1, \dots, T_k$ are nonempty.

By constructing, $S_i \cup \{y\}$ dominates X whenever $y \in T_{i+1}$. Thus $S_i \cup \{x, y\}$ is a connected dominating set, so $|S_i| \geq \gamma_c(G) - 2$. For $0 \leq i \leq k-1$, Let x_i be a vertex of X that is not dominated by S_i and let x_k be a vertex of X that is not dominated by T_k . Since $N(x) = (\cup S_i) \cup T_k$, the set $\{x, x_0, \dots, x_k\}$ is a connected dominating set of G^c , so $k \geq \gamma_c(G^c) - 2$. Since $|S_0| = \delta(G) - |T_k| - \sum_{i=1}^{k-1} |S_i|$ and $|T_k| \geq 1$.

$$\begin{aligned} \gamma_c(G) + \gamma_c(G^c) &\leq (|S_0| + 2) + (k + 2) \\ &= (\delta(G) - |T_k| - \sum_{i=1}^{k-1} |S_i| + 2) + (k + 2) \\ &\leq \delta(G) - |T_k| + 4 - (k-1)(\gamma_c(G) - 2) + k \\ &\leq \delta(G) + 4 - (k-1)(\gamma_c(G) - 3) \\ &\leq \delta(G) + 4 - \gamma_c(G^c) - 3)(\gamma_c(G) - 3) \end{aligned}$$

By symmetry, $\gamma_c(G) + \gamma_c(G^c) \leq \delta(G) + 4 - (\gamma_c(G) - 3)(\gamma_c(G^c) - 3)$.

2.2 Theorem

The bound of Theorem 1 is sharp for each value of $\delta^*(G)$ at least 2.

Proof:

For each integer r with $r \geq 2$, We construct a connected graph G_r with $\delta(G_r) = r < \delta(G_r^c), \gamma_c(G^c) = 3$, and $\gamma_c(G_r) + \gamma_c(G_r^c) = r + 4$, thereby achieving the bound.

The graph G_r will have $r^2 + r + 1$ vertices, with $\gamma_c(G_r^c)$ kept small by making $\delta(G_r^c)$ large: $\delta(G_r^c) = r^2 + r + 1$.

Form the graph G_r as follows,

Let $H_1 = H_2 = K_r$, with $V(H_2) = \{V_1, V_2, \dots, V_r\}$.

To the Cartesian product $H_1 \square H_2$, add a star with $r+1$ new vertices, having center y and leaves $x_1, x_2, \dots, x_r$.

For $1 \leq i \leq r$, add edges joining x_i to all vertices of $H_1 \square H_2$ with second coordinate v_i .

The resulting graph is G_r .

Note that $\text{diam}(G_r) = \text{diam}(G_r^c) = 2$ and that $\delta^*(G_r)$; the degrees in G of y, x_i and vertices of $H_1 \square H_2$ are $r, r+1$ and $2r-1$ respectively.

It suffices to show that $\gamma_c(G_r^c) = 3$ and $\gamma_c(G_r) = r+1$.

Since $\text{diam}(G) = 2$.

We have $\gamma_c(G^c) \geq 3$.

Equality holds using $\{y, u, w\}$, where u and w are neighbors of x_1 and x_2 in G other than y .

To see that $\gamma_c(G_r) = r+1$, note first that $\{y, x_1, x_2, \dots, x_r\}$ is a connected dominating set.

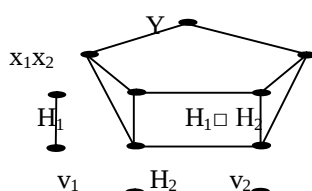
For the lower bound, let S be a connected dominating set, and let $T_i = N[x_i] - \{y\}$.

If S does not intersect T_i , which includes x_i and a copy of $V(H_1)$, then dominating T_i requires S to contain y and a vertex in each copy of H_2 .

This requires $r+1$ vertices.

Thus $|S| \geq r+1$ unless S intersects each of the r disjoint sets $T_1, T_2, \dots, T_r$ exactly once.

Dominating y without reaching size $r+1$ requires S to contain some x_i , but now x_i has no neighbors in



2.3 Theorem

If G and G^c are connected n -vertex graphs with $n \geq 7$, then $\gamma_c(G)\gamma_c(G^c) \leq 2n - 4$, with equality only when G or G^c is a path or cycle. The bound fails for C_5 and for one pair of complementary 6-vertex graphs.

Proof

For a path or cycle with at least 6 vertices, $\gamma_c(G) = n - 2$ and $\gamma_c(G^c) = 2$; hence the bound is sharp.

Since $\gamma_c(C_5) = \gamma_c(C_5^c) = 3$, the bound fails for C_5 .

The same is true for the 6-vertex graph obtained by subdividing one edge of K_4 twice.

Now consider the upper bound.

Since G and G^c are connected, $\gamma_c(G), \gamma_c(G^c) \geq 2$.

By symmetry, we may assume that $\gamma_c(G) \leq \gamma_c(G^c)$.

If $\gamma_c(G) = 2$, then $\gamma_c(G^c) \leq n - 2$ yields $\gamma_c(G)\gamma_c(G^c) \leq 2n - 4$.

Equality requires that G^c has no spanning tree with than two leaves; this happens only when G^c is a path or cycle.

For $\gamma_c(G) \geq 4$, We rewrite Theorem 1 to isolate $\gamma_c(G)\gamma_c(G^c)$ and then apply Theorem 5 and the fact that $n \geq 4$ when G and G^c are connected to compute

$$\gamma_c(G)\gamma_c(G^c) \leq \delta^*(G) - 5 + 2[\gamma_c(G) + \gamma_c(G^c)] \\ \leq 3\delta^*(G) - 1 \leq 3(n - 1/2) - 1 < 2n - 4.$$

Hence we may assume that $\gamma_c(G) = 3$ and $\gamma_c(G^c) \geq 3$.

This requires $\text{diam}(G) = \text{diam}(G^c) = 2$, and hence $\gamma_c(G^c) \leq \delta(G^c) + 1$.

Note also that $\delta^*(G) \geq 2$.

If $\delta(G^c) = 2$, then $\gamma_c(G)\gamma_c(G^c) \leq 9$, which suffices when $n \geq 7$.

If $\delta(G^c) \geq 4$, then Theorem A yields $\gamma_c(G^c) \leq (3n - 8)/5$; always $3(3n - 8)/5 < 2n - 4$ for positive n .

This leaves the case $\delta(G^c) = 3$ and $\gamma_c(G^c) = 4$, so $\gamma_c(G)\gamma_c(G^c) = 12$.

Since $12 < 2n - 4$ when $n > 8$, we are left with $n \in \{7, 8\}$.

To eliminate $n=7$ we use Theorem B.

Since $\gamma_c(G^c) \leq n - \Delta(G^c)$, having $\gamma_c(G^c) = 4$ requires $\Delta(G^c) \leq 3$.

Hence G^c is 3-regular with 7 vertices, but no such graph exists.

This leaves graphs with 8 vertices.

Each vertex of G^c has degree 3 or 4 (again by theorem B), and we have

$$\gamma_c(G^c) = 4, \gamma_c(G) = 3, \text{ and } \text{diam}(G^c) = \text{diam}(G) = 2.$$

Let x be a vertex of maximum degree in G^c , and let $U = V(G) - N[x]$.

If x has degree 4, then $|U| = 3$.

If any neighbor of x has two neighbors in U , then $\gamma_c(G^c) \leq 3$. Since $\text{diam}(G^c) = 2$.

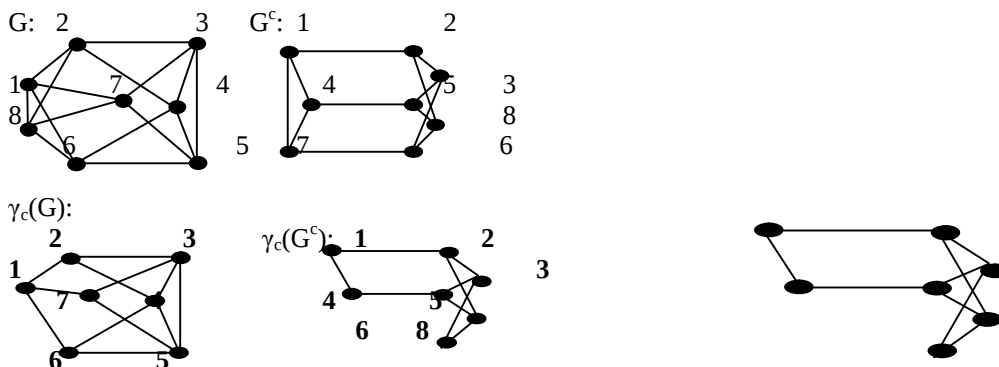
Hence at most four edges join U to $N(x)$.

This leaves degree _ sum at least 5 for edges within U , so $G[U]$ is a triangle.

Now $\gamma_c(G^c) \leq 3$, using x , a vertex of $N(x)$ having a neighbor in U , and that neighbor.

Hence we may assume that G^c is 3-regular and G is 4-regular.

By the result, there is no such graph with $\gamma_c(G) = 3$ and $\gamma_c(G^c) = 4$.



III. CONCLUSION

- In this dissertation, we discussed that when G and G^c are both connected, $\gamma_c(G) + \gamma_c(G^c) \leq \delta^*(G) + 4 - (\gamma_c(G) - 3)(\gamma_c(G^c) - 3)$.
- We also proved that $\gamma_c(G) + \gamma_c(G^c) \leq \delta(G) + 2$, when $\gamma_c(G), \gamma_c(G^c) \geq 4$.
- This bound is sharp when $\delta^*(G) = 6$ and equality can only hold when $\delta^*(G) = 6$.
- Finally, we proved that $\gamma_c(G), \gamma_c(G^c) \leq 2n - 4$ when $n \geq 7$, with equality only for paths and cycles.
- We also discussed that some bounds for $\gamma_c'(G)$.

ACKNOWLEDGEMENT

The authors thank to the respective management for their constant support.

REFERENCES

- [1] E.A. Nordhaus and J. W. Gaddum, *On Complementary graphs*, Amer. Math. Monthly 63 (1956) pp. 175 _ 177.
- [2] J. P. Joseph and S. Arumugam, *Domination in graphs*, Internet. J. Management Systems 11 (1995) 177 _ 182.
- [3] F. Jaeger and C. Payan, *Relations du type Nordhaus _ Gaddum pour le nombre d'absorption d'un graphe simple*, C. R. Acad. Sci. Paris Ser. A 274 (1972) 728 _ 730.
- [4] S. T. Hedetniemi and R. C. Laskar, *Connected domination in graphs*, In B. Bollobas, editor, Graph Theory and Combinatorics. Academic Press, London, 1984.
- [5] J. R. Griggs and M. Wu, *Spanning trees in graphs of minimum degree 4 or 5*, Discrete Math. 104 (1992), pp. 167- 183.
- [6] D. J. Kleitman and D. B. West, *Spanning trees with many leaves*, SIAM J. Discrete Math. 4 (1991), pp. 99 _ 106.
- [7] Y. Caro, R. Yuster, and D.B. West, *Connected domination and spanning trees*, SIAM J. Discrete Math. 13 (2000), 202 _ 211.
- [8] J. E. Dunbar, T. W. Haynes and S. T. Hedetniemi, *Nordhaus _ Gaddum bounds for domination sum in graphs with specified minimum degree*, Util.math. 67 (2005) 97 _ 105.
- [9] E. Sampathkumar and H.B. Walikar, *The connected domination number of a graph*, J. Math. Phys. Sci. 13(6) (1979), pp. 607 _ 613.
- [10] F. Harary, *Graph Theory*, Addison -Wesley, Reading Mass (1969).
- [11] R. P. Gupta, In : *Proof Techniques in Graph Theory* (Ed. F. Harary). Academic Press. New York (1969), 61 _ 62.
- [12] S. Mitchell and S. T. Hedetniemi, *Edge domination in trees* In: *proc. 8th S. E. Conf. on combinatorics*, Graph Theory and Computing, *Utilitas Mathematica*. Winnipeg 19 (1977), 489 _ 509.
- [13] V. R. Kulli and S. C. Sigarkanti, *Nar. Acad. Sci. Lett* 14 (1991), 473 _ 75.
- [14] S.R. Jayaram, *Graphs and Combinatorics* 3 (1987), 357 _ 63.