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Analytical Study of D'Alembert's Solution of the Wave Equation

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ABSTRACT : In this paper we study that there is another way to solve the wave equation, found by Jean Le Rond D'Alembert. The solution to the wave equation using the separation of variables technique, we also made use of the boundary conditions in finding the solutions to the partial differential equation. The functions of several variables are used to work through the concept of a partial derivatives. The solutions of wave equations will involve functions not just of single variable, but of several variables. It is easier and more instructive to derive this solution by making an appropriate change of variables to obtain an equation that can be solved by simple integration. We conclude that the solution of these partial differential equations are second order, and are linear.

KEYWORDS : Wave Equations, Transformation, Several Variables, Boundary Conditions, Initial Velocity Function.

I. INTRODUCTION

The wave equation is the second-order linear partial differential equation for the description of waves as they are in classical physics such as mechanical waves, light waves etc. There is important method to solve the wave equation, found by Jean Le Rond D'Alembert in the 18th century. There are various approaches to solve the wave equation. The first one we will work with, using a technique called separation of variables, again, demonstrates one of the most widely used solution techniques for partial differential equations. The idea behind it is to split up the original partial differential equations into a series of simpler ordinary differential equations, each of which we should be able to solve readily using tricks already learned [1]. The second technique, uses a transformation trick that also reduces the complexity of the original partial differential equations, but in a very different manner. We worked over the issue of the initial conditions, until the very end when we claimed that one could make use of something called Fourier Series to build up combinations of solutions.

II. RELATED WORK

2.1 D'Alembert's Solution of the Wave Equation is glossed over the issue of the initial conditions, until the very end when we claimed that one could make use of something called Fourier Series to build up combinations of solutions [2].

We give specific initial conditions meant being given both the shape of the string at time $t = 0$, i.e. the function $u(x,0) = f(x)$, and also the initial velocity, $u_t(x,0) = g(x)$ note that these two initial condition functions are functions of single variable x , as another variable t is set equal to 0 [8]. In the separation of variables solution, we ended up with an infinite set, or family, of solutions, $u_n(x,t)$ that we said could be combined in such a way as to satisfy any reasonable initial conditions.

III. METHODOLOGY

3.1 Special Transformation of Variables : The technique involves changing the original partial differential equations into the equation that can be solved by a series of two simple single variable integrations by using a special transformation of variables [4]. Suppose first that instead of thinking that the original partial differential equations

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (3.1, 1)$$

in terms of the two variables x , and t , we rewrite this equation to reflect two new variables

$$v = x + ct \text{ and } z = x - ct \quad (3.1, 2)$$

This then means that the function u , originally a function of x , and t , now becomes a function of v and z , instead. We see that how does this work? Note that we can solve for x and t in (3.1, 2), we get that

$$x = \frac{1}{2}(v + z) \quad \text{and} \quad t = \frac{1}{2c}(v - z) \quad (3.1, 3)$$

By using the chain rule for multivariable functions, you know that

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial v} \frac{\partial v}{\partial t} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial t} = c \frac{\partial u}{\partial v} - c \frac{\partial u}{\partial z} \quad (3.1, 4)$$

since $\frac{\partial v}{\partial t} = c$ and $\frac{\partial z}{\partial t} = -c$, and that similarly

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial x} = \frac{\partial u}{\partial v} + \frac{\partial u}{\partial z} \quad (3.1, 5)$$

since $\frac{\partial v}{\partial x} = 1$ and $\frac{\partial z}{\partial x} = 1$. [1, 6] Working up to second derivatives, another, more involved application of the chain rule yields that

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial t} \left(c \frac{\partial u}{\partial v} - c \frac{\partial u}{\partial z} \right) = c \left(\frac{\partial^2 u}{\partial v^2} \frac{\partial v}{\partial t} + \frac{\partial^2 u}{\partial z \partial v} \frac{\partial z}{\partial t} \right) - c \left(\frac{\partial^2 u}{\partial z^2} \frac{\partial z}{\partial t} + \frac{\partial^2 u}{\partial v \partial z} \frac{\partial v}{\partial t} \right) \\ &= c^2 \left(\frac{\partial^2 u}{\partial v^2} - \frac{\partial^2 u}{\partial z \partial v} \right) + c^2 \left(\frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 u}{\partial v \partial z} \right) = c^2 \left(\frac{\partial^2 u}{\partial v^2} - 2 \frac{\partial^2 u}{\partial z \partial v} + \frac{\partial^2 u}{\partial z^2} \right) \end{aligned} \quad (3.1, 6)$$

Similarly we use the chain rule to get results in the fact that

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial v} + \frac{\partial u}{\partial z} \right) = \left(\frac{\partial^2 u}{\partial v^2} \frac{\partial v}{\partial x} + \frac{\partial^2 u}{\partial z \partial v} \frac{\partial z}{\partial x} \right) + \left(\frac{\partial^2 u}{\partial z^2} \frac{\partial z}{\partial x} + \frac{\partial^2 u}{\partial v \partial z} \frac{\partial v}{\partial x} \right) \\ &= \frac{\partial^2 u}{\partial v^2} + 2 \frac{\partial^2 u}{\partial z \partial v} + \frac{\partial^2 u}{\partial z^2} \end{aligned} \quad (3.1, 7)$$

Again, we revisit the original wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (3.1, 8)$$

and substitute in what we have calculated for second partial derivatives $\frac{\partial^2 u}{\partial t^2}$ and $\frac{\partial^2 u}{\partial x^2}$ in terms of $\frac{\partial^2 u}{\partial v^2}$, $\frac{\partial^2 u}{\partial z^2}$ and

$\frac{\partial^2 u}{\partial z \partial v}$. Doing this, yields the following equation, ripe with cancellations:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial v^2} - 2 \frac{\partial^2 u}{\partial z \partial v} + \frac{\partial^2 u}{\partial z^2} \right) = c^2 \frac{\partial^2 u}{\partial x^2} = c^2 \left(\frac{\partial^2 u}{\partial v^2} + 2 \frac{\partial^2 u}{\partial z \partial v} + \frac{\partial^2 u}{\partial z^2} \right) \quad (3.1, 9)$$

Dividing by c^2 and canceling the terms involving $\frac{\partial^2 u}{\partial v^2}$ and $\frac{\partial^2 u}{\partial z^2}$ reduces the series of equations to

$$-2 \frac{\partial^2 u}{\partial z \partial v} = +2 \frac{\partial^2 u}{\partial z \partial v} \quad (3.1, 10)$$

which means that

$$\frac{\partial^2 u}{\partial z \partial v} = 0 \quad (3.1, 11)$$

So what, we ask, after all, we still have a second order partial differential equation, and there are still several variables involved [3-5]. But wait, think about what equation ((3.1, 11)) implies. Picture (3.1, 11) as it gives you information about the partial derivative of a partial derivative:

$$\frac{\partial}{\partial z} \left(\frac{\partial u}{\partial v} \right) = 0 \quad (3.1, 12)$$

In this form, this implies that $\frac{\partial u}{\partial v}$ considered as a function of z and v is a constant in terms of the variable z , so

that $\frac{\partial u}{\partial v}$ can only depend on v , i.e.

$$\frac{\partial u}{\partial v} = M(v) \quad (3.1, 13)$$

We integrate this equation with respect to v which yields that

$$u(v, z) = \int M(v) dv \quad (3.1, 14)$$

This equation is as an indefinite integral, which results in a constant of integration, in this case is just constant from the standpoint of the variable v . Thus, it can be any arbitrary function of single variable z , so that actually we get

$$u(v, z) = \int M(v) dv + N(z) = P(v) + N(z) \quad (3.1, 15)$$

where $P(v)$ is the function of single variable v , and the function $N(z)$ is of z alone, as the notation indicates.

Substituting back the original change of variable equations for v and z in (3.1, 2) yields that

$$u(x, t) = P(x + ct) + N(x - ct) \quad (3.1, 16)$$

where the function P and N are arbitrary single variable functions. Which is called D'Alembert's solution to the wave equation. Except for the somewhat annoying but easy enough to use chain rule computations, it was a straight forward solution technique. The reason it worked so well in this case was the fact that the change of variables used in (3.1, 2) were carefully selected so as to turn the original partial differential equation into one in which the variables basically had no interaction, so that the original second order partial differential equation could be solved by a series of integrations, which was easy to integrate [7, 9].

These solutions, you can take any functions for P and N such as $P(v) = v^2$ & $N(v) = v + 2$. Then

$$u(x, t) = (x + ct)^2 + (x - ct) + 2 = x^2 + x + ct + c^2 t^2 + 2 \quad (3.1, 17)$$

$$\frac{\partial^2 u}{\partial t^2} = 2c^2 \quad (3.1, 18)$$

and that

$$\frac{\partial^2 u}{\partial x^2} = 2 \quad (3.1, 19)$$

so that indeed

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (3.1, 20)$$

and therefore this is in fact the solution of original wave equation.

This similar transformation trick can be used for solving the fairly wide range of partial differential equations. For instance one can solve the equation

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y^2} \quad (3.1, 21)$$

by using the transformation of variables

$$v = x \text{ and } z = x + y \quad (3.1, 22)$$

Note that in our solution ((3.1, 16) to the wave equation, nothing has been specified about the initial and boundary conditions yet, and we said we would take care of this time around [2, 10]. Now we take a look at these conditions imply for our choices for these two functions P and N .

3.2 Velocity Function : If we were given an initial function $u(x,0) = f(x)$ along with initial velocity function $u_t(x,0) = g(x)$ then we can match up these conditions with our solution by simply substituting in $t=0$ into ((3.1, 16) and follow along. We start first with a simplified set-up, where we assume that we are given the initial displacement function $u(x,0) = f(x)$, and that the initial velocity function $g(x)$ is equal to 0. If somebody stretched the string and simply released it without imparting any extra velocity over this string tension alone).

Now the first initial condition implies that

$$u(x,0) = P(x+c \cdot 0) + N(x-c \cdot 0) = P(x) + N(x) = f(x) \quad (3.2, 1)$$

We next figure out what choosing the second initial condition implies[11]. By working with an initial condition that $u_t(x,0) = g(x) = 0$, we see that by using the chain rule again on the functions P and N

$$u_t(x,0) = \frac{\partial}{\partial t} (P(x+ct) + N(x-ct)) = cP'(x+ct) - cN'(x-ct) \quad (3.2, 2)$$

remember that these two function P and N are just single variable functions, so the derivative indicated is just a simple single variable derivative with respect to their input. Thus in the case where $u_t(x,0) = g(x) = 0$, then

$$cP'(x+ct) - cN'(x-ct) = 0 \quad (3.2, 3)$$

substituting in $t=0$ and dividing out by the the constant factor c, we get

$$P'(x) = N'(x) \quad (3.2, 4)$$

and so $P(x)+k = N(x)$ for some constant k. Combining this with the fact that $P(x)+N(x) = f(x)$, means that $2P(x)+k = f(x)$, so that $P(x) = (f(x)-k)/2$ and likewise $N(x) = (f(x)+k)/2$. Combining these leads to the solution

$$u(x,t) = P(x+ct) + N(x-ct) = \frac{1}{2} (f(x+ct) + f(x-ct)) \quad (3.2, 5)$$

IV. RESULTS AND DISCUSSION

4.1 We apply the boundary conditions to meet the solution, we have

$$u(0,t) = 0 \text{ and } u(l,t) = 0 \text{ for every values of } t \quad (4.1, 1)$$

This first boundary condition implies that

$$u(0,t) = \frac{1}{2} (f(ct) + f(-ct)) = 0 \quad (4.1, 2)$$

or

$$f(-ct) = -f(ct) \quad (4.1, 3)$$

so that to meet the above condition, then the initial condition function f must be selection of an odd function.

The second boundary condition that $u(l,t) = 0$ implies that

$$u(l,t) = \frac{1}{2} (f(l+ct) + f(l-ct)) = 0 \quad (4.1, 4)$$

so the function $f(l+ct) = -f(l-ct)$. Next, since we have seen that f has to be an odd function, then we have $-f(l-ct) = f(-l+ct)$. Putting this all together, it means that

$$f(l+ct) = f(-l+ct) \text{ for every values of } t \quad (4.1, 5)$$

which means that the function f must have period $2l$, since the inputs vary by that amount[8]. Remember that this means that the function repeats itself every time $2l$ is added to the input, the same way that the sine function and cosine function have period 2π . What happens if the initial velocity isn't equal to 0? Thus suppose $u_t(x,0) = g(x) \neq 0$ [3]. Tracing through the same types of arguments as the above leads to the solution function as follows



$$u(x, t) = \frac{1}{2} (f(x+ct) + f(x-ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds \quad (4.1, 6)$$

V. CONCLUSIONS

The D'Alembert's approach to solve this wave equation, we can solving these equations without using Fourier series to build up the solution from the initial conditions. We are able to explicitly construct the solutions to this wave equation for any reasonable given initial boundary condition functions $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$. The wave equation alone doesn't specify the physical solution; an unique solution is usually obtained by setting this problem with another conditions, such as initial boundary conditions, which prescribe the amplitude and the phase of the wave. Another important class of these problems occurs in enclosed spaces specified by initial and other boundary conditions, for which the solutions represent standing waves or harmonics.

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