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The Diagonal Equivalence of A Non-Negative Quaternion Matrix to A Doubly Stochastic Matrix

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ABSTRACT: Let A be a positive square quaternion matrix. Then there exists two diagonal matrices D_1, D_2 whose diagonal elements are positive such that $D_1 A D_2$ is quaternion doubly stochastic matrices.

Marcus and Newman [6] and maxfield and minc [7] also studied their problem. Perfect and mirsky have shown that given a fully indecomposable matrix B, there exists a doubly stochastic non-negative matrix with the same zero pattern.

Morishima [2] and Thompson [3] have studied such operators in extending the theorems of perron frobenius theory.

In this last two sections of the paper the diagonal equivalence of a non-negative quaternion matrix to a doubly stochastic matrix . We consider the diagonal equivalence of a non-negative quaternion matrix to a row doubly stochastic matrix. Given any square non-negative matrix A with atleast one positive element in each row it is clear that there exists a row doubly stochastic matrix C having zeros in exactly the same position as A.

Recently, sinkhorn [1] proved the existence of D sure that DAD is row stochastic under the assumption A is a positive matrix.

KEYWORDS: quaternion doubly stochastic matrices, fully indecomposable matrices, Frobenius-Konig theorem, eigen values.

1. INTRODUCTION

FULLY INDECOMPOSABLE MATRICES

1.1 Notation:

Let $A = [a_{st}]$ be an $m \times n$ quaternion doubly stochastic matrix. If $a_{st} > 0$ for each s and t then we write $A \gg 0$; if $a_{st} \geq 0$, then we write $A \geq 0$; if $A \geq 0$ but $A \neq 0$, then we write $A > 0$.

1.2 Definition:

Let $A \geq 0$ be an $n \times n$ quaternion doubly stochastic matrix. Then A is called reducible provided there exists a permutation matrix P such that PAP^T has the form

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$$\begin{bmatrix} A_1 & A_2 \\ 0 & B_2 \end{bmatrix} \quad \text{--- (2.2.1)}$$

Where A_1 and A_2 are non-empty square matrices. If A is not reducible, it is irreducible. A $n \times n$ matrix $A > 0$ is called fully indecomposable. If there do not exist permutation matrices P and Q such that PAQ has the form (2.2.1). By convention every 1×1 matrix is fully indecomposable if and only if single entry is positive.

An $n \times n$ matrix $A > 0$ is quaternion column stochastic if all its (column sums) are one. Then $n \times n$ matrix $A > 0$ is quaternion doubly stochastic provided it is both row stochastic & column stochastic.

Lemma 2.3:

Let $A \geq 0$ be an $n \times n$ quaternion doubly stochastic matrix. Then A is fully indecomposable if and only if there are permutation matrices P and Q such that PAQ has a positive main diagonal and is irreducible.

Proof:

Let A be fully indecomposable. Then by Frobenius-Konig theorem [7,P.97] there exists permutation matrices P and Q such that PAQ has a positive main diagonal. But obviously PAQ is fully indecomposable and thus irreducible.

Conversely, suppose that $C = PAQ$ has a positive main diagonal (positive main diagonal) and is irreducible. Since C is fully indecomposable if and only if A is, we may assume $C = A$. Suppose A is not fully indecomposable, and let P_1 and Q_1 be two permutation matrices such that P_1AQ_1 is of the form (2.2.1). Suppose A_1 has r columns, and A_2 has $n-r$ columns. Then we may write $P_1AQ_1 = A'Q'$ where $A' = P_1AP^T$ is again a matrix with a positive main diagonal and $Q' = P_1Q_1$ is a permutation matrix. But then it follows that Q' permutes the first r columns of A' among themselves and the last $n-r$ columns of A' among themselves. Hence A' is of the form (2.2.1) and A is reducible, which is contradiction.

Theorem 2.4:

If D_1 and D_2 are diagonal matrices such that $D_1AD_2 = S$ is a quaternion doubly stochastic matrices, then for any permutation matrices P and Q we have $D'_1(PAQ)D'_2 = S$. Where $D'_1 = PD_1P^T$ and $D'_2 = Q^TD_2Q$ are diagonal matrices.

Proof:

Let A is irreducible with positive main diagonal, then by the Frobenius-Konig theorem [7,P.97] there exists permutation matrices P and Q such that PAQ has a positive main diagonal.

But obviously PAQ is irreducible with positive main diagonal and thus fully indecomposable.

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Conversely, suppose that $C=PAQ$ with has a fully indecomposable. Since C is irreducible positive main diagonal, and let P_1 and Q_1 be two permutation matrices such that P_1AQ_1 is of the form (2.2.1). Suppose A_1 has r rows and A_2 has $n-r$ rows. Then we may write $P_1AQ_1 = A'Q'$ where $A' = P_1AP_1^T$ is again a matrix with a positive main diagonal and $Q' = P_1Q_1$ is a permutation matrix. But then it follows that Q' permutes the first r rows of A' among themselves and the last $n-r$ rows of A' of among themselves. Hence A' is of the form (2.2.1) A and D is fully indecomposable which is contradiction. This proves the above lemma.

Example:

$A \geq 0$, Since A fully indecomposable.

$$A = \begin{bmatrix} 2+i+2j+k & 3-i+j+k & -4-3j-2k \\ 3-i+j+k & 2+i-j+2k & -4-3k \\ -4-3j-2k & -4-3k & +9+3j+5k \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad Q = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$PAQ = C, C \geq 0,$

$$PAQ = \begin{bmatrix} 2+i+2j+k & 3-i+j+k & -4-3j-2k \\ 3-i+j+k & 2+i-j+2k & -4-3k \\ -4-3j-2k & -4-3k & 9+3j+5k \end{bmatrix}$$

The $D_1'AD_2'$ Theorems

Theorems 3.1:

Let $A>0$ be an $n \times n$ fully indecomposable quaternion matrix. Then there exist diagonal matrices D_1 and D_2 with positive main diagonals such that $D_1'AD_2'$ is quaternion doubly stochastic. Moreover D_1' and D_2' are uniquely determine up to scalar multiples.

Proof:

By(2.4), E is enough to prove this theorem for an irreducible matrix A with a positive main diagonal . let $x \gg 0$ and set $D_2' = \text{diag}(x_1', x_2', \dots, x_n')$ and $D_1' = \text{diag}((\alpha x)_1, \dots, (\alpha x)_n)$. Then $D_1'AD_2'$ is quaternion doubly stochastic if and only if $Tx = x$, where T is the operator associated with A . Hence the results from

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“Let $A > 0$ be an $n \times n$ irreducible matrix with a positive main diagonal. Let T be the operator associated with the matrix A . Then 1 is an eigen value of T with a unique eigen vector u in P . Further more $u >> 0$.

We now show that the condition that $A > 0$ fully indecomposable is essential necessary for (3.1) to hold.

Theorem 3.2:

Let $A > 0$ be an $n \times n$ quaternion matrix. Then there exist diagonal matrices D_1 and D_2 such that $D_1'AD_2'$ is quaternion doubly stochastic matrices if and only if after independent permutations of rows and columns A is the direct sum of fully indecomposable matrices.

Proof:

Suppose the matrix A' obtained from A by independent permutations of rows and columns is equal to $A^{(1)} + \dots + A^{(K)}$ where each $A^{(i)}$ is fully indecomposable. By there exists diagonal matrices $D_1^{(1)} \dots D_1^{(K)}$ and $D_2^{(1)} \dots D_2^{(K)}$ such that $D_1^{(i)}A^{(i)}D_2^{(i)}$ is quaternion doubly stochastic. If

$$D_1' = D_1^{(1)} + \dots + D_1^{(K)}; D_2' = D_2^{(1)} + \dots + D_2^{(K)}$$

Then $D_1'AD_2'$ is quaternion doubly stochastic. Hence (2.4) we can find D_1 and D_2 such that D_1AD_2 is quaternion doubly stochastic.

Conversely, suppose that there exists D_1 and D_2 such that $D_1AD_2 = S$ is quaternion doubly stochastic

$$S = \begin{bmatrix} S_1' & 0 \dots & 0 \\ S_{12}' & S_2' \dots & 0 \\ S_{k_1}' & S_{k_2}' \dots & S_k' \end{bmatrix}$$

Where each S_i is either fully indecomposable or a 1×1 zero matrix. But since S' is quaternion doubly stochastic, the row sums of S_i' are all one and hence the column sums of S_i' being less than or equal to 1 must in fact be equal to 1. Hence $S_{12}' = 0, \dots, S_k' = 0$. Repeating this argument we obtain $S' = S_1' + S_2' + \dots + S_k'$, where each S_i' is a quaternion doubly stochastic and hence fully indecomposable.

Theorem 3.3:

The Kronecker product of two quaternion fully indecomposable double stochastic matrices if and only if both are fully indecomposable doubly stochastic matrices.

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Proof:

Suppose that $A \otimes B$ is fully indecomposable, but A is not fully indecomposable quaternion doubly stochastic matrices, then there exist permutation matrices P and Q such that

$$PAQ = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \quad \text{--- (3.3.1)}$$

Where A_{11} and A_{22} are as in (1), Now

$$(P \otimes I)(A \otimes B)(Q \otimes I) = PAQ \otimes B = \begin{bmatrix} A_{11} \otimes B & A_{12} \otimes B \\ 0 & A_{22} \otimes B \end{bmatrix}$$

Where the Kronecker product of non-square matrices is defined in the obvious way since $(P \otimes I)$ and $(Q \otimes I)$ are permutation matrices this gives a contradiction quaternion doubly stochastic. Similarly, if B is not fully indecomposable then there permutation matrix R, where $[R]_{(i,k)(j,l)} \dots S_{il} S_{kj}$, satisfies $B \otimes A = R(A \otimes B)R^T$, from which the result follows as above.

II. CONCLUSION

In this paper, we can discuss fully indecomposable doubly stochastic matrices to a diagonal equivalence of a non-negative quaternion matrix.

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