

Function- ε -Chain in Bitopological Spaces

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ABSTRACT:In this paper, the concept of ε – **chainability** that was introduced using continuous functions in [1] for topological spaces has been extended to bitopological spaces. Function – ε – chainability has been defined in terms of real valued continuous functions on the space. Several well known results prove prerequisites to results established in the paper.

KEYWORDS: **function** – ε – chain, uniform **function** – ε – chainability, uniform **function** chainability, **function** – ε – chain preserving map, **function** chain preserving map, strongly **function** – ε – chain preserving map, strong **function** chain preserving map.

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I. INTRODUCTION

As long back as in 1883, Cantor defined connectedness in metric spaces through ε -chains which have been studied extensively by many mathematicians. These ε -chains were defined between two given points and comprised points of the metric space under consideration. ε -Chainability characterizes connected sets among compact sets in the setting of metric spaces. In the same setting Beer [1] has characterized compact sets among the connected sets. Different definitions of chains in metric spaces have been provided by various Mathematicians. A simple chain joining two points p, q of the metric space X is a finite sequence A_1, A_2, \dots, A_m of subsets of X such that A_1 (and only A_1) contains p , A_m (and only A_m) contains q and $A_i \cap A_{i+1} \neq \emptyset$ if and only if $|i-j| > 1$. However, Piotr Minc [10] has called such a chain by the name weak chain, whereas Newmann [9] has called it a chain of sets. S. A. Naimpally and C.M. Pareek [7] have generalized the concept of ε -chainability in metric spaces to topological spaces by the use of open covers and have obtained a characterization of connectedness in terms of open cover. D.P. Bellamy [2] defined the concept of connectedness by uniformly short paths. Kiran Shrivastava and Geeta Agrawal [11] have defined chainable sets using chains between points. Vijeta Iyer and Kiran Shrivastava [12] have extended these concepts to topological spaces through continuous functions. Function – ε – chainability has been defined in terms of real valued continuous functions on the space. In this article, the same concept [12] has been extended to bitopological spaces.

Throughout this paper $[0, \infty)$ has usual metric topology and ε is positive real number unless mentioned otherwise. Also τ is the intersection of two topologies τ_1 and τ_2 on X . If (X, τ_1, τ_2) is a bitopological space and A is any subset of X then $(A, \tau_{1A}, \tau_{2A})$ is a bitopological subspace of (X, τ_1, τ_2) where (X, τ) is a topological space and τ_{1A} and τ_{2A} are relative topologies on the A .

II. PRELIMINARIES

Definition 2.1: A topological space (X, τ) is said to be *function – f – ε – chainable* if for every $\varepsilon > 0$ there exists a non-constant continuous function $f: X \rightarrow [0, \infty)$ such that for every pair of elements x, y of X , there is a sequence $x = x_0, x_1, \dots, x_n = y$ of elements in X with $f(x) \neq f(y)$ and $|f(x_i) - f(x_{i-1})| < \varepsilon$; $1 \leq i \leq n$.

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Definition 2.2: Let (X, τ) be a topological space and let there exist a non-constant continuous function $f: X \rightarrow [0, \infty)$ such that X is function $-f - \varepsilon$ -chainable for every $\varepsilon > 0$. Then X is said to be *function $-f - \varepsilon$ -chainable*.

Definition 2.3: A topological space (X, τ) is said to be *uniformly function $-f - \varepsilon$ -chainable* if for every $\varepsilon > 0$ there exists a positive integer $l_\varepsilon(f)$ and a non-constant continuous function $f: X \rightarrow [0, \infty)$ such that for every pair of elements x, y of X there is a sequence $x = x_0, x_1, \dots, x_n = y$ in X with $n \leq l_\varepsilon(f)$ and $|f(x_i) - f(x_{i-1})| < \varepsilon; 1 \leq i \leq n$.

III. FUNCTION- $bi - \varepsilon$ -CHAINS

Definition 3.1: A bitopological space (X, τ_1, τ_2) is said to be *function $-bi - f - \varepsilon$ -chainable* if for any $\varepsilon > 0$ there exists a non-constant continuous function $f: (X, \tau_1 \cap \tau_2) \rightarrow [0, \infty)$ such that for every pair of points x, y of X , there exists a sequence $x = x_0, x_1, \dots, x_n = y$ of points in X with $|f(x_i) - f(x_{i-1})| < \varepsilon; 1 \leq i \leq n$.

Definition 3.2: Let (X, τ_1, τ_2) be a bitopological space and let there exist a non-constant continuous function $f: (X, \tau) \rightarrow [0, \infty)$, where $\tau = \tau_1 \cap \tau_2$ such that X is function $-bi - f - \varepsilon$ -chainable for every $\varepsilon > 0$. Then X is said to be *function $-bi - f - \varepsilon$ -chainable*.

Definition 3.3: Let (X, τ_1, τ_2) be a bitopological space, $\tau = \tau_1 \cap \tau_2$ and $A \subset X$. Then $(A, \tau_{1A}, \tau_{2A})$ is said to be *function $-bi - f - \varepsilon$ -chainable* if (A, τ_A) is function $-bi - f - \varepsilon$ -chainable subspace of (X, τ) .

Definition 3.4: Let (X, τ_1, τ_2) be a bitopological space, $\tau = \tau_1 \cap \tau_2$ and $A \subset X$. Let there exist a non-constant continuous function $f: (A, \tau_A) \rightarrow [0, \infty)$ such that (A, τ_A) is function $-bi - f - \varepsilon$ -chainable for every $\varepsilon > 0$. Then $(A, \tau_{1A}, \tau_{2A})$ is said to be *function $-bi - f - \varepsilon$ -chainable*.

Definition 3.5: A bitopological space (X, τ_1, τ_2) is said to be *uniformly function $-bi - f - \varepsilon$ -chainable* if for $\varepsilon > 0$ there exists a positive integer $l_\varepsilon(f)$ and a non-constant continuous function $f: X \rightarrow [0, \infty)$ such that for every pair of elements x, y of X there is a sequence $x = x_0, x_1, \dots, x_n = y$ in X with $n \leq l_\varepsilon(f)$ and $|f(x_i) - f(x_{i-1})| < \varepsilon; 1 \leq i \leq n$.

Definition 3.6: Let (X, τ_1, τ_2) be a bitopological space and let there exist a non-constant continuous function $f: X \rightarrow [0, \infty)$ such that X is uniformly function $-bi - f - \varepsilon$ -chainable for every $\varepsilon > 0$. Then X is said to be *uniformly function $-bi - f - \varepsilon$ -chainable*.

Theorem 3.1: Let (X, τ_1, τ_2) be a bitopological space and let τ be the intersection of τ_1 and τ_2 . Then if $f: (X, \tau) \rightarrow [0, \infty)$ is continuous onto function then (X, τ_1, τ_2) is function $-bi - f - \varepsilon$ -chainable.

Proof: Let $x, x' \in X$ and $\varepsilon > 0$. Let n be the least positive integer greater than $(|f(x) - f(x')|) / \varepsilon$. Without loss of generality, let $f(x') > f(x)$.

Choose

$$y_0 = f(x), y_1 = y_0 + \frac{|f(x) - f(x')|}{n}, y_2 = y_0 + \frac{2(|f(x) - f(x')|)}{n}, \dots, y_n = f(x') \text{ in } [0, \infty).$$

Also $|y_i - y_{i-1}| < \varepsilon; 1 \leq i \leq n$.

Then there exists a sequence of points $x, x_1, x_2, \dots, x_{n-1}, x_n$ in X such that $|f(x_i) - f(x_{i-1})| = |y_i - y_{i-1}| < \varepsilon$. or (X, τ_1, τ_2) is function $-bi - f - \varepsilon$ -chainable.

Theorem 3.2: The relation of function $-bi - f - \varepsilon$ -chainability in a bitopological space is an equivalence relation.

Proof: Obvious.

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Theorem 3.3: Let (X, τ_1, τ_2) be a bitopological space and $A \subset X$. If for every $\varepsilon > 0$ there exists a continuous function $f : (X, \tau) \rightarrow [0, \infty)$, where τ be the intersection of τ_1 and τ_2 such that A is function - $bi - f_A - \varepsilon$ - chainable, then $(\overline{A}, \tau_{1\overline{A}}, \tau_{2\overline{A}})$ is function - $bi - f_{\overline{A}} - \varepsilon$ - chainable.

Proof: Let $x, y \in \overline{A}$. As $f(\overline{A}) \subset f(A)$, $f(x), f(y) \in f(A)$ or there exist $x', y' \in A$ such that $|f(x) - f(x')| < \varepsilon$ and $|f(y) - f(y')| < \varepsilon$. Hence there exist a sequence of elements $x' = x_1, x_2, \dots, x_{n-1} = y'$ in A such that

$$|f_A(x_i) - f_A(x_{i-1})| = |f(x_i) - f(x_{i-1})| < \varepsilon, \quad 2 \leq i \leq n-1.$$

or there exist a sequence of elements $x = x_0, x' = x_1, x_2, \dots, x_{n-1} = y', x_n = y$ in \overline{A} such that

$$|f_{\overline{A}}(x_i) - f_{\overline{A}}(x_{i-1})| = |f(x_i) - f(x_{i-1})| < \varepsilon, \quad 1 \leq i \leq n.$$

Hence the result.

Theorem 3.4: Let (X, τ_1, τ_2) be a bitopological space and $\tau = \tau_1 \cap \tau_2$. Let A be a dense subset of a topological space (X, τ) and for every $\varepsilon > 0$ and let there exist a continuous function $f : (X, \tau) \rightarrow [0, \infty)$ such that A is function - $f_A - \varepsilon$ - chainable. Then the bitopological space X is function - $bi - f - \varepsilon$ - chainable.

Proof: Follows from Theorem 3.3

Theorem 3.5: Let (X, τ_1, τ_2) and (Y, u_1, u_2) be two bitopological spaces and $\tau = \tau_1 \cap \tau_2$ and $u = u_1 \cap u_2$. Let f be an onto continuous function from topological space (X, τ) to topological space (Y, u) . Then if the bitopological space Y is function - $bi - g - \varepsilon$ - chainable, the bitopological space X is function - $bi - g \circ f - \varepsilon$ - chainable.

Proof: Let $x, x' \in X$. Then there is a sequence of points $f(x) = y_0, y_1, \dots, y_n = f(x')$ in a bitopological space Y such that $|g(y_i) - g(y_{i-1})| < \varepsilon; 1 \leq i \leq n$.

or there is a sequence of points $x = x_0, x_1, \dots, x_n = x'$ in the bitopological space X such that $f(x_i) = y_i; 1 \leq i \leq n$ and $|g \circ f(x_i) - g \circ f(x_{i-1})| < \varepsilon; 1 \leq i \leq n$.

Hence bitopological space (X, τ_1, τ_2) is function - $bi - g \circ f - \varepsilon$ - chainable.

Theorem 3.6: Let $f : (X, \tau) \rightarrow (Y, u)$ be bijective open map where (X, τ_1, τ_2) and (Y, u_1, u_2) are two bitopological spaces and $\tau = \tau_1 \cap \tau_2, u = u_1 \cap u_2$. If the bitopological space (X, τ_1, τ_2) is function - $bi - g - \varepsilon$ - chainable then the bitopological space (Y, u_1, u_2) is function - $bi - g \circ f^{-1} - \varepsilon$ - chainable.

Proof: Let $f(x), f(x') \in Y$ where $x, x' \in X$. Now there exists a sequence of points $x = x_0, x_1, \dots, x_n = x'$ in X such that $|g(x_i) - g(x_{i-1})| < \varepsilon; 1 \leq i \leq n$.

or there is a sequence of points $f(x) = f(x_0), f(x_1), \dots, f(x_n) = f(x')$ in Y such that

$$|g \circ f^{-1}(f(x_i)) - g \circ f^{-1}(f(x_{i-1}))| = |g(x_i) - g(x_{i-1})| < \varepsilon; \quad 1 \leq i \leq n.$$

or Y is function - $bi - g \circ f^{-1} - \varepsilon$ - chainable.

Theorem 3.7: Let (X, τ_1, τ_2) and (Y, u_1, u_2) be two bitopological spaces and $\tau = \tau_1 \cap \tau_2$ and $u = u_1 \cap u_2$. Let $f : (X, \tau) \rightarrow (Y, u)$ be a homeomorphism. Then the bitopological space (X, τ_1, τ_2) is function - $bi - g - \varepsilon$ - chainable iff the bitopological space (Y, u_1, u_2) is function - $bi - g \circ f^{-1} - \varepsilon$ - chainable.

Proof: Follows from Theorem 3.5 and 3.6.

Theorem 3.8: Let (X, τ_1, τ_2) and (Y, u_1, u_2) be two bitopological spaces and $\tau = \tau_1 \cap \tau_2$ and $u = u_1 \cap u_2$. Let f be an injective open map from a topological space (X, τ) to a topological space (Y, u) . Let A be any subset of X and $f(A) = B$. If the bitopological subspace A is function - $g - \varepsilon$ - chainable then the bitopological subspace B of (Y, u_1, u_2) is function - $bi - g \circ f_A^{-1} - \varepsilon$ - chainable.

Proof: Follows from Problem 24, chapter 7 [6] and Theorem 3.6.

Theorem 3.9: Let (X, τ_1, τ_2) and (Y, u_1, u_2) be two bitopological spaces and $\tau = \tau_1 \cap \tau_2$ and $u = u_1 \cap u_2$. Let f be an homeomorphism from a topological space (X, τ) to a topological space (Y, u) . Let A be any subset of X and $f(A) = B$. Then bitopological subspace A is function - $g - \varepsilon$ - chainable iff the bitopological subspace B is function - $bi - g \circ f_A^{-1} - \varepsilon$ - chainable.

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Proof: Follows from Problem 25, chap. 7 [6] and Theorem 3.9.

Theorem 3.10: Let (X, τ_1, τ_2) be a bitopological space and (X, τ) is a topological space where $\tau = \tau_1 \cap \tau_2$ and let $X = A \cup B$ where A and B are closed sets in topological space X . Let A be function $-bi - f - \varepsilon$ -chainable and B be function $-bi - g - \varepsilon$ -chainable such that $f(x) = g(x)$ for every $x \in A \cap B$. Then the bitopological space X is function $-bi - h - \varepsilon$ -chainable where

$$h(x) = \begin{cases} f(x), & x \in A \\ g(x), & x \in B \end{cases}$$

Proof: By pasting lemma the function $h : X \rightarrow [0, \infty)$ is continuous.

Then the bitopological space X is function $-bi - h - \varepsilon$ -chainable follows directly from definition of h and function $-f - \varepsilon$ -chainability of A and function $-bi - g - \varepsilon$ -chainability of B .

And hence the bitopological space X is function $-bi - h - \varepsilon$ -chainable.

Theorem 3.11: Let (X, τ_1, τ_2) be a bitopological space, $\tau = \tau_1 \cap \tau_2$ and $A, B \subset X$ such that $A \sim B$ and $B \sim A$ are separated sets and $X = A \cup B$. Let for every $\varepsilon > 0$ there exist a function $f : (X, \tau) \rightarrow [0, \infty)$ such that $f_A : A \rightarrow [0, \infty)$ and $f_B : B \rightarrow [0, \infty)$ are continuous. If A is function $-bi - f_A - \varepsilon$ -chainable and B is function $-bi - f_B - \varepsilon$ -chainable then bitopological space X is function $-bi - f - \varepsilon$ -chainable.

Proof: Now $f_A : A \rightarrow [0, \infty)$ and $f_B : B \rightarrow [0, \infty)$ are continuous functions.

By Problem B, chap. 3[5], f is continuous on X .

Again $f(x) = \begin{cases} f_A(x), & x \in A \\ f_B(x), & x \in B \end{cases}$ and $f_A(x) = f_B(x)$ for every $x \in A \cap B$.

Then by Theorem 3.10, X is function $-bi - f - \varepsilon$ -chainable.

Theorem 3.12: Let (X, τ_1, τ_2) and (Y, u_1, u_2) be two bitopological spaces and $\tau = \tau_1 \cap \tau_2$ and $u = u_1 \cap u_2$. Let (X, τ) be a compact space and (Y, u) be a Hausdorff space and $f : (X, \tau) \rightarrow (Y, u)$ be a continuous bijection. Then bitopological space X is function $-bi - g - \varepsilon$ -chainable iff bitopological space Y is function $-bi - g \circ f^{-1} - \varepsilon$ -chainable.

Proof: Now f is a homeomorphism by Corollary 2.4, chap. 7[4].

The result then follows from Theorem 3.7.

Theorem 3.13: Let (X, τ_1, τ_2) be a bitopological space, $\tau = \tau_1 \cap \tau_2$ and $\{f_n\}$ be a sequence of continuous functions from (X, τ) to $[0, \infty)$ such that $\{f_n\}$ uniformly converges to a function $f : (X, \tau) \rightarrow [0, \infty)$. If bitopological space X is function $-f_n - \varepsilon$ -chainable for each $n \in \mathbb{N}$ then bitopological space X is function $-bi - f - \varepsilon$ -chainable.

Proof: Now $f : X \rightarrow [0, \infty)$ is continuous, by Theorem 4.4 [4].

By uniform convergence; there is $m \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $x \in X$ and for all $n \geq m$.

Let $n \geq m$ and let $x, y \in X$.

Then there is a sequence of points $x = x_0, x_1, \dots, x_n = y$ such that $|f_n(x_i) - f_n(x_{i-1})| < \varepsilon$; $1 \leq i \leq n$.

Also $|f_n(x_i) - f(x_i)| < \varepsilon$; and $|f_n(x_{i-1}) - f(x_{i-1})| < \varepsilon$.

Consequently, $|f(x_i) - f(x_{i-1})| < 3\varepsilon$; $1 \leq i \leq n$.

Hence the result.

Theorem 3.14: Let (X, τ_1, τ_2) be a bitopological space, $\tau = \tau_1 \cap \tau_2$. For every $\varepsilon > 0$, a normal space bitopological space X is function $-bi - f - \varepsilon$ -chainable for some function f on X .

Proof: Choose two non-negative real numbers a and b such that $b - a < \varepsilon$.

Let A and B be disjoint closed subsets of X .

By Urysohn's Lemma, there exists a continuous function $f : (X, \tau) \rightarrow [a, b]$ where $f(x) = a$ for all $x \in A$ and $f(x) = b$ for all $x \in B$.

or $|f(x) - f(y)| \leq b - a < \varepsilon$ for all $x, y \in X$.

As $f : X \rightarrow [0, \infty)$ is continuous, X is $bi - f - \varepsilon$ -chainable.

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Theorem 3.15: Let (X, τ_1, τ_2) be a bitopological space and $\tau = \tau_1 \cap \tau_2$. Let (X, τ) be a compact uniformly function – $bi - f -$ chainable space for some positive real valued function f on X . Then there exists a positive real number e such that

$$l_e(f) + 1 > \frac{f(\bar{x})}{f(\bar{y})} \text{ for some } \bar{x}, \bar{y} \in X.$$

Proof: Let $\bar{x}, \bar{y} \in X$ such that

$$f(\bar{x}) = \inf_{x \in X} f(x) \text{ and } f(\bar{y}) = \sup_{x \in X} f(x)$$

By Problem A(b) and (c), chap. 5 [5], there is an $e > 0$ such that $f(x) > e$ for all $x \in X$.

Now there is a sequence of elements $\bar{x} = x_0, x_1, \dots, x_n = \bar{y}$ in X with $|f(x_i) - f(x_{i-1})| < \varepsilon$ and $n \leq l_e(f)$ or $f(\bar{y}) - f(\bar{x}) < l_e(f)e$

Putting $f(\bar{y}) = k f(\bar{x})$ for some $k > 1$, $f(\bar{x}) < \frac{e l_e(f)}{k-1} \not\leq e$

Hence $l_e(f) + 1 > \frac{f(\bar{x})}{f(\bar{y})}$.

EXAMPLES

- Let (X, τ_1, τ_2) be a bitopological space where X is the set of natural numbers, τ_1 is a discrete topology on X and τ_2 is an odd-even topology which is partition topology generated by $P = \{\{1,2\}, \{3,4\}, \{5,6\}, \dots\}$ on X and $\tau = \tau_1 \cap \tau_2$

(i) Let $f : (X, \tau) \rightarrow [0, \infty)$ defined by $f(2k) = k$, $f(2k - 1) = k$ is continuous.

Then the bitopological space X is function – $bi - f - \varepsilon -$ chainable for $\varepsilon > 1$.

(ii) Let $f : X \rightarrow [0, \infty)$ defined by $f(2k) = 1/k$, $f(2k - 1) = 1/k$ is continuous.

Then the bitopological space X is function – $bi - f - \varepsilon -$ chainable for $\varepsilon > 0.5$.

(iii) Let $\varepsilon > 0$ choose $n \in \mathbb{N}$ with $n > 1/\varepsilon$.

Define $f_\varepsilon : X \rightarrow [0, \infty)$ by $f_\varepsilon(2k) = 1/n^k$, $f_\varepsilon(2k - 1) = 1/n^k$ is continuous.

Then the bitopological space X is function – $bi - f_\varepsilon - \varepsilon -$ chainable for any $\varepsilon > 0$.

- Let (X, τ_1, τ_2) be a bitopological space with τ_1 and τ_2 be two partition topology on space $X \subset [0, \infty) \times \mathbb{R}$ generated by the sets $A_\alpha = \{(\alpha - 1, \beta) : \beta \in \mathbb{R}\}; \alpha \geq 1$ and $B_\alpha = \{(\alpha + 1, \beta) : \beta \in \mathbb{R}\}; \alpha \geq -1$.

Then the bitopological space X is function – $bi - \pi -$ chainable where π is the projection map given by

$$\pi(\alpha, \beta) = \alpha; (\alpha, \beta) \in [0, \infty) \times \mathbb{R}.$$

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