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## Trivial Colouring and Non-Trivial Colouring for Graph's Vertices

M.E. Hassan

Assistant Professor, Department of Mathematics, Faculty of Science, Sudan

University of Science and Technology, Khartoum, Sudan

**ABSTRACT:** In this paper we used colouring of families of disjoint sets technique, to colour vertices of the graph depend on the clique number, and we introduce some results of vertex's colouring (trivial colouring and nontrivial colouring) depend on clique number. We introduce the concept maximum and minimum number of edges in the graph to colour  $k$  vertices by  $m$  colours, and we used this concept to prove a result about number of all possible ways to colour  $k$  vertices by  $m$  colours, and a result about the range of edges in which we can colour  $k$  vertices by  $m$  colours.

**KEYWORDS:** clique number, colouring of families of disjoint sets technique, trivial colouring, nontrivial colouring.

### I. INTRODUCTION

In paper [1] we introduced the concept of colouring of families of disjoint sets technique as a topic of set theory. We also introduce definitions of degree of intersection, meeting and standard of meeting, minimum number of families of disjoint sets and maximum number of families of disjoint sets. In paper [2] we introduce some concepts in graph theory to introduce new method for colouring edges and vertices of graph. This method has two types first related to vertices of the graph and second related to cycles. This method depend on the theory in our paper "Family of disjoint sets and Its Applications", with suitable changes. In paper [3] we prove minimum number of families of disjoint sets with known degree of intersection (minimum number of colours) and we introduce another proof for maximum number of families of disjoint sets (maximum number of colours). Also in that paper we introduce the concept types of families of disjoint sets (types of colouring) and prove some results related to this concept, and we introduce the concept restricted families of disjoint sets (restricted colouring), restricted colouring is colouring in which we add a condition. Suppose we have small identical cubes, we want to colour these cubes such that any two adjacent small cubes coloured by two different colours, and any big cube consists of twenty seven small identical cubes coloured by twenty seven different colours, how many colours will be used. This problem of cubes can be solved by using restricted colouring. In paper [4] we assume both sets and graphs are finite. Results in that paper use colouring of families of disjoint sets technique for finite sets and finite graphs. Using results in that paper we can compare between colouring of families of disjoint sets technique and colouring of graphs in graph theory.

In paper [5] we prove some results in graph theory using colouring of families of disjoint sets technique. We introduced the concept of set labeled by families of disjoint sets and the degree of this set, as generalization to the degree of intersection and proved some results related to this concept. We introduce sorts of families of disjoint sets (sorts of colour classes) as a concept related to set labeled by families of disjoint sets, also we compare between familiar and unfamiliar sorts of families of disjoint sets and explain that by some examples.

In paper [6] we prove some results of edge's colouring (trivial colouring and nontrivial colouring). We prove some results of edge's colouring for many graphs connected by bridges. In many examples we determine minimum colours to colour edges of the graphs only by calculation when we know number of the edges, number of the vertices and largest degree of vertex, whatever the graph is huge. We used the concept of trivial colouring and nontrivial colouring to prove results of edge's colouring for trees, bipartite graphs and complete graphs.

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In paper [7] we modify the definitions of local maximum families of disjoint sets and local minimum families of disjoint sets, which we introduced in paper [1], and prove some propositions related to these two concepts. Also in paper [7] we define local minimum and local maximum colour classes to colour edges of the graph, and prove some results related to these two concepts, in that paper for any graph with known number of vertices, known maximum degree of vertex, and known number of edges, such that number of vertices not exceed twice maximum degree of vertex, we can determine minimum and maximum number of colour classes to colour edges of the graph, for any huge number of edges, and give many examples.

In this paper we introduce the concept maximum and minimum number of edges in the graph to colour  $k$  vertices by  $m$  colours, and introduce some results related to this concept. In section four we used colouring of families of disjoint sets technique, to colour vertices of the graph depend on the clique number, and we introduce some results of vertex colouring (trivial colouring and nontrivial colouring) depend on clique number. In section five we prove a result about number of all possible ways to colour  $k$  vertices by  $m$  colours. In section six we introduced a result about the range of edges in which we can colour  $k$  vertices by  $m$  colours. By this result we can determine the range of edges in which we can colour a huge number of vertices by a huge number of colours.

## II. RELATED WORK

In this section we recall some definitions and results related to colouring in graph theory and colouring of families of disjoint sets technique.

Theorem 2.1. The sum of degrees of vertices equals twice the number of edges [13].

Definition 2.2. The clique number is the greatest number of mutually adjacent vertices in the graph  $G$  [12].

Definition 2.3. If we have  $m$  families of disjoint sets and we have a set all element of this set represent some of these  $m$  families of disjoint sets, such that any two elements of this set represent two different families of disjoint sets and there is a relation between these elements of the set, then we call this set, set labeled by families of disjoint sets [5].

Definition 2.4. Suppose we have  $m$  families of disjoint sets and many sets labeled by families of disjoint sets, if maximum number of elements for each set labeled by families of disjoint sets is  $n$  where  $n < m$ , then  $n$  is called the degree of set labeled by families of disjoint sets. If  $n = m$  then the degree is called trivial degree or trivial case [5].

Definition 2.5. If  $S_1, S_2, S_3, \dots, S_{k-1}$  are sets (families of disjoint) each labeled by one label such that  $S_i \neq S_{i+1}$ , where  $1 \leq i \leq k-2$ ,  $k \geq 3$ , and any set  $S_i$  represent family of disjoint sets  $\chi_i$ , and if  $X_1, X_2, X_3, \dots, X_k$  are sets labeled by  $l$  labels, such that  $2 \leq l \leq n$ , at least there is one of degree  $n$ , and  $X_i$  labeled by  $l$  labels include label of  $S_{i-1}$  and include label of  $S_i$ , where  $2 \leq i \leq k-1$ ,  $X_1$  labeled by  $l$  labels include label of  $S_1$ , and  $X_k$  labeled by  $l$  labels include label of  $S_{k-1}$ , then  $X_1, S_1, X_2, S_2, X_3, S_3, \dots, X_{k-1}, S_{k-1}, X_k$  is called a sequence of intersections and sets [6].

Definition 2.6. If  $S_1, S_2, S_3, \dots, S_{k-1}$  are sets (families of disjoint) such that  $S_i \neq S_{i+1}$ , where  $1 \leq i \leq k-2$ ,  $k \geq 3$ , and any set  $S_i$  represent family of disjoint sets  $\chi_i$ , and if  $X_1, X_2, X_3, \dots, X_k$  are intersections each of degree  $l$  such that  $2 \leq l \leq n$ , at least there is one of degree  $n$ , and  $X_i$  intersection of  $l$  sets include  $S_{i-1}$  and  $S_i$ , where  $2 \leq i \leq k-1$ ,  $X_1$  can be intersection of  $l$  sets include  $S_1$ , and  $X_k$  can be intersection of  $l$  sets include  $S_{k-1}$ , then  $X_1, S_1, X_2, S_2, X_3, S_3, \dots, X_{k-1}, S_{k-1}, X_k$  is called a sequence of sets labeled by  $n$  families of disjoint sets or sequence of sets labeled by  $n$  labels [6].

Definition 2.7. A walk of intersections and sets is an alternating sequence  $X_1, S_1, X_2, S_2, X_3, S_3, \dots, X_k, S_k$  of intersections and sets where  $S_1, S_2, S_3, \dots, S_k$  are sets such that  $S_i \neq S_{i+1}$ ,  $S_k \neq S_1$ , and any set  $S_i$  represent family of disjoint sets  $\chi_i$  where  $1 \leq i \leq k$ , and  $X_1, X_2, X_3, \dots, X_k$  are intersections each of degree  $n$  such that  $X_i$  is intersection of  $n$  sets include  $S_{i-1}$  and  $S_i$ , where  $2 \leq i \leq k$  and  $X_1$  is intersection of  $n$  sets include  $S_1$  and  $S_k$ . The walk of intersections and sets can also denoted by  $X_1, X_2, \dots, X_{k-1}, X_k$ . A walk of  $k$  intersections  $S_0, X_1, S_1, X_2, S_2, X_3, S_3, \dots, X_k, S_k$  is called cycle of intersections if any two intersections meet each other at these  $k$  intersections,  $S_0 = S_k$  and  $k \geq 3$  [6].

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Definition 2.8. If  $S_0, X_1, S_1, X_2, S_2, X_3, S_3, \dots, X_k, S_k$  is a sequence of sets labeled by  $n$  families of disjoint sets or sequence of sets labeled by  $n$  labels, then these  $k$  sets labeled by  $n$  labels called cycle of sets labeled by  $n$  labels if  $S_0 = S_k$  and  $k \geq 3$  [6].  
Theorem 2.9. Let  $w$  intersections each with degree of intersection equal  $n$ ,  $m$  be number of local maximum families of disjoint sets, meet each other at  $w$  intersections, then we have  $m(\frac{w+1}{2}) \leq wn$  for  $w$  odd, and  $m(\frac{w}{2} + 1) \leq wn$  for  $w$  even [1].

Remark 2.10. In this paper we mean by different families of disjoint sets (colour classes), different values of  $m$  from relation  $\sum_{i=1}^m t_i = wn$ , where  $n$  is constant (can be clique number) and both  $w$  and  $t_i$  are variables, and we mean by different types of families of disjoint sets, different values of  $m$  from relation  $\sum_{i=1}^m t_i = wn$ , where  $n$  is variable, both  $w$  and  $t_i$  can be variables or constant [4].

Proposition 2.11. For the regular graph has  $k$  vertices (and  $k$  edges) where  $k$  is number number, then minimum number to colour edges of this graph is nontrivial colouring [6].

Proposition 2.12. For the regular graph has  $k$  vertices (and  $k$  edges) where  $k$  is odd number, then minimum number to colour edges of this graph is nontrivial colouring [6].

Definition 2.13. Let  $G$  be a graph,  $m$  is called local minimum colour classes to colour edges of graph  $G$  at  $w$  vertices  $v_1, v_2, v_3, \dots, v_w$ , if we use  $m-1$  colour classes to colour edges of graph  $G$ , then exists at least two adjacent edges belong to one colour class or satisfy trivial colouring (every colour class appear  $w$  times at  $w$  vertices) [7].

Definition 2.14. Let  $G$  be a graph,  $m$  is called local maximum colour classes to colour edges of graph  $G$  at  $w$  vertices  $v_1, v_2, v_3, \dots, v_w$ , if we use  $m+1$  colour classes to colour edges of graph  $G$ , then exists a colour class  $c_i$  does not meet colour class  $c_j$  at these  $w$  vertices, where  $1 \leq i < j \leq m+1$  [7].

## III. MAXIMUM AND MINIMUM NUMBER OF EDGES

In this section we introduce the concept maximum and minimum number of edges in the graph to colour  $k$  vertices by  $m$  colours, and introduce some results related to this concept.

Definition 3.1. If the graph  $G$  of  $k$  vertices, these vertices coloured by  $m$  colours, then  $G$  is called complete graph of  $k$  vertices with respect of  $m$  colours, as abbreviation  $G$  is called complete graph with respect to colours. We will use the symbol  $K_{k,m}$ .

Definition 3.2. We called  $l$  maximum number of edges in the graph  $G$  to colour  $k$  vertices of this graph by  $m$  colours, if we add any edge (without addition of any vertex) there will be two adjacent vertices have same colour.

Definition 3.3. We called  $l$  minimum number of edges in the graph  $G$  to colour  $k$  vertices of this graph by  $m$  colours, if we delete any edge, then there is a vertex colourless (of degree zero), or minimum colours to colour  $k$  vertices becomes  $m-1$ .

Remark 3.4. If the vertices of the graph  $G$  coloured by  $m$  colours, these  $m$  colour classes labeled by  $1, 2, 3, \dots, m$ . Let  $a_i$  be number of vertices labeled by the colour class  $i$ , where  $1 \leq i \leq m$ , therefore  $\sum_{i=1}^m a_i = k$  where  $k$  is the number of vertices in

the graph  $G$ . We exchange colour classes labeled by  $1, 2, 3, \dots, m$  such that  $a_1 \leq a_2 \leq a_3 \leq \dots \leq a_m$  using this arrangement the number

$[a_1(a_2 + a_3 + a_4 + \dots + a_m) - h_1] + [a_2(a_3 + a_4 + a_5 + \dots + a_m) - h_2] + [a_3(a_4 + a_5 + a_6 + \dots + a_m) - h_3] + \dots + [a_{m-2}(a_{m-1} + a_m) - h_{m-2}] + [a_{m-1}(a_m) - h_{m-1}] = l$ , where  $h_i$  is number of edges not connected the following vertices:  $v_i$  with  $v_{i+1}$ ,  $v_i$  with  $v_{i+2}$ ,  $v_i$  with  $v_{i+3}$  and till  $v_i$  with  $v_m$ ,  $1 \leq i \leq m-1$ , and  $v_i$  vertex labeled by  $i$ . Maximum number of edges such that there is no vertex of degree zero equal  $a_1(a_2 + a_3 + a_4 + \dots + a_m) + a_2(a_3 + a_4 + a_5 + \dots + a_m) +$

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$a_3(a_4 + a_5 + a_6 + \dots + a_m) + \dots + a_{m-2}(a_{m-1} + a_m) + a_{m-1}(a_m) = l$ . Maximum value of  $\sum_{i=1}^{m-1} a_i(a_{i+1} + a_{i+2} + a_{i+3} + \dots + a_m)$  when

we minimize  $a_j - a_i$  for all  $a_i \neq a_j$ , where  $1 \leq i < j \leq m$ .

Minimum number of edges such that any two vertices in two different colours are adjacent, this mean there is no vertex of degree one ( $m > 2$ ) or degree zero, for  $\sum_{i=1}^{m-1} a_i(a_{i+1} + a_{i+2} + a_{i+3} + \dots + a_m)$  we have  $a_1 = a_2 = a_3 = \dots = a_{m-1} = 1$ , and

$a_m = k + 1 - m$ . In case  $a_1 = a_2 = a_3 = \dots = a_m = 1$ , the maximum number of edges equal minimum number of edges in the graph  $G$ , such that any two vertices in two different colours are adjacent. The symbol  $a_1$  means number of vertices labeled by the colour class 1, and the  $a_2$  means number of vertices labeled by the colour class 2, the symbol  $a_1 a_2$  means edge join vertex labeled by the colour class 1, and vertex labeled by the colour class 2, then number  $a_1 a_2$  means number of edges joining vertices labeled by the colour classes 1 and 2.

Proposition 3.5. If the vertices of the graph  $G$  coloured by  $m$  colours, these colour classes labeled by  $1, 2, 3, \dots, m$ . Let  $a_i$  be number of vertices labeled by the colour  $i$ , where  $1 \leq i \leq m$ , therefore  $\sum_{i=1}^m a_i$  equal the number of vertices in the graph  $G$ ,

and  $a_1(a_2 + a_3 + a_4 + \dots + a_m) + a_2(a_3 + a_4 + a_5 + \dots + a_m) + a_3(a_4 + a_5 + a_6 + \dots + a_m) + \dots + a_{m-2}(a_{m-1} + a_m) + a_{m-1}(a_m) = l$ , where  $l$  is the number of edges in the graph  $G$ .

Proof: Since  $a_i$  be number of vertices labeled by the colour class  $i$ , where  $1 \leq i \leq m$ , and every vertex labeled by a colour class  $i$ , where  $1 \leq i \leq m$ , therefore  $\sum_{i=1}^m a_i$  equal the number of vertices in the graph  $G$ . Let  $l$  be the number of edges in the

graph  $G$ , since  $a_1$  is the number of vertices labeled by the colour 1, and  $a_2$  is the number of vertices labeled by the colour 2, then  $a_1 a_2$  is the number of edges connected vertex labeled by 1, and vertex labeled by 2. therefore  $a_1(a_2 + a_3 + a_4 + \dots + a_m) + a_2(a_3 + a_4 + a_5 + \dots + a_m) + a_3(a_4 + a_5 + a_6 + \dots + a_m) + \dots + a_{m-2}(a_{m-1} + a_m) + a_{m-1}(a_m) = l$ .

Example 3.6. Let the graph  $G$  be the triangle, the three vertices labeled by the colour classes 1, 2, 3.  $a_1, a_2, a_3$ , are numbers of vertices labeled by the colour classes 1, 2, 3 respectively.  $a_1 = 1, a_2 = 1, a_3 = 1$ , using Proposition 3.5. we have  $a_1(a_2 + a_3) + a_2(a_3) = 1(1 + 1) + 1(1) = 3$ . The number of edges in graph  $G$  equal three.

Example 3.7. Let the graph  $G$  be the complete graph  $K_4$ , the four vertices labeled by the colour classes 1, 2, 3, 4.

$a_1, a_2, a_3, a_4$ , are numbers of vertices labeled by the colour classes 1, 2, 3, 4 respectively.  $a_1 = 1, a_2 = 1, a_3 = 1, a_4 = 1$ , using Proposition 3.4. we have  $a_1(a_2 + a_3 + a_4) + a_2(a_3 + a_4) + a_3(a_4) = 1(1 + 1 + 1) + 1(1 + 1) + 1(1) = 6$ . The number of edges in graph  $G$  equal three.

Example 3.8. Let  $G$  be a graph with five vertices, its vertices coloured by three colours find maximum and minimum number of edges.

Solution: Let  $a_1, a_2, a_3$ , are numbers of vertices labeled by the colour classes 1, 2, 3 respectively, and such that

$a_1 \leq a_2 \leq a_3$ , and  $a_1 + a_2 + a_3 = 5$ , then  $a_1, a_2, a_3$  have the value 1, 1, 3, or the value 1, 2, 2. For the value 1, 1, 3 maximum number of edges equal  $a_1(a_2 + a_3) + a_2(a_3) = 1(1 + 3) + 1(3) = 7$ . For the value 1, 2, 2 maximum number of edges equal  $a_1(a_2 + a_3) + a_2(a_3) = 1(2 + 2) + 2(2) = 8$ .

Example 3.9. Let  $G$  be a graph with 20 vertices, its vertices coloured by 13 colours, we have 14 ways, in each way we coloured the graph by 13 colours, and in each way we can find maximum number of edges / where

$l = a_1(a_2 + a_3 + a_4 + \dots + a_{13}) + a_2(a_3 + a_4 + a_5 + \dots + a_{13}) + a_3(a_4 + a_5 + a_6 + \dots + a_{13}) + \dots + a_{11}(a_{12} + a_{13}) + a_{12}(a_{13})$ , and  $a_i$  equal

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number of vertices coloured by the colour class  $i$ ,  $1 \leq i \leq 13$ ,  $1 \leq a_i \leq 8$ , and  $1 \leq i \leq 13$ . Table (I) explains these 14 ways with more details.

	$a_1$	$a_2$	$a_3$	$a_4$	$a_5$	$a_6$	$a_7$	$a_8$	$a_9$	$a_{10}$	$a_{11}$	$a_{12}$	$a_{13}$
1	1	1	1	1	1	1	2	2	2	2	2	2	2
2	1	1	1	1	1	1	1	2	2	2	2	2	3
3	1	1	1	1	1	1	1	1	2	2	2	2	4
4	1	1	1	1	1	1	1	1	1	2	2	2	5
5	1	1	1	1	1	1	1	1	1	1	2	2	6
6	1	1	1	1	1	1	1	1	1	1	1	2	7
7	1	1	1	1	1	1	1	1	1	1	1	1	8
8	1	1	1	1	1	1	1	1	2	2	2	3	3
9	1	1	1	1	1	1	1	1	1	2	3	3	3
10	1	1	1	1	1	1	1	1	1	2	2	3	4
11	1	1	1	1	1	1	1	1	1	1	2	4	4
12	1	1	1	1	1	1	1	1	1	1	2	3	5
13	1	1	1	1	1	1	1	1	1	1	1	4	5
14	1	1	1	1	1	1	1	1	1	1	1	3	6

Table (I)

Example 3.10. Let  $a_1(a_2 + a_3 + a_4 + a_5) + a_2(a_3 + a_4 + a_5) + a_3(a_4 + a_5) + a_4(a_5) = l_1$ , and  $b_1(b_2 + b_3 + b_4 + b_5) + b_2(b_3 + b_4 + b_5) + b_3(b_4 + b_5) + b_4(b_5) = l_2$ , where  $a_i, b_i$ , are integers, such that  $\sum_{i=1}^5 a_i = \sum_{i=1}^5 b_i$ ,  $l_1$  have  $\binom{5}{2}$  terms, each can be written as  $a_i a_j$  where  $1 \leq a_i < a_j \leq 5$ , and  $l_2$  have  $\binom{m}{2}$  terms, each can be written as  $b_i b_j$  where  $1 \leq b_i < b_j \leq 5$ . Let

$a_1 = x - 2$ ,  $a_2 = x - 1$ ,  $a_3 = x$ ,  $a_4 = x + 1$ ,  $a_5 = x + 2$ ,  $b_1 = b_2 = b_3 = b_4 = b_5 = x$ ,  $l_1$  have  $\binom{5}{2} = 10$  terms of  $a_i a_j$ , also  $l_2$  have  $\binom{5}{2} = 10$  terms of  $b_i b_j$  (with the same subscript),

$$l_1 = a_1 a_2 + a_1 a_3 + a_1 a_4 + a_1 a_5 + a_2 a_3 + a_2 a_4 + a_2 a_5 + a_3 a_4 + a_3 a_5 + a_4 a_5 = 10x^2 - 1 - 4,$$

$$l_2 = b_1 b_2 + b_1 b_3 + b_1 b_4 + b_1 b_5 + b_2 b_3 + b_2 b_4 + b_2 b_5 + b_3 b_4 + b_3 b_5 + b_4 b_5 = 10x^2, \text{ then we have } l_1 < l_2.$$

Lemma 3.11. Let  $\sum_{i=1}^m a_i = \sum_{i=1}^m b_i$ ,  $a_1(a_2 + a_3 + a_4 + \dots + a_m) + a_2(a_3 + a_4 + a_5 + \dots + a_m) + a_3(a_4 + a_5 + a_6 + \dots + a_m) + \dots$

$+ a_{m-2}(a_{m-1} + a_m) + a_{m-1}(a_m) = l_1$ , and  $b_1(b_2 + b_3 + b_4 + \dots + b_m) + b_2(b_3 + b_4 + b_5 + \dots + b_m) + b_3(b_4 + b_5 + b_6 + \dots + b_m) + \dots$

$+ b_{m-2}(b_{m-1} + b_m) + b_{m-1}(b_m) = l_2$ , such that  $l_1$  have  $\binom{m}{2}$  terms, each can be written as  $a_i a_j$  where  $1 \leq i < j \leq m$ ,  $m$  is odd, and

such that  $l_2$  have  $\binom{m}{2}$  terms, each can be written as  $b_i b_j$  where  $1 \leq i < j \leq m$ . Let  $a_i = x - \frac{m+1-2i}{2}$ , and  $a_j = x - \frac{m+1-2j}{2}$ ,

where  $1 \leq i < j \leq m$ , and  $b_1 = b_2 = b_3 = \dots = b_m = x$ , then  $l_1 < l_2$ .

Proof: Let us take the terms in the form of  $a_i a_j$  as follows  $(x+c)(x-c)$ ,  $(x+d)(x-d)$ ,  $(x+c)(x+d)$ ,  $(x-c)(x-d)$ ,

$(x+c)(x-d)$ ,  $(x-c)(x+d)$ , where  $1 \leq c < d \leq \frac{m-1}{2}$ , then we have

$$(x+c)(x-c) + (x+d)(x-d) + (x+c)(x+d) + (x-c)(x-d) + (x+c)(x-d) + (x-c)(x+d) = 6x^2 - c^2 - d^2 \quad (3.1)$$

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Let us take the terms in the form of  $a_i a_j$  as  $(x+c)(x+g)$ ,  $(x-c)(x+g)$ ,  $(x+d)(x+g)$ ,  $(x-d)(x+g)$ , where  $1 \leq g \leq \frac{m-1}{2}$ ,

then we can write

$$(x+c)(x+g) + (x-c)(x+g) + (x+c)(x-g) + (x-c)(x-g) = 4x^2 \quad (3.2)$$

Let us take the terms in the form of  $a_i a_j$  as  $x(x+c)$ ,  $x(x-c)$ , then we can write

$$x(x+c) + x(x-c) = 2x^2 \quad (3.3)$$

From view of (3.1), (3.2) and (3.3) we can write

$$l_1 = c^2 x^2 - 2 \sum_{i=1}^{\frac{m-1}{2}} i^2 \quad (3.4)$$

Let us take the terms in the form of  $b_i b_j$  where  $b_i = b_j = x$ , and  $1 \leq i < j \leq m$ , then we have

$$l_2 = \sum_{i=1}^m b_i b_j = c^2 x^2 \quad (3.5)$$

from (3.4) and (3.5) then  $l_1 < l_2$ .

Example 3.12. Let  $a_1(a_2 + a_3 + a_4 + a_5 + a_6) + a_2(a_3 + a_4 + a_5 + a_6) + a_3(a_4 + a_5 + a_6) + a_4(a_5 + a_6) + a_5(a_6) = l_1$ , and

$b_1(b_2 + b_3 + b_4 + b_5 + b_6) + b_2(b_3 + b_4 + b_5 + b_6) + b_3(b_4 + b_5 + b_6) + b_4(b_5 + b_6) + b_5(b_6) = l_2$ , where  $a_i, b_i$ , are integers, such

that  $\sum_{i=1}^6 a_i = \sum_{i=1}^6 b_i$ ,  $a_1 \leq a_2 \leq a_3 \leq \dots \leq a_6$ , and  $b_1 \leq b_2 \leq b_3 \leq \dots \leq b_6$ ,  $l_1$  have  $\binom{6}{2}$  terms, each can be written as  $a_i a_j$  where

$1 \leq a_i < a_j \leq 6$ , and  $l_2$  have  $\binom{6}{2}$  terms, each can be written as  $b_i b_j$  where  $1 \leq b_i < b_j \leq 6$ . Let  $a_1 = x-2$ ,  $a_2 = x-1$ ,  $a_3 = x$ ,

$a_4 = x+1$ ,  $a_5 = x+2$ ,  $a_6 = x+3$ ,  $b_1 = b_2 = b_3 = x$ ,  $b_4 = b_5 = b_6 = x+1$ , satisfying  $\sum_{i=1}^6 a_i = \sum_{i=1}^6 b_i$ ,  $l_1$  have  $\binom{6}{2} = 15$  terms of  $a_i a_j$ ,

also  $l_2$  have  $\binom{6}{2} = 15$  terms of  $b_i b_j$  (both  $a_i a_j$  and  $b_i b_j$  have the same subscript),

$$l_1 = a_1 a_2 + a_1 a_3 + a_1 a_4 + a_1 a_5 + a_1 a_6 + a_2 a_3 + a_2 a_4 + a_2 a_5 + a_2 a_6 + a_3 a_4 + a_3 a_5 + a_3 a_6 + a_4 a_5 + a_4 a_6 + a_5 a_6 = 15x^2 + 15x - 5,$$

$$l_2 = b_1 b_2 + b_1 b_3 + b_1 b_4 + b_1 b_5 + b_1 b_6 + b_2 b_3 + b_2 b_4 + b_2 b_5 + b_2 b_6 + b_3 b_4 + b_3 b_5 + b_3 b_6 + b_4 b_5 + b_4 b_6 + b_5 b_6 = 15x^2 + 15x + 3, \text{ then we have}$$

$$l_1 < l_2.$$

Lemma 3.13. Let  $t \sum_{i=1}^m a_i = \sum_{i=1}^m b_i$ ,  $a_1(a_2 + a_3 + a_4 + \dots + a_m) + a_2(a_3 + a_4 + a_5 + \dots + a_m) + a_3(a_4 + a_5 + a_6 + \dots + a_m) + \dots$

$+ a_{m-2}(a_{m-1} + a_m) + a_{m-1}(a_m) = l_1$ , and  $b_1(b_2 + b_3 + b_4 + \dots + b_m) + b_2(b_3 + b_4 + b_5 + \dots + b_m) + b_3(b_4 + b_5 + b_6 + \dots + b_m) + \dots$

$+ b_{m-2}(b_{m-1} + b_m) + b_{m-1}(b_m) = l_2$ , such that  $l_1$  have  $\binom{m}{2}$  terms, each can be written as  $a_i a_j$  where  $1 \leq i < j \leq m$ ,  $m$  is even, and

such that  $l_2$  have  $\binom{m}{2}$  terms, each can be written as  $b_i b_j$  where  $1 \leq i < j \leq m$ ,  $1 \leq i \leq m-1$ ,  $2 \leq j \leq m$ . Let  $a_i = x - \frac{m-2t}{2}$ , where

$1 \leq t \leq m$ , and  $b_1 = b_2 = b_3 = \dots = b_{\frac{m}{2}} = x$ ,  $b_{\frac{m}{2}+1} = b_{\frac{m}{2}+2} = b_{\frac{m}{2}+3} = \dots = b_m = x+1$ , then  $l_1 < l_2$ .

Proof: Let us take the terms in the form of  $a_i a_j$  as follows  $(x+c)(x-c-2)$ ,  $(x+d)(x-d)$ ,  $(x+c)(x+d)$ ,  $(x-c)(x-d)$ ,

$(x+c)(x-d)$ ,  $(x-c)(x+d)$ , where  $1 \leq c < d \leq \frac{m-2}{2}$ , then we have

$$(x+c)(x-c) + (x+d)(x-d) + (x+c)(x+d) + (x-c)(x-d) + (x+c)(x-d) + (x-c)(x+d) = 6x^2 - c^2 - d^2 \quad (3.6)$$

Let us take the terms  $(x+c)(x+\frac{m}{2})$ ,  $(x-c)(x+\frac{m}{2})$ ,  $(x+d)(x+\frac{m}{2})$ ,  $(x-d)(x+\frac{m}{2})$ , then we can write

$$(x+c)(x+\frac{m}{2}) + (x-c)(x+\frac{m}{2}) + (x+d)(x+\frac{m}{2}) + (x-d)(x+\frac{m}{2}) = 4x^2 + 2mx \quad (3.7)$$



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Let us take the terms  $x(x+c)$ ,  $x(x-c)$ , then we can write

$$x(x+c) + x(x-c) = 2x^2 \quad (3.8)$$

We have  $a_1 = x - \frac{m-2}{2}$ ,  $a_{\frac{m}{2}} = x$ ,  $a_m = x + \frac{m}{2}$ , from view of (3.6), (3.7) and (3.8) we can write

$$l_1 = \sum_{i=1}^m c x^2 + \frac{m}{2} (m-2)x - \sum_{i=1}^{\frac{m-2}{2}} i^2 \quad (3.9)$$

Let us take the terms in the form of  $b_i b_j$  such that  $b_i = b_j = x$ , where  $1 \leq i < j \leq \frac{m}{2}$ , then we have

$$\sum_{i=1}^{\frac{m-2}{2}} b_i b_j = \frac{m}{2} x^2 \quad (3.10)$$

Let us take the terms in the form of  $b_i b_j$  such that  $b_i = b_j = x+1$ , where  $\frac{m+2}{2} \leq i < j \leq m$ , then we have

$$\sum b_i b_j = \frac{m^2}{4} (x^2 + x) \quad (3.11)$$

Let us take the terms in the form of  $b_i b_j$  such that  $b_i = x$ , and  $b_j = x+1$ , where  $1 \leq i \leq \frac{m}{2}$ , and  $\frac{m+2}{2} \leq j \leq m$ , then we have

$$\sum b_i b_j = \frac{m^2}{4} (x^2 + x) \quad (3.12)$$

From view of (3.10), (3.11) and (3.12) we can write

$$l_2 = \sum_{i=1}^m c x^2 + (2 \frac{m}{2} + \frac{m^2}{4}) x + \frac{m}{2} \quad (3.13)$$

From (3.9) and (3.13) we have  $l_1 < l_2$ .

**Proposition 3.14.** If  $G$  a graph of  $k$  vertices, coloured by  $m$  colours, these  $m$  colour labeled by  $1, 2, 3, \dots, m$ . Let  $a_i$  and  $b_i$  any one of them is the number of vertices labeled by the colour class  $i$ , where  $1 \leq i \leq m$ , if  $\frac{k}{m} = a$  is integer,

$$a_1 = a_2 = a_3 = \dots = a_{m-1} = a_m = a \text{ and } a_1(a_2 + a_3 + a_4 + \dots + a_m) + a_2(a_3 + a_4 + a_5 + \dots + a_m) + a_3(a_4 + a_5 + a_6 + \dots + a_m) + \dots + a_{m-2}(a_{m-1} + a_m) + a_{m-1}(a_m) = l_1, \text{ and if } b_1 \leq b_2 \leq b_3 \leq \dots \leq b_{m-1} \leq b_m, b_1(b_2 + b_3 + b_4 + \dots + b_m) + b_2(b_3 + b_4 + b_5 + \dots + b_m) + b_3(b_4 + b_5 + b_6 + \dots + b_m) + \dots + b_{m-2}(b_{m-1} + b_m) + b_{m-1}(b_m) = l_2, \text{ then } l_2 \leq l_1.$$

**Proof:** The proof directly from Lemma 3.11.

**Proposition 3.15.** If  $G$  a graph of  $k$  vertices, coloured by  $m$  colours, these  $m$  colour labeled by  $1, 2, 3, \dots, m$ . Let  $a_i$  and  $b_i$  any one of them is the number of vertices labeled by the colour class  $i$ , where  $1 \leq i \leq m$ , if  $\frac{k}{m}$  is not integer,

$$a_1 \leq a_2 \leq a_3 \leq \dots \leq a_{m-1} \leq a_m, a_m - a_1 = 1, \text{ and } a_1(a_2 + a_3 + a_4 + \dots + a_m) + a_2(a_3 + a_4 + a_5 + \dots + a_m) + a_3(a_4 + a_5 + a_6 + \dots + a_m) + \dots + a_{m-2}(a_{m-1} + a_m) + a_{m-1}(a_m) = l_1, \text{ and if } b_1 \leq b_2 \leq b_3 \leq \dots \leq b_{m-1} \leq b_m, b_1(b_2 + b_3 + b_4 + \dots + b_m) + b_2(b_3 + b_4 + b_5 + \dots + b_m) + b_3(b_4 + b_5 + b_6 + \dots + b_m) + \dots + b_{m-2}(b_{m-1} + b_m) + b_{m-1}(b_m) = l_2, \text{ then } l_2 \leq l_1.$$

**Proof:** The proof directly from Lemma 3.13.

**Proposition 3.16.** Minimum number of edges to colour  $k$  vertices of the graph  $G$  with clique number  $n$ , by  $n$  colours equal

$$\frac{n(n-1)}{2} + (k-n).$$

**Proof:** Since edges of a clique with clique number  $n$  not less than  $\frac{n(n-1)}{2}$  edges, if the graph consist of a clique with

clique number  $n$  and  $k-n$  vertices each of degree zero, then minimum number for sum of vertices degree equal  $n(n-1)$ , if the graph consist of a clique with clique number  $n$  and  $k-n$  vertices each of degree one and connected to a vertex of

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the clique, then minimum number for sum of vertices degree equal  $n(n-1) + 2(k-n)$ , from

Theorem 2.1. therefore minimum number of edges to colour  $k$  vertices of the graph by  $n$  colours equal  $\frac{n(n-1)}{2} + (k-n)$ .

Remark 3.17. If the graph  $G$  of  $k$  vertices, these vertices coloured by  $m$  colours, then  $G$  is called complete graph of  $k$  vertices with respect of  $m$  colours, as abbreviation  $G$  is called complete graph with respect to colours. We use the symbol  $K_{k,m}$ .

## IV. SOME RESULTS DEPEND ON CLIQUE NUMBER

Using colouring of families of disjoint sets technique, to colour vertices of the graph depend on the clique number. In this section we prove some results of vertex's colouring (trivial colouring and nontrivial colouring) depend on clique number.

Remark 4.1. Suppose we have a clique of mutually adjacent  $n$  vertices  $v_1, v_2, v_3, \dots, v_n$ , these vertices labeled by  $1, 2, 3, \dots, n$  respectively, where each number represent colour class, coloured the vertex. We label the clique by  $(1, 2, 3, \dots, n)$ , same as in edge colouring, (if the degree of vertex  $v$  equal  $n$ , and these edges incident with the vertex  $v$  labeled by  $1, 2, 3, \dots, n$ , we labeled the vertex  $v$  by  $(1, 2, 3, \dots, n)$ ).

Remark 4.2. Colouring of families of disjoint sets technique depend on the number  $n$ , generally in case of sets  $n$  is degree of intersection or  $n$  is the degree of set labeled by families of disjoint sets, for colouring edges of the graph  $n$  is the largest degree of vertex (maximum number of edges such that any two are adjacent). If minimum number of colours  $m$  equal  $n$  it is trivial case, if  $m = n + 1$  it is nontrivial. For colouring vertices of the graph  $n$  is the clique number (maximum number of vertices such that any two are adjacent). In Example 3.9. we have 14 choices, in each we coloured the graph by 13 colours, all these 14 choices are trivial colouring. Suppose we have a graph consist of three cliques  $C_1, C_2, C_3$ , each with clique number  $n$ , minimum number of colours to colour vertices of this graph either equal  $n$ , (trivial case) or equal  $n + 1$ , (nontrivial case).

Proposition 4.3. Suppose we have a graph consist of two cliques  $C_1, C_2$ , each with clique number  $n$ , and the two cliques have  $n - 1$  common vertices, also there is an edge connects two nonadjacent vertices, then number of colours to colour these vertices is nontrivial colouring.

Proof: Suppose number of colours to colour these vertices equal  $n$ , and suppose the  $n - 1$  common vertices labeled by  $1, 2, 3, \dots, n - 1$ , let the vertex different from  $n - 1$  common vertices in the first clique  $C_1$  labeled by  $n$ , and let the vertex different from  $n - 1$  common vertices in the second clique  $C_2$  labeled by  $n$ , the edge connects two nonadjacent vertices, connects two vertices both labeled by  $n$ , a contradiction, we can not connect two vertices have same colour. Therefore number of colours to colour these vertices equal  $n + 1$ , where first clique labeled by  $(1, 2, 3, \dots, n - 1, n)$ , second clique labeled by  $(1, 2, 3, \dots, n - 1, n + 1)$ , the third clique (the edge connects the two vertices) labeled by  $(n, n + 1)$ , and the  $n - 1$  common vertices labeled by  $(1, 2, 3, \dots, n - 1)$ , each colour class appear twice at these three cliques.

Proposition 4.4. Suppose we have a graph consist of two cliques  $C_1, C_2$ , each with clique number  $n$ , and the two cliques have  $n - 2$  common vertices, also there is an edge connects two nonadjacent vertices, then number of colours to colour these vertices is trivial colouring.

Proof: Suppose number of colours to colour these vertices equal  $n$ , and suppose the  $n - 2$  common vertices labeled by  $1, 2, 3, \dots, n - 2$ , therefore the two vertices different from  $n - 2$  common vertices in the first clique  $C_1$  labeled by  $n - 1, n$ , and the two vertices different from  $n - 2$  common vertices in the second clique  $C_2$  also labeled by  $n - 1, n$ , we connect the vertex labeled by  $n - 1$  in the first clique  $C_1$  with the vertex labeled by  $n$  in the second clique  $C_2$ , therefore number of colours to colour these vertices equal  $n$ , where first clique  $C_1$  labeled by  $(1, 2, 3, \dots, n - 1, n)$ , second clique  $C_2$  labeled by  $(1, 2, 3, \dots, n - 1, n)$ , the third clique (the edge connects the two vertices) labeled by  $(n - 1, n)$ .



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Example 4.5. If we have a graph  $G$  consist of three cliques  $C_1, C_2, C_3$ , each with clique number  $n = 4$ , such that any two cliques has one common point, number of vertices in this graph equal  $n + (n - 1) + (n - 2) = 3n - 3$ , in case of  $n = 4$ , number of vertices in this graph equal  $3n - 3 = 9$ , using Theorem 2.9.  $m(\frac{w+1}{2}) \leq wn$  where  $w = 3$ ,  $wn = 3n - 3 = 9$ ,  $m = 5$ , we have  $5(\frac{3+1}{2}) \leq 9$ , a contradiction, number of colours can not be five, it is four, therefore colouring is trivial colouring.

Remark 4.6. We have directly from definition of clique the following:

- (i) If both  $C_1$  and  $C_2$  are cliques each with clique number 3, and both labeled by (1,2,3) where 1,2,3 labels of colour classes,  $C_1$  clique of vertices  $v_1, v_2, v_3$ , and  $C_2$  clique of vertices  $v_3, v_4, v_5$ ,  $v_3$  is common vertex of  $C_1$  and  $C_2$ . The vertices  $v_1$  labeled by 1,  $v_2$  labeled by 2,  $v_3$  labeled by 3,  $v_4$  labeled by 1,  $v_5$  labeled by 1. For the labels (1,2,3) and (1,2,3) the label 3 in the first and the label 3 in the second represent common vertex  $v_3$ , the label 1 in the first and the label 1 in the second represent non adjacent vertices  $v_1$  and  $v_4$ , the label 2 in the first and the label 2 in the second represent non adjacent vertices  $v_2$  and  $v_5$ .
- (ii) If the two cliques  $C_1$  and  $C_2$  have many common vertices, for non common vertices, any two vertices not in same clique, are non adjacent vertices.
- (iii) If the two cliques  $C_1$  and  $C_2$  have not any common vertex, and connected by an edge  $v_i v_j$ ,  $v_i$  in  $C_1$  and  $v_j$  in  $C_2$ , the vertices  $v_i$  and  $v_j$  are only adjacent vertices and not in same clique.
- (iv) If the two cliques  $C_1$  and  $C_2$  have not common vertex, and not connected by any edge, each clique with clique number  $n$ , and each clique labeled by labels (1,2,3,...,n), then any two vertices  $v_i$  and  $v_j$  not in same clique, for all  $i = j$  and for all  $i \neq j$ , vertices  $v_i$  and  $v_j$  are not adjacent.
- (v) If the two cliques  $C_1$  and  $C_2$  have one common vertex, and there are edges join pairs of vertices, let the number of vertices equal  $k$ , and the number of colours to colour these  $k$  vertices equal  $n$ , then  $\sum_{i=1}^n a_i = k$ . For all  $i$ , we have  $1 \leq a_i \leq 2$ , and if  $a_i = 3$ , is a contradiction because there are no three vertices every two of them non adjacent, since the three vertices either in one clique or in the two cliques. Since we have one common vertex, there is one vertex of degree equal  $k - 1$ , therefore exists  $i$  such that  $a_i = 1$ .

- (vi) If the three cliques  $C_1, C_2, C_3$  have one common vertex, and there are edges join pairs of vertices, let the number of vertices equal  $k$ , and the number of colours to colour these  $k$  vertices equal  $n$ , then  $\sum_{i=1}^n a_i = k$ . For all  $i$ , we have  $1 \leq a_i \leq 3$ , and if  $a_i = 4$ , is a contradiction because there are no four vertices every two of them non adjacent, since the four vertices either in one clique or in two cliques or in the three cliques.

Proposition 4.7. If  $G$  is a graph consists of one clique with clique number  $n$  if we connect this clique to one vertex of degree two, then number of colours to colour these  $n + 1$  vertices equal  $n$ .

Proof: From addition of one vertex of degree two, we have three cliques first clique with clique number equal  $n$ , the two other cliques, each of clique number equal 2. Using Theorem 2.9.  $w$  is number of cliques,  $w = 3$ , suppose  $m = n + 1$  is number of colours to colour  $n + 1$  vertices, using  $m(\frac{w+1}{2}) \leq wn$  we have  $2(n + 1) \leq n + 2$ , which is a contradiction, therefore the number of colours to colour  $n + 1$  vertices equal  $n$ .

Definition 4.8. If  $v_1, v_2, v_3, \dots, v_{k-1}$  are vertices labeled by 1,2,3,... where 1,2,3,... are labels of colour classes, such that labels of  $v_i$  is not identical to the labels of  $v_{i+1}$ , where  $1 \leq i \leq k - 2$ ,  $k \geq 3$ , and each of  $C_1, C_2, C_3, \dots, C_k$  is mutually adjacent  $l$  vertices  $v_1, v_2, v_3, \dots, v_l$ , where  $2 \leq l \leq n$ , if  $C_i$  labeled by  $n$  labels,  $C_i$  is a clique, both vertices  $v_{i-1}$  and  $v_i$  are in  $C_i$ , then

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$C_1, v_1, C_2, v_2, C_3, v_3, \dots, C_{k-1}, v_{k-1}, C_k$  is called a sequence of cliques and vertices. Definition 4.9. A walk of cliques and vertices is an alternating sequence  $v_0, C_1, v_1, C_2, v_2, C_3, v_3, \dots, C_{k-1}, v_{k-1}, C_k, v_k$  of cliques and vertices where  $v_1, v_2, v_3, \dots, v_k$  are vertices such that  $v_i \neq v_{i+1}$ ,  $v_k \neq v_1$ , and each of  $C_1, C_2, C_3, \dots, C_{k-1}, C_k$  is mutually adjacent  $l$  vertices  $v_1, v_2, v_3, \dots, v_l$ , where  $2 \leq l \leq n$ , if  $C_i$  labeled by  $n$  labels,  $C_i$  is a clique, then  $v_0, C_1, v_1, C_2, v_2, C_3, v_3, \dots, C_{k-1}, v_{k-1}, C_k, v_k$  is called a cycle of  $k$  cliques and vertices if  $v_0 = v_k$  and  $k \geq 3$ . The walk and cycle of cliques and vertices can also denoted by  $C_1, C_2, C_3, \dots, C_{k-1}, C_k$ .

Example 4.10. Let  $C_1$  and  $C_2$ , be two cliques with clique number equal 3, and they have one common vertex, if we add two edges, each one join pair of vertices not in same clique, then the number of colours to colour vertices of this graph equal trivial colouring.

Solution: The graph consists of five vertices, six edges and two cliques with clique number equal 3, we labeled the common vertex by label 1, we label the two vertices (except the common vertex) in same clique by the label 2, and by the label 3, we have three colours from Remark 4.6.  $a_1 = 1, a_2 = 1, a_3 = 2$ , since  $l = a_1(a_2 + a_3) + a_2(a_3) = 8$ , where  $l$  is maximum number of edges to colour five vertices by three colours. If we add two edges to the graph, the edges of the graph become eight, therefore the number of colours to colour vertices of this graph equal trivial colouring.

Example 4.11. Let  $C_1$  and  $C_2$ , be two cliques with clique number equal 3, and they have one common vertex, if we add four edges, each one join pair of vertices not in same clique, then the number of colours to colour vertices of this graph equal non trivial colouring.

Solution: The graph consists of five vertices, six edges and two cliques with clique number equal 3, we labeled the common vertex by label 1, we label the two vertices (except the common vertex) in same clique by the label 2, and by the label 3, we have three colours from Remark 4.6.  $a_1 = 1, a_2 = 1, a_3 = 2$ , since  $l = a_1(a_2 + a_3) + a_2(a_3) = 8$ , where  $l$  is maximum number of edges to colour five vertices by three colours. If we add four edges to the graph, the edges of the graph become ten, and maximum number of edges to colour five vertices by four colours equal  $l = a_1(a_2 + a_3 + a_4) + a_2(a_3 + a_4) + a_3(a_4) = 9$ , therefore the number of colours to colour vertices of this graph equal 5, the graph become one clique with clique number equal 5.

Example 4.12. Let  $C_1, C_2, C_3$ , be three cliques with clique number equal 3, and they have one common vertex, if we add seven edges, each edge join pair of vertices not in same clique, then the number of colours to colour vertices of this graph equal 4.

Solution: The graph consists of seven vertices, nine edges and three cliques with clique number equal 3, we labeled the common vertex by label 1, we label the two vertices (except the common vertex) in same clique by the label 2, and by the label 3, we have three colours from Remark 4.6.  $a_1 = 1, a_2 = 3, a_3 = 3$ , since  $l = a_1(a_2 + a_3) + a_2(a_3) = 15$ , where  $l$  is maximum number of edges to colour seven vertices by three colours. If we add seven edges to the graph, the edges of the graph become sixteen, therefore the number of colours to colour vertices of this graph equal 4.

Proposition 4.13. If  $G$  is a graph of walk of  $k$  cliques and vertices  $C_1, v_1, C_2, v_2, C_3, v_3, \dots, C_{k-1}, v_{k-1}, C_k, v_k$ , each clique with clique number equal  $n$ , then the number of colours to colour vertices of this graph equal  $n$ .

Proof: If  $G$  is a graph of  $k$  cliques, and  $C_1, v_1, C_2, v_2, C_3, v_3, \dots, C_{k-1}, v_{k-1}, C_k, v_k$  is a walk of  $k$  cliques and vertices, from definition of walk,  $v_1$  in  $C_1$ , and  $v_1$  in  $C_2$ , also  $v_2$  in  $C_2$ , and  $v_2$  in  $C_3$ ,  $v_3$  in  $C_3$ , and  $v_3$  in  $C_4$ ,  $v_k$  in  $C_k$ , both  $v_1$  and  $v_2$  in  $C_2$ , from definition of clique  $v_1$  and  $v_2$  are adjacent,  $v_2$  and  $v_3$  are adjacent, so any two successive vertices of the walk  $v_1, v_2, v_3, \dots, v_{k-1}, v_k$  are adjacent, for any  $k$  odd or even we coloured  $v_i$  by  $n$  and  $v_{i+1}$  by  $n-1$ , where  $i$  is odd  $1 \leq i \leq k$ , and other vertices in any clique coloured by  $1, 2, 3, \dots, n-2$ , so any two successive vertices of the cycle  $v_1, v_2, v_3, \dots, v_{k-1}, v_k$  have two different colours, therefore the number of colours to colour vertices of this graph equal  $n$ .

Example 4.14. Let  $G$  be a graph of 6 vertices, with clique number equal 4, two vertices of degree 4, two vertices of degree 3, and two vertices of degree 1, find minimum number of edges to colour 6 vertices of the graph by minimum number of colours.

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Solution: This graph satisfies condition of Proposition 4.13. therefore colouring of vertices of this graph is trivial, since the clique number equal 4, we apply Proposition 3.16. therefore minimum number of edges to colour the 6 vertices of the graph by 4 colours equal

$$\frac{4(3)}{2} + 2 = 8,$$

Corollary 4.15. If the graph  $G$  consists of one clique (with clique number  $n$ ) and  $r$  edges, each edge connected to one vertex of the clique and the other vertex of degree one, the minimum number of colour to colour vertices of the graph  $G$  equal  $n$ .

Proof: Let  $G$  be a graph of  $n + r$  vertices with clique number  $n$ , from Proposition 3.16. therefore minimum number of colour to colour vertices of the graph  $G$  equal  $n$ .

## V. POSSIBLE WAYS TO COLOUR VERTICES

In this section we prove a result about number of all possible ways to colour  $k$  vertices by  $m$  colours

Theorem 5.1. A graph  $G$  of  $k$  vertices by  $m$  colours there are  $X$  ways to colour these vertices by  $m$  colours, where  $X$  equal partition of  $k - m + 1$  such that the partition of  $k - m + 1$  can not consists of more than  $m$  integers.

Proof: To colour  $k$  vertices by  $m$  colours, we label these  $m$  colours by labels  $1, 2, 3, \dots, m$ , and  $a_i$  is number of vertices labeled by  $i$ ,  $a_i$  is number of vertices labeled by  $i$ , we exchange colour classes labeled by  $1, 2, 3, \dots, m$  such that  $a_1 \leq a_2 \leq a_3 \leq \dots \leq a_m$ , from Proposition

3.5. we have  $\sum_{i=1}^m a_i = k$ , and  $1 \leq a_1 \leq a_2 \leq a_3 \leq \dots \leq a_{m-1} \leq a_m \leq k - m + 1$ . Let the integers  $b_1, b_2, b_3, \dots, b_{m-1}, b_m$  defined as

$b_1 + b_2 + b_3 + \dots + b_{m-1} + b_m = k - m$ , and  $0 \leq b_1 \leq b_2 \leq b_3 \leq \dots \leq b_{m-1} \leq b_m \leq k - m$ , from definition of these integers,

$b_1, b_2, b_3, \dots, b_{m-1}, b_m$  can be a partition of the integer  $k - m$ , such that the partition of  $k - m$  can not consists of more than  $m$  integers. Let  $f: B \rightarrow A$  defined as  $f(b_1, b_2, b_3, \dots, b_{m-1}, b_m) = (a_1, a_2, a_3, \dots, a_m) = (b_1 + 1, b_2 + 1, b_3 + 1, \dots, b_m + 1)$ , where

$B = \{(b_1, b_2, b_3, \dots, b_{m-1}, b_m) : b_1 + b_2 + b_3 + \dots + b_m = k - m, 0 \leq b_1 \leq b_2 \leq b_3 \leq \dots \leq b_m \leq k - m \text{ and}$

$A = \{(a_1, a_2, a_3, \dots, a_m) : 1 \leq a_1 \leq a_2 \leq a_3 \leq \dots \leq a_m \leq k - m + 1, a_1 + a_2 + a_3 + \dots + a_m = k\}$ . Let

$R = \{f(b_1, b_2, b_3, \dots, b_{m-1}, b_m) : (b_1, b_2, b_3, \dots, b_m) \in B\}$ . Therefore from definition of the set  $B$  and definition of the set  $R$ , any element of  $R$ , can be a partition of the integer  $k - m + 1$ , such that the partition of  $k - m + 1$  can not consists of more than  $m$  integers.

Example 5.2. Let  $G$  be a graph with  $k$  vertices, its vertices coloured by  $m$  colours, here we find possible ways to colour these vertices by  $m$  colours. Table (II) explain how this example satisfies Theorem 5.1.

Number of vertices $k$	Number of colours $m$	Possible colouring $(a_1, a_2, a_3, \dots, a_m)$	Number possible colouring
7	3	(1,1,5) (1,2,4) (1,3,3) (2,2,3)	4
7	4	(1,1,1,4) (1,1,2,3) (1,2,2,2)	3
8	5	(1,1,1,1,4) (1,1,1,2,3) (1,1,2,2,2)	3
9	3	(1,1,7) (1,2,6) (1,3,5) (2,2,5) (1,4,4) (2,3,4)	6
9	4	(1,1,1,6) (1,1,2,5) (1,1,3,4) (1,2,2,4) (1,2,3,3) (2,2,2,3)	6
9	5	(1,1,1,1,5) (1,1,1,2,4) (1,1,1,3,3) (1,1,2,2,3) (1,2,2,2,2)	5

Table (II)

## VI. RANGE OF EDGES IN WHICH WE CAN COLOUR VERTICES

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In this section we prove a result about the range of edges in which we can not colour  $k$  vertices by  $m$  colours. By this result we can determinethe range of edges in which we can not colour a huge number of vertices by a huge number of colours.

**Theorem 6.1.** Let  $G$  be a graph, the range of edges in which we can colour  $k$  vertices by  $m$  colours is number of edges not less than  $\frac{m(m-1)}{2} + (k-m)$ , and not greater than  $a_1(a_2 + a_3 + a_4 + \dots + a_m) + a_2(a_3 + a_4 + a_5 + \dots + a_m) + a_3(a_4 + a_5 + a_6 + \dots + a_m) + \dots + a_{m-2}(a_{m-1} + a_m) + a_{m-1}(a_m) = l$ , where  $\frac{k}{m} = a$  is an integer, and  $a_1 = a_2 = a_3 = \dots = a_{m-1} = a_m = a$ , or  $a_1 \leq a_2 \leq a_3 \leq \dots \leq a_{m-1} \leq a_m$ , such that  $a_m - a_1 = 1$  if  $\frac{k}{m} = a$  is not an integer.

**Proof:** Using Proposition 3.16. minimum number of edges to colour  $k$  vertices of any graph by  $m$  number of colours equal  $\frac{m(m-1)}{2} + (k-m)$ . Using Proposition 3.15. the maximum number of edges to colour  $k$  vertices of the graph  $G$  by  $m$  colours equal  $a_1(a_2 + a_3 + a_4 + \dots + a_m) + a_2(a_3 + a_4 + a_5 + \dots + a_m) + a_3(a_4 + a_5 + a_6 + \dots + a_m) + \dots + a_{m-2}(a_{m-1} + a_m) + a_{m-1}(a_m) = l$ , where  $\frac{k}{m} = a$  is integer, and  $a_1 = a_2 = a_3 = \dots = a_{m-1} = a_m = a$ , or  $a_1 \leq a_2 \leq a_3 \leq \dots \leq a_{m-1} \leq a_m$ , such that  $a_m - a_1 = 1$ , if  $\frac{k}{m} = a$  is not an integer. Hence the theorem holds.

**Example 6.2.** Find range of edges in which we can colour a graph of 7 vertices by 4 colours.

**Solution:** From Proposition 3.16. minimum number of edges to colour 7 vertices of the graph by 4 colours equal  $\frac{4(4-1)}{2} + (7-4) = 9$ . From Proposition 3.15. maximum number of edges to colour 7 vertices of any graph by 4 colours equal  $a_1(a_2 + a_3 + a_4) + a_2(a_3 + a_4) + a_3(a_4) = 18$ , where  $a_1 = 1, a_2 = 2, a_3 = 2, a_4 = 2$ . Therefore maximum number of edges to colour 7 vertices of any graph by 4 colours equal 18. The range of edges in which we can colour a graph of 7 vertices by 4 colours is, from 9 edges to 18 edges.

**Example 6.3.** Find range of edges in which we can colour a graph of 1000 vertices by 25 colours.

**Solution:** From Proposition 3.16. minimum number of edges to colour 1000 vertices of the graph by 25 colours equal  $\frac{25(25-1)}{2} + (1000-25) = 675$ . From Proposition 3.14. maximum number of edges to colour 1000 vertices of any graph by 25 colours equal  $a_1(a_2 + a_3 + a_4 + \dots + a_{25}) + a_2(a_3 + a_4 + a_5 + \dots + a_{25}) + a_3(a_4 + a_5 + a_6 + \dots + a_{25}) + \dots + a_{24}(a_{25}) = l = 480000$ , where  $a_1 = a_2 = a_3 = \dots = a_{25} = 40$ . Therefore maximum number of edges to colour 1000 vertices of any graph by 25 colours equal 480000. The range of edges in which we can colour a graph of 1000 vertices by 25 colours is, from 675 edges to 480000 edges.

**Example 6.4.** We have two graphs  $G_1$  of 7 vertices and 11 edges coloured by 4 colours, and also  $G_2$  of 7 vertices and 11 edges coloured by 4 colours.  $G_1$  consists of 4 cliques each with clique number equal 3. First clique and second clique have two common vertices (common edge), second clique and third clique have one common vertex, third clique and fourth clique have two common vertices (common edge), fourth clique and third clique connected by an edge. In  $G_1$  we have 6 vertices each of degree equal 3, and one vertex of degree equal 4.  $G_2$  has one clique with clique number equal 4. There are 3 vertices each connected by an edge to the clique. In  $G_2$  we have 2 vertices each of degree equal 2, and 2 vertices each of degree equal 3, and 3 vertices each of degree equal 4.  $G_1$  is a graph with clique number equal 3, coloured by 4 colours, colouring is nontrivial.  $G_2$  is a graph with clique number equal 4, coloured by 4 colours, colouring is trivial.

**Remark 6.5.** For two graphs with same number of vertices and same number of edges, both coloured by  $m$  colours, colouring may be trivial colouring or nontrivial colouring. This means the range of edges in which we can colour  $k$  vertices by  $m$  colours, may be trivial colouring or nontrivial colouring, as we have been seen in Example 6.4.

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